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INVESTIGATION OF TWO-DIMENSIONAL MODELS OF ELASTIC PRISMATIC SHELL

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Abstract. Statical and dynamical two-dimensional models of a prismatic elastic shell are constructed. The existence and uniqueness of solutions of the corresponding boundary and initial boundary value problems are proved, the rate of approximation of the solution of a three-dimensional problem by the vector-function restored from the solution of a two-dimensional one is estimated.

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1. Introduction

In mathematical physics and theory of elasticity one of the urgent issues is constructing and investigating lower-dimensional models. I. Vekua proposed one of the methods of constructing two-dimensional models of prismatic shells in [1]. It must be pointed out that in [1] boundary value problems are considered in C^k spaces and the convergence of sequences of approximations to exact solutions of the corresponding three-dimensional problems is not investigated. In the statical case the existence and uniqueness of a solution of the reduced two-dimensional problem in Sobolev spaces were investigated in [2] and the rate of approximation of an exact solution of a three-dimensional problem by the vector-function restored from the solution of the reduced problem in C^k spaces was estimated in [3]. Later, various types of lower-dimensional models were constructed and investigated in [4–18].

In the present paper we consider static equilibrium of a prismatic elastic shell and a dynamical problem of vibration of a shell. Due to I. Vekua's reduction method we construct a statical two-dimensional model of the plate and investigate the obtained boundary value problem. Moreover, if the solution of the original problem satisfies additional regularity properties, we estimate the accuracy of its approximation by the vector-function restored from the solution of a two-dimensional problem. We reduce the dynamical three-dimensional problem for a prismatic shell to the two-dimensional one, prove the existence and uniqueness of the solution of the corresponding initial boundary value problem and show that the vector-function restored from the latter problem approximates the solution of the original problem. Also, under the regularity conditions on the solution of the original problem we obtain the rate of its approximation.

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Let us consider a prismatic elastic shell of variable thickness and initial configuration $\overline{\Omega} \subset \mathbb{R}^3$; $x = (x_1, x_2, x_3) \in \overline{\Omega}$, $\overline{\Omega}$ denotes the closure of the domain $\Omega \subset \mathbb{R}^3$ (the domain is a bounded open connected set with a Lipschitz-continuous boundary, the set being locally on one side of the boundary [25]). Assume that the ruling of the lateral surface Γ of the plate is parallel to the Ox_3-axis ($Ox_1x_2x_3$ is a system of Cartesian coordinates in $\mathbb{R}^3$) and the upper Γ^+, lower Γ^- surfaces of the plate are defined by the equations $x_3 = h^+(x_1, x_2)$, $x_3 = h^-(x_1, x_2)$, $h^+(x_1, x_2) > h^-(x_1, x_2)$, $x_1, x_2 \in \bar{\omega}$, $h^-, h^+ \in C^1(\bar{\omega})$, where $\omega \subset \mathbb{R}^2$ is a domain with boundary γ. Let γ_0 denote the Lipschitz-continuous part of γ with positive length.

In order to simplify the notation, we assume that the indices i, j, p, q take their values in the set $\{1, 2, 3\}$, while the indices α, β vary in the set $\{1, 2\}$ and the summation convention with respect to repeated indices is used. Also, the partial derivative with respect to the p-th argument $\dfrac{\partial}{\partial x_p}$ is denoted by ∂_p. For any domain $D \subset \mathbb{R}^s$, $L^2(D)$ denote the space of square-integrable functions in D in the Lebesgue sense, $H^m(D) = W^{m,2}(D)$ denotes the Sobolev space of order m and the space of vector-functions denote with $\mathbf{H}^m(D) = [H^m(D)]^3$, $\mathbf{L}^2(D) = [L^2(D)]^3$, $s, m \in \mathbb{N}$.

Let us suppose that the material of the plate is elastic, homogeneous, isotropic and λ, μ are the Lamé constants of the material. The applied body force density is denoted by $\boldsymbol{f} = (f_i) : \Omega \times (0, T) \to \mathbb{R}^3$ and the surface force densities on the surfaces Γ^+, Γ^- by $\boldsymbol{g}^+$ and $\boldsymbol{g}^-$, respectively, $\boldsymbol{g}^\pm = (g_i^\pm) : \Gamma^\pm \times [0, T] \to \mathbb{R}^3$. The plate is clamped on the part $\Gamma^0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3;\ (x_1, x_2) \in \gamma_0,\ h^-(x_1, x_2) \le x_3 \le h^+(x_1, x_2)\}$ of its lateral surface Γ, while the surface $\Gamma^1 = \Gamma \backslash \Gamma^0$ is free. For the stress-strain state of the plate we have the following initial boundary value problem:

$$\frac{\partial^2 u_i}{\partial t^2} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \big\{\lambda e_{pp}(\boldsymbol{u}) \delta_{ij} + 2\mu e_{ij}(\boldsymbol{u})\big\} = f_i(x, t),\ (x, t) \in \Omega_T, \tag{1.1}$$

$$\boldsymbol{u}(x, 0) = \boldsymbol{\varphi}(x), \quad \frac{\partial \boldsymbol{u}}{\partial t}(x, 0) = \boldsymbol{\psi}(x), \quad x \in \Omega, \tag{1.2}$$

$$\boldsymbol{u} = 0 \quad \text{on} \quad \Gamma_T^0 = \Gamma^0 \times [0, T],$$

$$\sum_{j=1}^{3} \big(\lambda e_{pp}(\boldsymbol{u}) \delta_{ij} + 2\mu e_{ij}(\boldsymbol{u})\big) n_j = \begin{cases} g_i^+ & \text{on} \quad \Gamma_T^+ = \Gamma^+ \times [0, T], \\ g_i^- & \text{on} \quad \Gamma_T^- = \Gamma^- \times [0, T], \\ 0 & \text{on} \quad \Gamma_T^1 = \Gamma^1 \times [0, T], \end{cases} \tag{1.3}$$

where $\Omega_T = \Omega \times (0, T)$, $\boldsymbol{u} = (u_i) : \overline{\Omega}_T \to \mathbb{R}^3$ is the unknown displacement vector-function, $\boldsymbol{\varphi}, \boldsymbol{\psi} : \Omega \to \mathbb{R}^3$ are initial displacement and velocity of the plate, $\boldsymbol{n} = (n_j)$ denotes the unit outer normal vector along the boundary $\partial\Omega$. δ_{ij} is the Kronecker symbol and $\boldsymbol{e}(\boldsymbol{u}) = \{e_{ij}(\boldsymbol{u})\}$ is the deformation tensor

$$e_{ij}(\boldsymbol{u}) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i, j = 1, 2, 3.$$

In Section 2 we study the statical case of problem (1.1)–(1.3) and construct a statical two-dimensional model of the plate, investigate the convergence of the sequence of vector-functions restored from the solutions of the corresponding boundary value problems to the solution of the original three-dimensional problem. In Section 3 we consider problem (1.1)–(1.3) in the suitable functional spaces, construct and investigate a hierarchic dynamical two-dimensional model of a prismatic shell.

2. Statical Boundary Value Problem

As we have mentioned in the introduction, in this section we study the statical case of problem (1.1)–(1.3). In this case the latter problem admits the following variational formulation: find a vector-function $\boldsymbol{u} \in V(\Omega) = \{\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega),$ $\boldsymbol{v} = 0 \ \text{ on } \Gamma^0\}$, which satisfies the equation

$$B^{\Omega}(\boldsymbol{u}, \boldsymbol{v}) = L^{\Omega}(\boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in V(\Omega), \tag{2.1}$$

where

$$B^{\Omega}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} \big(\lambda e_{pp}(\boldsymbol{u}) e_{qq}(\boldsymbol{v}) + 2\mu e_{ij}(\boldsymbol{u}) e_{ij}(\boldsymbol{v})\big) dx,$$

$$L^{\Omega}(\boldsymbol{v}) = \int_{\Omega} f_i v_i dx + \int_{\Gamma^+} g_i^+ v_i d\Gamma + \int_{\Gamma^-} g_i^- v_i d\Gamma.$$

The variational problem (2.1) has a unique solution if $\lambda \geq 0$, $\mu > 0$, $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $\boldsymbol{g}^{\pm} \in \mathbf{L}^2(\Gamma^{\pm})$, which is a unique solution of the following minimization problem: find $\boldsymbol{u} \in V(\Omega)$ such that

$$J^{\Omega}(\boldsymbol{u}) = \inf_{\boldsymbol{v} \in V(\Omega} J^{\Omega}(\boldsymbol{v}), \quad J^{\Omega}(\boldsymbol{v}) = \frac{1}{2} B^{\Omega}(\boldsymbol{v}, \boldsymbol{v}) - L^{\Omega}(\boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in V(\Omega).$$

In order to reduce the three-dimensional problem (2.1) to a two-dimensional one, let us consider equation (2.1) on the subspace of $V(\Omega)$, which consists of polynomials of degree N with respect to the variable x_3, i.e.,

$$\boldsymbol{v}_N = \sum_{r=0}^{N} a\Big(r + \frac{1}{2}\Big) \overset{r}{\boldsymbol{v}} P_r(a x_3 - b), \quad \overset{r}{\boldsymbol{v}} = (\overset{r}{v}_i), \quad r = \overline{0, N},$$

where $a = \dfrac{2}{h^+ - h^-}$, $\quad b = \dfrac{h^+ + h^-}{h^+ - h^-}$, and P_r is a Legendre polynomial of degree r. Thus we get the following problem:

$$B^{\Omega}(\boldsymbol{w}_N, \boldsymbol{v}_N) = L^{\Omega}(\boldsymbol{v}_N), \quad \forall\, \boldsymbol{v}_N \in V_N(\Omega), \tag{2.2}$$

$$V_N(\Omega) = \Big\{\boldsymbol{v}_N = \sum_{r=0}^{N} a\Big(r + \frac{1}{2}\Big) \overset{r}{\boldsymbol{v}} P_r(a x_3 - b);$$

$$\overset{r}{\boldsymbol{v}} \in \mathbf{H}^1(\omega), \overset{r}{\boldsymbol{v}} = 0 \ \text{ on } \ \gamma_0, \quad r = \overline{0, N}\Big\}.$$

In problem (2.2) the unknown function $\boldsymbol{w}_N \in V_N(\Omega)$ is of the form

$$\boldsymbol{w}_N = \sum_{r=0}^{N} a\Big(r+\frac{1}{2}\Big) \overset{r}{\boldsymbol{w}} P_r(ax_3-b),$$

so we have to find the vector-function $\vec{w}_N = (\overset{0}{\boldsymbol{w}}, \dots, \overset{N}{\boldsymbol{w}})$,

$$\vec{w}_N \in \vec{V}_N(\omega) = \big\{\vec{v}_N = (\overset{0}{\boldsymbol{v}}, \dots, \overset{N}{\boldsymbol{v}});\ \overset{r}{\boldsymbol{v}}\in \mathbf{H}^1(\omega),\ \ \overset{r}{\boldsymbol{v}}= 0 \quad \text{on} \quad \gamma_0,\ \ r=\overline{0,N}\big\},$$

such that the corresponding $\boldsymbol{w}_N$ is a solution of problem (2.2).

Taking into account properties of the Legendre polynomials [19] equation (2.2) can be written in the following form

$$\begin{aligned}
&\sum_{r=0}^{N}\Big(r+\frac{1}{2}\Big)\int_\omega a\big(\lambda \overset{r}{\theta}(\vec{w}_N)\overset{r}{\theta}(\vec{v}_N) + 2\mu \overset{r}{e}_{ij}(\vec{w}_N)\overset{r}{e}_{ij}(\vec{v}_N)\big)dx_1dx_2 \\
&\quad = \sum_{r=0}^{N}\Big(r+\frac{1}{2}\Big)\int_\omega a \overset{r}{f}_i \overset{r}{v}_i\, dx_1dx_2 + \sum_{r=0}^{N}\Big(r+\frac{1}{2}\Big)\int_\omega a\tilde{g}_i^{+}k_+ \overset{r}{v}_i\, dx_1dx_2 \\
&\quad + \sum_{r=0}^{N}\Big(r+\frac{1}{2}\Big)\int_\omega a\tilde{g}_i^{-}k_-(-1)^r \overset{r}{v}_i\, dx_1dx_2, \quad \forall\, \vec{v}_N \in \vec{V}_N(\omega), \qquad (2.3)
\end{aligned}$$

where $\tilde{g}_i^{\pm}(x_1,x_2) = g_i^{\pm}(x_1,x_2,h^{\pm}(x_1,x_2))$, $(x_1,x_2)\in\omega$, $i=\overline{1,3}$,

$$\overset{r}{\theta}(\vec{v}_N) = \overset{r}{e}_{ii}(\vec{v}_N),$$

$$\overset{r}{e}_{ij}(\vec{v}_N) = \frac{1}{2}\Big(\frac{\partial\, v^r{}_i}{\partial x_j} + \frac{\partial\, v^r{}_j}{\partial x_i}\Big) + \frac{1}{2}\sum_{s=r}^{N}(\overset{r}{b}_{is}\overset{s}{v}_j + \overset{r}{b}_{js}\overset{s}{v}_i),$$

$$\overset{r}{b}_{\alpha r} = -(r+1)\frac{\partial_\alpha h^+ - \partial_\alpha h^-}{h^+-h^-}, \qquad \overset{r}{b}_{3r} = 0,$$

$$\overset{r}{b}_{js} = \begin{cases} 0, & s<r, \\ -(2s+1)\dfrac{\partial_\alpha h^+ - (-1)^{s+r}\partial_\alpha h^-}{h^+-h^-}, & j=\alpha,\ \ s>r, \\ (2s+1)\dfrac{1-(-1)^{s+r}}{h^+-h^-}, & j=3,\ \ s>r, \end{cases}$$

$$k_\pm = \sqrt{1+\Big(\frac{\partial h^\pm}{\partial x_1}\Big)^2 + \Big(\frac{\partial h^\pm}{\partial x_2}\Big)^2}, \quad \overset{r}{f}_i = \int_{h^-}^{h^+} f_i P_r(ax_3-b)dx_3, \quad r,s=\overline{0,N}.$$

Thus three-dimensional problem (2.1) we have reduced to two-dimensional one. For the last problem (2.3) we obtain the existence and uniqueness of its solution. First we prove the inequalities of Korn's type in this case.

Theorem 2.1. *Assume that $\omega\subset\mathbb{R}^2$ is a bounded domain with Lipschitz boundary $\gamma=\partial\omega$.*

a) *There exists a constant* $\alpha > 0$*, which depends only on* ω*, such that*

$$\sum_{r=0}^{N}\left(r+\frac{1}{2}\right)\int_\omega a \overset{r}{e}_{ij}(\vec{v}_N)\overset{r}{e}_{ij}(\vec{v}_N)dx_1dx_2+\sum_{r=0}^{N}\int_\omega \overset{r}{v}_i\overset{r}{v}_i\, dx_1dx_2 \geq \alpha\|\vec{v}_N\|^2_{\mathfrak{H}^1(\omega)},$$

for any $\vec{v}_N \in \mathfrak{H}^1(\omega)$*, where* $\mathfrak{H}^1(\omega) = [\mathbf{H}^1(\omega)]^{N+1}$.

b) *There exists a constant* $\alpha > 0$*, which depends only on* ω *and* γ_0*, such that*

$$\sum_{r=0}^{N}\left(r+\frac{1}{2}\right)\int_\omega a \overset{r}{e}_{ij}(\vec{v}_N)\overset{r}{e}_{ij}(\vec{v}_N)dx_1dx_2 \geq \alpha\|\vec{v}_N\|^2_{\mathfrak{H}^1(\omega)}, \quad \forall\, \vec{v}_N \in \vec{V}_N(\omega).$$

Proof. Let us introduce the space

$$E(\omega) = \left\{\vec{v}_N = (\overset{0}{\boldsymbol{v}},\dots,\overset{N}{\boldsymbol{v}}) \in \mathfrak{L}^2(\omega) = [\mathbf{L}^2(\omega)]^{N+1},\ \overset{r}{e}_{ij}(\vec{v}_N) \in L^2(\omega),\ \ r=\overline{0,N}\right\}.$$

Then, equipped with the norm $\|.\|$ defined by

$$\|\vec{v}_N\| = \left\{|\vec{v}_N|^2_{0,\omega} + |e(\vec{v}_N)|^2_{0,\omega}\right\}^{1/2},$$

where

$$|\vec{v}_N|^2_{0,\omega} = \sum_{r=0}^{N}\int_\omega \overset{r}{v}_i\overset{r}{v}_i\, dx_1dx_2,$$

$$|e(\vec{v}_N)|^2_{0,\omega} = \sum_{r=0}^{N}\left(r+\frac{1}{2}\right)\int_\omega a \overset{r}{e}_{ij}(\vec{v}_N)\overset{r}{e}_{ij}(\vec{v}_N)dx_1dx_2,$$

the space $E(\omega)$ is a Hilbert space. Indeed, let us consider the Cauchy sequence $\{\vec{v}_N^{(k)}\}_{k=1}^{\infty}$ in the space $E(\omega)$. By the definition of the norm $\|.\|$ there exists $\overset{r}{v}_i \in L^2(\omega)$ and $\overset{r}{e}_{ij} \in L^2(\omega)$ such that

$$\overset{r}{v}{}_i^{(k)} \to \overset{r}{v}_i,\quad \overset{r}{e}_{ij}(\vec{v}_N^{(k)}) \to \overset{r}{e}_{ij} \quad \text{in} \quad L^2(\omega),\ \text{ as } k\to\infty.$$

Moreover, for any $\varphi \in D(\omega)$ ($D(\omega)$ is a space of infinitely differentiable functions with compact support in ω) the following equality is valid:

$$\int_\omega \overset{r}{e}_{ij}(\vec{v}_N^{(k)})\varphi dx_1dx_2 = \frac{1}{2}\int_\omega\Big(-\overset{r}{v}{}_i^{(k)}\partial_j\varphi - \overset{r}{v}{}_j^{(k)}\partial_i\varphi$$

$$+\sum_{s=r}^{N}(\overset{r}{b}_{is}\overset{s}{v}{}_j^{(k)} + \overset{r}{b}_{js}\overset{s}{v}{}_i^{(k)})\varphi\Big)dx_1dx_2, \quad \forall\, k\in\mathbb{N}.$$

Hence, passing to the limit as $k\to\infty$, we obtain $\overset{r}{e}_{ij} = \overset{r}{e}_{ij}(\vec{v}_N)$.

Let us show that the spaces $E(\omega)$ and $\mathfrak{H}^1(\omega)$ are isomorphic. It is clear that $\mathfrak{H}^1(\omega) \subset E(\omega)$. Moreover, if we take $\vec{v}_N \in E(\omega)$, then for any $1 \leq i,j,p \leq 3$ and $r=\overline{0,N}$ we get

$$\partial_p \overset{r}{v}_i \in H^{-1}(\omega),$$

$$\partial_j(\partial_p \overset{r}{v}_i) = \partial_j \overset{r}{e}_{ip}(\vec{v}_N) + \partial_p \overset{r}{e}_{ij}(\vec{v}_N) - \partial_i \overset{r}{e}_{jp}(\vec{v}_N)$$

$$-\frac{1}{2}\sum_{s=r}^{N}\left(\partial_j(\overset{r}{b}_{is}\overset{s}{v}_p+\overset{r}{b}_{ps}\overset{s}{v}_i)+\partial_p(\overset{r}{b}_{is}\overset{s}{v}_j+\overset{r}{b}_{js}\overset{s}{v}_i)-\partial_i(\overset{r}{b}_{ps}\overset{s}{v}_j+\overset{r}{b}_{js}\overset{s}{v}_p)\right)\in H^{-1}(\omega)$$

since if $y\in L^2(\omega)$, then $\partial_p y\in H^{-1}(\omega)$. By virtue of the lemma of Lions [20–22] we have $\partial_p\overset{r}{v}_i\in L^2(\omega)$, and therefore the spaces $E(\omega)$ and $\mathfrak{H}^1(\omega)$ coincide.

To prove the item a) of the theorem note that the identity mapping from $\mathfrak{H}^1(\omega)$ into $E(\omega)$ is injective, continuous and, by virtue of the preceding result, is surjective. Since both spaces are complete, the closed graph theorem [23] shows that the inverse mapping is also continuous, which proves the desired inequality.

Now we will prove the item b) of the theorem. Notice that the semi-norm $|.|$ defined by $|\vec{v}_N|=|e(\vec{v}_N)|_{0,\omega}$ is the norm in the space $\vec{V}_N(\omega)$ when the measure of γ_0 is positive. Indeed, if $|e(\boldsymbol{\vec{v}}_N)|^2_{0,\omega}=0$, then [4]

$$\begin{aligned}
&\overset{0}{v}_1(x_1,x_2)=\frac{1}{a}\Big(b_3x_2+\frac{1}{2}(h^++h^-)b_1+c_1\Big),\\
&\overset{0}{v}_2(x_1,x_2)=\frac{1}{a}\Big(-b_3x_1+\frac{1}{2}(h^++h^-)b_2+c_2\Big),\qquad\qquad (x_1,x_2)\in\omega,\\
&\overset{0}{v}_3(x_1,x_2)=\frac{1}{a}\big(-b_1x_1-b_2x_2+c_3\big),\\
&\overset{1}{v}_1(x_1,x_2)=\frac{b_1}{3a^2},\ \ \overset{1}{v}_2(x_1,x_2)=\frac{b_2}{3a^2},\ \ \overset{1}{v}_3=0,\ \ \overset{r}{v}_i=0,\ \ i=\overline{1,3},\ \ r=\overline{2,N},
\end{aligned}$$

for any real constants b_1,b_2,b_3,c_1,c_2,c_3. Since $\vec{v}_N=0$ on γ_0 and the measure of γ_0 is positive, we get $\vec{v}_N=0$ on ω.

To prove the inequality of the item b) we argue by contradiction. Then there exists a sequence $\{\vec{v}_N^k\}_{k=1}^\infty$, $\vec{v}_N^k\in\vec{V}_N(\omega)$ such that $\|\vec{v}_N^k\|_{\mathfrak{H}^1(\omega)}=1$ for all $k\in\mathbb{N}$, $\lim\limits_{k\to\infty}|e(\vec{v}_N^k)|_{0,\omega}=0$. Since the sequence $\{\vec{v}_N^k\}_{k=1}^\infty$ is bounded in the space $\mathfrak{H}^1(\omega)$, a subsequence $\{\vec{v}_N^{k_l}\}_{l=1}^\infty$ converges in $\mathfrak{L}^2(\omega)$ by the Rellich–Kondrašov theorem. Each sequence $\{\overset{r}{e}_{ij}(\vec{v}_N^{k_l})\}_{l=1}^\infty$, $r=\overline{0,N}$, also converges in $L^2(\omega)$. The subsequence $\{\vec{v}_N^{k_l}\}_{l=1}^\infty$ is thus a Cauchy sequence with respect to the norm $\|.\|$.

According to the inequality of the item a) we have that the subsequence $\{\vec{v}_N^{k_l}\}_{l=1}^\infty$ is a Cauchy sequence with respect to the norm $\|.\|_{\mathfrak{H}^1(\omega)}$ too. The space $\vec{V}_N(\omega)$ is complete as a closed subspace of $\mathfrak{H}^1(\omega)$ and hence there exists $\vec{v}_N\in\vec{V}_N(\omega)$ such that

$$\vec{v}_N^{k_l}\to\vec{v}_N\quad\text{in}\quad\mathfrak{H}^1(\omega),\quad\text{as}\quad l\to\infty,$$

and $|e(\vec{v}_N)|_{0,\omega}=\lim\limits_{l\to\infty}|(\vec{v}_N^{k_l})|_{0,\omega}=0$. Therefore $\vec{v}_N=0$, which contradicts the relations $\|\vec{v}_N^{k_l}\|_{\mathfrak{H}^1(\omega)}=1$ for all $l\geq 1$. □

On the basis of Theorem 2.1 we prove the theorem on the existence and uniqueness of a solution of problem (2.3).

Theorem 2.2. *Assume that the Lamé constants* $\lambda \geq 0, \mu > 0, \boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $\boldsymbol{g}^{\pm} \in \mathbf{L}^2(\Gamma^{\pm})$, *then the symmetric bilinear form* $B_N^{\Omega} : \vec{V}_N(\omega) \times \vec{V}_N(\omega) \to \mathbb{R}$,

$$B_N^{\Omega}(\vec{u}_N, \vec{v}_N) = \sum_{r=0}^{N} \Big(r + \frac{1}{2}\Big) \int_{\omega} a\big(\lambda \overset{r}{\theta} (\vec{u}_N) \overset{r}{\theta} (\vec{v}_N) + 2\mu \overset{r}{e}_{ij} (\vec{u}_N) \overset{r}{e}_{ij} (\vec{v}_N)\big) dx_1 dx_2$$

is continuous and coercive, the linear form $L_N^{\Omega} : \vec{V}_N(\omega) \to \mathbb{R}$,

$$L_N^{\Omega}(\vec{v}_N) = \sum_{r=0}^{N} \Big(r + \frac{1}{2}\Big) \int_{\omega} a \overset{r}{f}_i \overset{r}{v}_i \, dx_1 dx_2 + \sum_{r=0}^{N} \Big(r + \frac{1}{2}\Big) \int_{\omega} a \tilde{g}_i^{+} k_{+} \overset{r}{v}_i \, dx_1 dx_2$$
$$+ \sum_{r=0}^{N} \Big(r + \frac{1}{2}\Big) \int_{\omega} a \tilde{g}_i^{-} k_{-} (-1)^r \overset{r}{v}_i \, dx_1 dx_2$$

is continuous. The two-dimensional problem (2.3) *has a unique solution* $\vec{w}_N \in \vec{V}_N(\omega)$, *which is also a unique solution to the following minimization problem:*

$$\vec{w}_N \in \vec{V}_N(\omega), \quad J_N(\vec{w}_N) = \inf_{\vec{v}_N \in \vec{V}_N(\omega)} J_N(\vec{v}_N),$$
$$J_N(\vec{v}_N) = \frac{1}{2} B_N^{\Omega}(\vec{v}_N, \vec{v}_N) - L_N^{\Omega}(\vec{v}_N), \quad \forall\, \vec{v}_N \in \vec{V}_N(\omega).$$

Proof. By the inequality of the item b) of Theorem 2.1, the bilinear form B_N^{Ω} is coercive

$$B_N^{\Omega}(\vec{v}_N, \vec{v}_N) \geq 2\mu \sum_{r=0}^{N} \Big(r + \frac{1}{2}\Big) \int_{\omega} a \overset{r}{e}_{ij} (\vec{v}_N) \overset{r}{e}_{ij} (\vec{v}_N) dx_1 dx_2 \geq 2\mu\alpha \|\vec{v}_N\|^2_{\mathfrak{H}^1(\omega)}.$$

Therefore, applying the Lax–Milgram theorem we obtain that problem (2.3) has a unique solution $\vec{w}_N$, which can be equivalently characterized as a solution of the minimization problem of the energy functional $J_N(\vec{v}_N)$. $\square$

Thus we have reduced the three-dimensional problem (2.1) to the two-dimensional one and for the latter problem proved the existence and uniqueness of its solution. For the reduced two-dimensional problem (2.3) the following theorem is true.

Theorem 2.3. *If all the conditions of Theorem* 2.2 *hold, then the vector-function* $\boldsymbol{w}_N = \sum_{r=0}^{N} a\Big(r + \frac{1}{2}\Big) \overset{r}{\boldsymbol{w}} P_r(ax_3 - b)$ *corresponding to the solution* $\vec{w}_N = (\overset{0}{\boldsymbol{w}}, \dots, \overset{N}{\boldsymbol{w}})$ *of the reduced problem* (2.3) *tends to the solution* $\boldsymbol{u}$ *of the three-dimensional problem* (2.1) $\boldsymbol{w}_N \to \boldsymbol{u}$ *in the space* $\mathbf{H}^1(\Omega)$ *as* $N \to \infty$. *Moreover, if* $\boldsymbol{u} \in \mathbf{H}^s(\Omega)$, $s \geq 2$, *then the following estimate is valid:*

$$\|\boldsymbol{u} - \boldsymbol{w}_N\|^2_{\mathbf{H}^1(\Omega)} \leq \frac{1}{N^{2s-3}} q_1(h^+, h^-, N),$$
$$q_1(h^+, h^-, N) \to 0 \quad as \quad N \to \infty. \tag{2.4}$$

If, additionally, $\|\boldsymbol{u}\|_{\mathbf{H}^s(\Omega)} \leq c$, *where* c *is independent of* $h = \max_{(x_1,x_2)\in\bar{\omega}} (h^+(x_1,x_2) - h^-(x_1,x_2))$, *then the following estimate holds:*

$$\|\boldsymbol{u}-\boldsymbol{w}_N\|^2_{E(\Omega)} \leq \frac{h^{2(s-1)}}{N^{2s-3}} q_2(N), \quad q_2(N) \to 0 \quad as \quad N \to \infty,$$

where $\|\boldsymbol{v}\|_{E(\Omega)} = \sqrt{B^{\Omega}(\boldsymbol{v},\boldsymbol{v})}, \quad \forall\, \boldsymbol{v} \in V(\Omega).$

Proof. By virtue of Theorem 2.2 $\vec{w}_N$ is a solution of the minimization problem of the energy functional J_N, i.e.

$$J_N(\vec{w}_N) = \frac{1}{2} B^{\Omega}_N(\vec{w}_N,\vec{w}_N) - L^{\Omega}_N(\vec{w}_N)$$
$$\leq J_N(\vec{v}_N) = \frac{1}{2} B^{\Omega}_N(\vec{v}_N,\vec{v}_N) - L^{\Omega}_N(\vec{v}_N), \;\; \forall\, \vec{v}_N \in \vec{V}_N(\omega). \tag{2.5}$$

Taking into account that

$$B^{\Omega}_N(\vec{v}_N,\vec{v}_N) = B^{\Omega}(\boldsymbol{v}_N,\boldsymbol{v}_N), \quad L^{\Omega}_N(\vec{v}_N) = L^{\Omega}(\boldsymbol{v}_N), \quad \forall\, \vec{v}_N \in \vec{V}_N(\omega),$$

where $\boldsymbol{v}_N = \sum_{r=0}^{N} a\Big(r+\frac{1}{2}\Big) \overset{r}{\boldsymbol{v}} P_r(ax_3-b)$, and applying (2.5), we obtain

$$B^{\Omega}(\boldsymbol{u}-\boldsymbol{w}_N,\boldsymbol{u}-\boldsymbol{w}_N) \leq B^{\Omega}(\boldsymbol{u},\boldsymbol{u}) - 2L^{\Omega}(\boldsymbol{v}_N) + B^{\Omega}(\boldsymbol{v}_N,\boldsymbol{v}_N).$$

From the latter inequality we have

$$B^{\Omega}(\boldsymbol{u}-\boldsymbol{w}_N,\boldsymbol{u}-\boldsymbol{w}_N) \leq B^{\Omega}(\boldsymbol{u}-\boldsymbol{v}_N,\boldsymbol{u}-\boldsymbol{v}_N), \quad \forall\, \boldsymbol{v}_N \in V_N(\Omega). \tag{2.6}$$

Since γ_0 is Lipschitz-continuous, by the trace theorems for Sobolev spaces [24], for any $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$, $\boldsymbol{v} = 0$ on Γ^0, there exists a continuation $\tilde{\boldsymbol{v}} \in \mathbf{H}^1_0(\Omega_1)$ of the function $\boldsymbol{v}$, where $\Omega_1 \supset \Omega$, $\partial\Omega_1 \supset \Gamma^0$. From the density of $D(\Omega_1)$ in $\mathbf{H}^1_0(\Omega_1)$ we obtain that the space of infinitely differentiable functions in Ω, which are equal to zero on Γ^0, is dense in $V(\Omega)$, and since the set of polynomials is dense in $L^2(-1,1)$, we conclude that $\bigcup_{N\geq 1} V_N(\Omega)$ is dense in $V(\Omega)$, and therefore $\boldsymbol{w}_N \to \boldsymbol{u}$ in the space $\mathbf{H}^1(\Omega)$ as $N \to \infty$.

Let us prove estimate (2.4). Suppose that $\boldsymbol{u} \in \mathbf{H}^s(\Omega)$, $s \geq 2$, then

$$\boldsymbol{\varepsilon}_N = \boldsymbol{u} - \boldsymbol{u}_N = \boldsymbol{u} - \sum_{r=0}^{N} a\Big(r+\frac{1}{2}\Big) \overset{r}{\boldsymbol{u}} P_r(ax_3-b), \quad \overset{r}{\boldsymbol{u}} = \int_{h^-}^{h^+} \boldsymbol{u} P_r(ax_3-b)dx_3.$$

Applying the properties of Legendre polynomials [19], we obtain

$$\|\boldsymbol{\varepsilon}_N\|^2_{\mathbf{L}^2(\Omega)} = \sum_{r=N+1}^{\infty} \int_{\omega} a\Big(r+\frac{1}{2}\Big)(\overset{r}{u}_i)^2 dx_1dx_2,$$

$$\Big\|\frac{\partial \boldsymbol{\varepsilon}_N}{\partial x_3}\Big\|^2_{\mathbf{L}^2(\Omega)} = \sum_{r=N}^{\infty} \int_{\omega} a\Big(r+\frac{1}{2}\Big)(\overset{r}{\partial_3 u_i})^2 dx_1dx_2 + \int_{\omega} a\frac{N(N-1)}{4}(\overset{N}{\partial_3 u_i})^2 dx_1dx_2$$
$$+ \int_{\omega} a\frac{N(N+1)}{4}(\overset{N+1}{\partial_3 u_i})^2 dx_1dx_2,$$

$$\Big\|\frac{\partial \boldsymbol{\varepsilon}_N}{\partial x_\alpha}\Big\|^2_{\mathbf{L}^2(\Omega} \le 2\Big(\sum_{r=N+1}^{\infty}\int_\omega a\Big(r+\frac{1}{2}\Big)(\partial_\alpha \overset{r}{u_i})^2 dx_1dx_2$$
$$+\int_\omega a\frac{N+1}{4}\Big(N\Big(\frac{\partial \tilde{h}}{\partial x_\alpha}\Big)^2+(N+2)\Big(\frac{\partial \bar{h}}{\partial x_\alpha}\Big)^2\Big)(\partial_3 \overset{N}{u_i})^2 dx_1dx_2$$
$$+\int_\omega a\frac{N+1}{4}\Big((N+2)\Big(\frac{\partial \tilde{h}}{\partial x_\alpha}\Big)^2+N\Big(\frac{\partial \bar{h}}{\partial x_\alpha}\Big)^2\Big)(\partial_3 \overset{N+1}{u_i})^2 dx_1dx_2\Big),$$

where $\alpha=1,2$, $\bar{h}=\frac{1}{2}(h^+ + h^-)$, $\tilde{h}=\frac{1}{2}(h^+ - h^-)$.

It should be mentioned that

$$\overset{r}{\boldsymbol{u}}=\int_{h^-}^{h^+}\boldsymbol{u}P_r(ax_3-b)dx_3=\frac{1}{a(2r+1)}\big(\overset{r-1}{\partial_3 \boldsymbol{u}}-\overset{r+1}{\partial_3 \boldsymbol{u}}\big),\quad r=1,2,\dots,N,$$

and thus we have

$$\|\overset{r}{\boldsymbol{u}}\|^2_{\mathbf{L}^2(\omega)}\le \frac{c}{r^{2s}}\sum_{k=r-s}^{r+s}\Big\|\frac{1}{a^s}\overset{k}{\underbrace{\partial_3\dots\partial_3}_{s}\boldsymbol{u}}\Big\|^2_{\mathbf{L}^2(\omega)},\tag{2.7}$$

where $c=const$ is independent of r,h^+,h^-. Therefore from (2.7) we get

$$\begin{aligned}\|\boldsymbol{\varepsilon}_N\|^2_{\mathbf{L}^2(\Omega)}&\le\frac{1}{N^{2s}}q(h^+,h^-,N),\\ \Big\|\frac{\partial\boldsymbol{\varepsilon}_N}{\partial x_i}\Big\|^2_{\mathbf{L}^2(\Omega)}&\le\frac{1}{N^{2s-3}}q(h^+,h^-,N),\end{aligned}\qquad q(h^+,h^-,N)\to 0\quad\text{as}\quad N\to\infty,$$

where $i=1,2,3$. Taking into account (2.6) and the coerciveness of B^Ω, we obtain

$$\|\boldsymbol{u}-\boldsymbol{w}_N\|^2_{\mathbf{H}^1(\Omega)}\le\frac{1}{N^{2s-3}}q_1(h^+,h^-,N), q_1(h^+,h^-,N)\to 0\quad\text{as}\quad N\to\infty.\tag{2.8}$$

From (2.7) we also have

$$\begin{aligned}\|\boldsymbol{\varepsilon}_N\|^2_{\mathbf{L}^2(\Omega)}&\le\frac{h^{2s}}{N^{2s}}\bar{q}(N),\\ \Big\|\frac{\partial\boldsymbol{\varepsilon}_N}{\partial x_i}\Big\|^2_{\mathbf{L}^2(\Omega)}&\le\frac{h^{2(s-1)}}{N^{2s-3}}\bar{q}(N),\end{aligned}\qquad \bar{q}(N)\to 0\quad\text{as}\quad N\to\infty,\ \ i=1,2,3,\tag{2.9}$$

where $h=\max\limits_{(x_1,x_2)\in\bar{\omega}}2\tilde{h}(x_1,x_2)$.

From inequalities (2.9) we obtain the second inequality of the theorem

$$\|\boldsymbol{u}-\boldsymbol{w}_N\|^2_{E(\Omega)}\le\frac{h^{2(s-1)}}{N^{2s-3}}q_2(N),\quad q_2(N)\to 0\quad\text{as}\quad N\to\infty.$$

3. Dynamical Initial Boundary Value Problem

Now we proceed to studying the dynamical problem (1.1)–(1.3), constructing and investigating a dynamical two-dimensional model of a prismatic shell. This problem admits the following variational formulation: find $\boldsymbol{u} \in C^0([0,T];V(\Omega))$, $\boldsymbol{u}' \in C^0([0,T];\mathbf{L}^2(\Omega))$, which satisfies the equation

$$\frac{d}{dt}(\boldsymbol{u}',\boldsymbol{v})_{\mathbf{L}^2(\Omega)} + B^\Omega(\boldsymbol{u},\boldsymbol{v}) = L^\Omega(\boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in V(\Omega) \tag{3.1}$$

in the sense of distributions in $(0,T)$ and the initial conditions

$$\boldsymbol{u}(0) = \boldsymbol{\varphi}, \quad \boldsymbol{u}'(0) = \boldsymbol{\psi}, \tag{3.2}$$

where $\boldsymbol{\varphi} \in V(\Omega)$, $\boldsymbol{\psi} \in \mathbf{L}^2(\Omega)$.

Note that the formulated three-dimensional dynamical problem (3.1), (3.2) has a unique solution $\boldsymbol{u}$ for $\lambda \geq 0, \mu > 0$, $\boldsymbol{f} \in \mathbf{L}^2(\Omega \times (0,T))$, $\boldsymbol{g}^\pm$, $\dfrac{\partial \boldsymbol{g}^\pm}{\partial t} \in \mathbf{L}^2(\Gamma^\pm \times (0,T))$, which satisfies the following energetical identity: $\forall\, t \in [0,T]$,

$$(\boldsymbol{u}'(t),\boldsymbol{u}'(t))_{\mathbf{L}^2(\Omega)} + B^\Omega(\boldsymbol{u}(t),\boldsymbol{u}(t)) = (\boldsymbol{\psi},\boldsymbol{\psi})_{\mathbf{L}^2(\Omega)} + B^\Omega(\boldsymbol{\varphi},\boldsymbol{\varphi}) + 2\tilde{L}^\Omega(\boldsymbol{u})(t),$$

where

$$\begin{aligned}\tilde{L}^\Omega(\boldsymbol{u})(t) = &\int_0^t \big(\boldsymbol{f}(\tau),\boldsymbol{u}'(\tau)\big)_{\mathbf{L}^2(\Omega)} d\tau + \big(\boldsymbol{g}^+(t),\boldsymbol{u}(t)\big)_{\mathbf{L}^2(\Gamma^+)} \\ &+ \big(\boldsymbol{g}^-(t),\boldsymbol{u}(t)\big)_{\mathbf{L}^2(\Gamma^-)} - \big(\boldsymbol{g}^+(0),\boldsymbol{u}(0)\big)_{\mathbf{L}^2(\Gamma^+)} - \big(\boldsymbol{g}^-(0),\boldsymbol{u}(0)\big)_{\mathbf{L}^2(\Gamma^-)} \\ &- \int_0^t \Big(\frac{\partial \boldsymbol{g}^+}{\partial t}(\tau),\boldsymbol{u}(\tau)\Big)_{\mathbf{L}^2(\Gamma^+)} d\tau - \int_0^t \Big(\frac{\partial \boldsymbol{g}^-}{\partial t}(\tau),\boldsymbol{u}(\tau)\Big)_{\mathbf{L}^2(\Gamma^-)} d\tau, \ \forall\, t \in [0,T].\end{aligned}$$

As in the statical case, to reduce the three-dimensional problem (3.1), (3.2) to a two-dimensional one, let us consider equation (3.1) on the subspace $V_N(\Omega)$ ($V_N(\Omega)$ is defined in Section 2), and choose $\boldsymbol{\varphi},\boldsymbol{\psi}$ as elements of $V_N(\Omega)$ and $H_N(\Omega)$, respectively, where

$$H_N(\Omega) = \Big\{\boldsymbol{v}_N = \sum_{k=0}^N a\Big(k+\frac{1}{2}\Big) \overset{k}{\boldsymbol{v}} P_k(ax_3-b);\ \overset{k}{\boldsymbol{v}} \in \mathbf{L}^2(\omega),\ k=0,\dots,N\Big\}.$$

Consequently, we consider the following variational problem: find $\boldsymbol{w}_N \in C^0([0,T];V_N(\Omega))$, $\boldsymbol{w}_N' \in C^0([0,T];H_N(\Omega))$, which satisfies the equation

$$\frac{d}{dt}(\boldsymbol{w}_N',\boldsymbol{v}_N)_{\mathbf{L}^2(\Omega)} + B^\Omega(\boldsymbol{w}_N,\boldsymbol{v}_N) = L^\Omega(\boldsymbol{v}_N), \quad \forall\, \boldsymbol{v}_N \in V_N(\Omega), \tag{3.3}$$

in the sense of distributions in $(0,T)$ and the initial conditions

$$\boldsymbol{w}_N(0) = \boldsymbol{\varphi}_N, \quad \boldsymbol{w}_N'(0) = \boldsymbol{\psi}_N, \tag{3.4}$$

where $\boldsymbol{\varphi}_N \in V_N(\Omega)$, $\boldsymbol{\psi}_N \in H_N(\Omega)$.

It must be pointed out that in problem (3.3), (3.4) the unknown is the vector-function $\boldsymbol{w}_N(t) = \sum\limits_{k=0}^N a\Big(k+\dfrac{1}{2}\Big) \overset{k}{\boldsymbol{w}}(t) P_k(ax_3-b)$ and so this problem is equivalent to the following one: find $\vec{w}_N = (\overset{0}{\boldsymbol{w}}, \dots, \overset{N}{\boldsymbol{w}}) \in C^0([0,T]; \vec{V}_N(\omega))$, $\vec{w}'_N \in C^0([0,T]; [\mathbf{L}^2(\omega)]^{N+1})$, which satisfies the equation

$$\frac{d}{dt}(P\vec{w}'_N, \vec{v}_N)_{[\mathbf{L}^2(\omega)]^{N+1}} + B^{\Omega}_N(\vec{w}_N, \vec{v}_N) = L^{\Omega}_N(\vec{v}_N), \quad \forall\, \vec{v}_N \in \vec{V}_N(\omega), \tag{3.5}$$

in the sense of distributions in $(0,T)$ and the initial condition

$$\vec{w}_N(0) = \vec{\varphi}_N, \quad \vec{w}'_N(0) = \vec{\psi}_N, \tag{3.6}$$

where $\vec{\varphi}_N = (\overset{0}{\boldsymbol{\varphi}}, \dots, \overset{N}{\boldsymbol{\varphi}}) \in \vec{V}_N(\omega)$, $\vec{\psi}_N = (\boldsymbol{\psi}^0, \dots, \boldsymbol{\psi}^N) \in [\mathbf{L}^2(\omega)]^{N+1}$,

$$\boldsymbol{\varphi}_N = \sum_{k=0}^{N} a\Big(k+\frac{1}{2}\Big) \overset{k}{\boldsymbol{\varphi}}\, P_k(ax_3 - b), \boldsymbol{\psi}_N = \sum_{k=0}^{N} a\Big(k+\frac{1}{2}\Big) \overset{k}{\boldsymbol{\psi}}\, P_k(ax_3 - b),$$

$$P = (P_{rl}),\ P_{rl} = a\Big(r+\frac{1}{2}\Big)\delta_{rl}, \quad r,l = 0,\dots,N,$$

and $B^{\Omega}_N, L^{\Omega}_N$ are defined in Section 2.

Thus we get a two-dimensional model of the prismatic shell. To investigate problem (3.5), (3.6) let us consider a more general variational problem and formulate the theorem on the existence and uniqueness of its solution, from which we obtain the corresponding result for (3.5), (3.6).

Let us suppose that V and H are Hilbert spaces, V is dense in H and continuously imbedded into it. The dual space of V is denoted by V' and H is identified with its dual with respect to the scalar product, then

$$V \hookrightarrow H \hookrightarrow V'$$

with continuous and dense imbeddings. The scalar product and the norm in the space V is denoted by $((.,.))$, $\|.\|$, while in the space H by $(.,.)$ and $|.|$. Denote the norm in the space V' by $\|.\|_*$, and the dual relation between the spaces V' and V by $\langle .,. \rangle$.

Assume that A, B, L are linear continuous operators such that

$$B = B_1 + B_2,\ B_1 \in L(V,V'),\ B_2 \in L(V,H) \cap L(H,V'),\ A, L \in L(H,H),$$

B_1 is self-adjoint and $B_1 + \lambda I$ is coercive for some real number λ, A is self-adjoint and coercive, i.e.,

$$\begin{aligned}
&\left.\begin{aligned} &b_1(u,v) = b_1(v,u),\ |b_1(u,v)| \le c_{b_1}\|u\|\|v\|,\\ &b_1(u,u) \ge \beta\|u\|^2 - \lambda|u|^2,\ \beta > 0, \end{aligned}\right\} \quad \forall\, u,v \in V,\\
&|b_2(\tilde{u},\tilde{v})| \le \begin{cases} c_{b_2}\|\tilde{u}\||\tilde{v}|, & \forall\, \tilde{u} \in V, \tilde{v} \in H,\\ c_{b_2}|\tilde{u}|\|\tilde{v}\|, & \forall\, \tilde{u} \in H, \tilde{v} \in V, \end{cases}\\
&\left.\begin{aligned} &a(u_1,v_1) = a(v_1,u_1),\ a(u_1,u_1) \ge \alpha|u_1|^2,\ \alpha > 0,\\ &|a(u_1,v_1)| \le c_a|u_1||v_1|, |l(u_1,v_1)| \le c_l|u_1||v_1|, \end{aligned}\right\} \quad \forall\, u_1,v_1 \in H,
\end{aligned} \tag{3.7}$$

where $b_1(u,v) = \langle B_1 u, v\rangle$, $b_2(u,v) = \langle B_2 u, v\rangle$, $l(u_1,v_1) = (Lu_1,v_1)$, $a(u_1,v_1) = (Au_1,v_1)$, $b(u,v) = b_1(u,v) + b_2(u,v)$, $\forall\, u,v \in V,\ u_1,v_1 \in H$.

Let us consider the following variational problem: find $z \in C^0([0,T];V), z' \in C^0([0,T];H)$, which satisfies the equation

$$\frac{d}{dt}a(z',v) + b(z,v) + l(z',v) = (F,v) + \langle G,v\rangle, \quad \forall\, v \in V, \tag{3.8}$$

in the sense of distributions in $(0,T)$ and the initial conditions

$$z(0) = z_0, \quad z'(0) = z_1, \tag{3.9}$$

where $z_0 \in V,\ z_1 \in H,\ F \in L^2(0,T;H),\ G, G' \in L^2(0,T;V')$.

For the latter problem the following theorem is true.

Theorem 3.1. *If all the conditions* (3.7) *hold, then problem* (3.8), (3.9) *has a unique solution which satisfies the equality*

$$\begin{aligned} a(z'(t),z'(t)) + b_1(z(t),z(t)) + 2\int_0^t b_2(z(\tau),z'(\tau))d\tau + 2\int_0^t l(z'(\tau),z'(\tau))d\tau \\ = a(z_1,z_1) + b_1(z_0,z_0) + 2\int_0^t (F(\tau),z'(\tau))d\tau + 2\langle G(t),z(t)\rangle - 2\langle G(0),z_0\rangle \\ -2\int_0^t \langle G'(\tau),z(\tau)\rangle d\tau, \quad \forall\, t \in [0,T]. \end{aligned}$$

In Theorem 3.1 the existence and uniqueness of the solution can be proved in a standard way applying the Faedo–Galerkin method [25], while the energetical equality is obtained by regularization and passing to the limit.

For the reduced problem (3.5), (3.6), from Theorem 3.1 we obtain

Theorem 3.2. *Assume that Lamé constants* $\lambda \geq 0, \mu > 0, \boldsymbol{f} \in \boldsymbol{L}^2(\Omega \times (0,T)),\ \boldsymbol{g}^{\pm}, \dfrac{\partial \boldsymbol{g}^{\pm}}{\partial t} \in \mathbf{L}^2(\Gamma^{\pm} \times (0,T)),\ \vec{\varphi}_N \in \vec{V}_N(\omega),\ \vec{\psi}_N \in [\mathbf{L}^2(\omega)]^{N+1}$, *then problem* (3.5), (3.6) *has a unique solution* $\vec{w}_N(t)$ *and the following energetical identity is valid:*

$$\begin{aligned} \big(\boldsymbol{w}'_N(t),\boldsymbol{w}'_N(t)\big)_{\mathbf{L}^2(\Omega)} + B^{\Omega}\big(\boldsymbol{w}_N(t),\boldsymbol{w}_N(t)\big) = (\boldsymbol{\psi}_N,\boldsymbol{\psi}_N)_{\mathbf{L}^2(\Omega)} \\ +B^{\Omega}(\boldsymbol{\varphi}_N,\boldsymbol{\varphi}_N) + 2\tilde{L}^{\Omega}(\boldsymbol{w}_N)(t), \quad \forall\, t \in [0,T]. \end{aligned} \tag{3.10}$$

Proof. This theorem is just a consequence of Theorem 3.1. Indeed, it is sufficient to take $V = \vec{V}_N(\omega)$, $H = [\mathbf{L}^2(\omega)]^{N+1}$, $z(t) = \vec{w}_N(t)$, $v = \vec{v}_N$,

$$a(\vec{w}_N,\vec{v}_N) = (P\vec{w}_N,\vec{v}_N)_{[\mathbf{L}^2(\omega)]^{N+1}}, \quad b_1(\vec{w}_N,\vec{v}_N) = B^{\Omega}_N(\vec{w}_N,\vec{v}_N),$$

$$b_2 \equiv 0, \quad l \equiv 0, \quad z_0 = \vec{\varphi}_N, \quad z_1 = \vec{\psi}_N,$$

$$\langle G,\vec{v}_N\rangle = (\vec{Q}^{+},\vec{v}_N)_{[\mathbf{L}^2(\omega)]^{N+1}} + (\vec{Q}^{-},\vec{v}_N)_{[\mathbf{L}^2(\omega)]^{N+1}}, \quad \forall\, \vec{v}_N \in \vec{V}_N(\omega),$$

$$\vec{Q}^{\pm} = (\overset{r}{\boldsymbol{Q}}{}^{\pm})_{r=0}^N, \quad \overset{r}{\boldsymbol{Q}}{}^{+} = a\Big(r+\frac{1}{2}\Big)\tilde{\boldsymbol{g}}^{+}k_{+}, \quad \overset{r}{\boldsymbol{Q}}{}^{-} = a(-1)^r\Big(r+\frac{1}{2}\Big)\tilde{\boldsymbol{g}}^{-}k_{-},$$

$$\tilde{\boldsymbol{g}}^{\pm}(x_1,x_2,t) = \boldsymbol{g}^{\pm}(x_1,x_2,h^{\pm}(x_1,x_2),t)\ ,\ (x_1,x_2,t) \in \omega \times (0,T),$$

$$F = P\vec{f}_N, \quad \vec{f}_N = (\overset{r}{\boldsymbol{f}})_{r=0}^N, \quad \overset{r}{\boldsymbol{f}} = \int_{h^-}^{h^+} \boldsymbol{f} P_r(ax_3 - b)dx_3, \quad r = \overline{0,N}.$$

It should be mentioned that all the conditions of Theorem 3.1 are valid and therefore problem (3.5), (3.6) has a unique solution which satisfies the energetical identity

$$\begin{aligned}
&\big(P\vec{w}_N'(t),\vec{w}_N'(t)\big)_{[\mathbf{L}^2(\omega)]^{N+1}}+B_N^{\Omega}\big(\vec{w}_N(t),\vec{w}_N(t)\big)=\big(P\vec{\psi}_N,\vec{\psi}_N\big)_{[\mathbf{L}^2(\omega)]^{N+1}}\\
&+2\int_0^t\big(P\vec{f}_N(\tau),\vec{w}_N'(\tau)\big)_{[\mathbf{L}^2(\omega)]^{N+1}}d\tau+2\big(\vec{Q}^+(t)+\vec{Q}^-(t),\vec{w}_N(t)\big)_{[\mathbf{L}^2(\omega)]^{N+1}}\\
&+B_N^{\Omega}(\vec{\varphi}_N,\vec{\varphi}_N)-2\big(\vec{Q}^+(0)+\vec{Q}^-(0),\vec{\varphi}_N\big)_{[\mathbf{L}^2(\omega)]^{N+1}}\\
&-2\int_0^t\Big(\frac{\partial\vec{Q}^+}{\partial t}(\tau)+\frac{\partial\vec{Q}^-}{\partial t}(\tau),\vec{w}_N(\tau)\Big)_{[\mathbf{L}^2(\omega)]^{N+1}}d\tau,\quad\forall\, t\in[0,T],
\end{aligned}$$

which is equivalent to identity (3.10). □

Thus we have reduced the three-dimensional problem (3.1), (3.2) to the two-dimensional one (3.5), (3.6) and proved that it has a unique solution. Now, let us estimate the rate of approximation of the exact solution $\boldsymbol{u}$ of the three-dimensional problem by the vector-function $\boldsymbol{w}_N(t)$ restored from the solution $\vec{w}_N(t)$ of the reduced problem. In order to simplify the notation, the norms in spaces $V(\Omega)$ and $\mathbf{L}^2(\Omega)$ are denoted by $\|.\|$ and $|.|$, respectively, and the scalar product in $\mathbf{L}^2(\Omega)$ is denoted by $(.,.)$. The following theorem is true.

Theorem 3.3. *If all the conditions of Theorem* 3.2 *are valid and* $\boldsymbol{\varphi}_N,\boldsymbol{\psi}_N$ *corresponding to* $\vec{\varphi}_N,\vec{\psi}_N$ *tend to* $\boldsymbol{\varphi},\boldsymbol{\psi}$ *in the spaces* $V(\Omega)$ *and* $\mathbf{L}^2(\Omega)$*, respectively, then the vector-function* $\boldsymbol{w}_N(t)$ *corresponding to the solution* $\vec{w}_N(t)=(\overset{0}{\boldsymbol{w}}(t),\dots,\overset{N}{\boldsymbol{w}}(t))$ *of the two-dimensional problem* (3.5), (3.6) *tends to the solution* $\boldsymbol{u}(t)$ *of the three-dimensional problem* (3.1), (3.2) *in the space* $V(\Omega)$

$$\left.\begin{aligned}&\boldsymbol{w}_N(t)\to\boldsymbol{u}(t)\quad\text{strongly in}\quad V(\Omega),\\&\boldsymbol{w}_N'(t)\to\boldsymbol{u}'(t)\quad\text{strongly in}\quad \mathbf{L}^2(\Omega)\end{aligned}\right.\quad\text{as}\quad N\to\infty,\quad\forall\, t\in[0,T].$$

Moreover, if components of $\vec{\varphi}_N,\vec{\psi}_N$ *are moments of* $\boldsymbol{\varphi},\boldsymbol{\psi}$ *with respect to Legendre polynomials, i.e.* $\vec{\varphi}_N=(\overset{0}{\boldsymbol{\varphi}},\dots,\overset{N}{\boldsymbol{\varphi}}),\vec{\psi}_N=(\overset{0}{\boldsymbol{\psi}},\dots,\overset{N}{\boldsymbol{\psi}})$,

$$\overset{k}{\boldsymbol{\varphi}}=\int_{h^-}^{h^+}\boldsymbol{\varphi}P_k(ax_3-b)dx_3,\quad \overset{k}{\boldsymbol{\psi}}=\int_{h^-}^{h^+}\boldsymbol{\psi}P_k(ax_3-b)dx_3,\quad k=\overline{0,N},$$

and $\boldsymbol{u}$ *satisfies additional regularity properties with respect to the spatial variables* $\boldsymbol{u}\in\mathbf{L}^2(0,T;\mathbf{H}^{s_0}(\Omega))$, $\boldsymbol{u}'\in\mathbf{L}^2(0,T;\mathbf{H}^{s_1}(\Omega))$, $\boldsymbol{u}''\in\mathbf{L}^2(0,T;\mathbf{H}^{s_2}(\Omega))$, $s_0\geq s_1\geq s_2\geq 1$, $s_1\geq 2$*, then the following estimate is valid:* $s=\min\{s_2,s_1-3/2\}$,

$$|\boldsymbol{u}'-\boldsymbol{w}_N'|^2+\|\boldsymbol{u}-\boldsymbol{w}_N\|^2\leq\frac{1}{N^{2s}}q(\Omega,\Gamma^0,N),\quad q(\Omega,\Gamma^0,N)\to 0\quad\text{as}\quad N\to\infty.$$

If additionally $\|\boldsymbol{u}\|_{L^2(0,T;\mathbf{H}^{s_0}(\Omega))}\leq c$, $\|\boldsymbol{u}'\|_{L^2(0,T;\mathbf{H}^{s_1}(\Omega))}\leq c$, $\|\boldsymbol{u}''\|_{L^2(0,T;\mathbf{H}^{s_2}(\Omega))}\leq c$, *where* c *is independent of* $h=\max\limits_{(x_1,x_2)\in\bar{\omega}}(h^+(x_1,x_2)-h^-(x_1,x_2))$*, then*

$$|\boldsymbol{u}'-\boldsymbol{w}_N'|^2+\|\boldsymbol{u}-\boldsymbol{w}_N\|^2\leq q_1(\Omega,\Gamma^0)\frac{h^{2\bar{s}}}{N^{2s}}q_2(N),\quad q_2(N)\to 0\quad\text{as}\quad N\to\infty,$$

where $\bar{s} = \min\{s_2, s_1 - 1\}$.

Proof. By virtue of Theorem 3.2 the vector-function $\boldsymbol{w}_N(t)$ corresponding to the solution $\vec{w}_N(t)$ of the reduced problem (3.5), (3.6) satisfies the energetical equality (3.10) and since $\boldsymbol{\varphi}_N \to \boldsymbol{\varphi}$ in $V(\Omega)$ and $\boldsymbol{\psi}_N \to \boldsymbol{\psi}$ in $\mathbf{L}^2(\Omega)$, we get: $\forall\, t \in [0,T]$,

$$|\boldsymbol{w}'_N(t)|^2 + \|\boldsymbol{w}_N(t)\|^2 \le c\bigg(|\boldsymbol{\psi}|^2 + \|\boldsymbol{\varphi}\|^2 + \int_0^t |\boldsymbol{f}(\tau)|^2 d\tau + \|\boldsymbol{g}^+(t)\|^2_{\mathbf{L}^2(\Gamma^+)}$$
$$+\|\boldsymbol{g}^-(t)\|^2_{\mathbf{L}^2(\Gamma^-)} + \|\boldsymbol{g}^+(0)\|^2_{\mathbf{L}^2(\Gamma^+)} + \|\boldsymbol{g}^-(0)\|^2_{\mathbf{L}^2(\Gamma^-)} + \int_0^t \Big\|\frac{\partial \boldsymbol{g}^+}{\partial t}(\tau)\Big\|^2_{\mathbf{L}^2(\Gamma^+)} d\tau$$
$$+ \int_0^t \Big\|\frac{\partial \boldsymbol{g}^-}{\partial t}(\tau)\Big\|^2_{\mathbf{L}^2(\Gamma^-)} d\tau\bigg) + c\int_0^t \big(|\boldsymbol{w}'_N(\tau)|^2 + \|\boldsymbol{w}_N(\tau)\|^2\big) d\tau.$$

Applying Gronwall's lemma [26], from the latter inequality we have

$$|\boldsymbol{w}'_N(t)|^2 + \|\boldsymbol{w}_N(t)\|^2 < c_1, \quad \forall\, N \in \mathbb{N}, \ \ t \in [0,T]. \tag{3.11}$$

Hence $\{\boldsymbol{w}_N(t)\}$ belongs to the bounded set of $L^\infty(0,T;V(\Omega)) \cap L^2(0,T;V(\Omega))$, while $\{\boldsymbol{w}'_N(t)\}$ belongs to the bounded set of $L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{L}^2(\Omega))$. Consequently, there exists a subsequence $\{\boldsymbol{w}_\nu\}$ of $\{\boldsymbol{w}_N\}$ such that when $\nu \to \infty$,

$$\begin{aligned} &\boldsymbol{w}_\nu \to \tilde{\boldsymbol{u}} \ \text{ weakly in } \ L^2(0,T;V(\Omega)), \ \ \text{weakly-}* \text{ in } \ L^\infty(0,T;V(\Omega)), \\ &\boldsymbol{w}'_\nu \to \tilde{\boldsymbol{u}}' \text{ weakly in } L^2(0,T;\mathbf{L}^2(\Omega)), \ \ \text{weakly-}* \text{ in } \ L^\infty(0,T;\mathbf{L}^2(\Omega)). \end{aligned} \tag{3.12}$$

Let us prove that $\tilde{\boldsymbol{u}}$ is a solution of problem (3.1), (3.2). Since $\bigcup\limits_{N\ge 0} V_N(\Omega)$ is dense in $V(\Omega)$, for any $\boldsymbol{v} \in V(\Omega)$ there exists $\boldsymbol{v}_N \in V_N(\Omega)$ such that $\boldsymbol{v}_N \to \boldsymbol{v}$ in $V(\Omega)$ as $N \to \infty$. For any $\zeta \in D((0,T))$, $\boldsymbol{\theta} = \zeta \boldsymbol{v}$, $\boldsymbol{\theta}_N = \zeta \boldsymbol{v}_N$, we have

$$\begin{aligned} &\boldsymbol{\theta}_N \to \boldsymbol{\theta} \quad \text{strongly in} \quad L^2(0,T;V(\Omega)), \\ &\boldsymbol{\theta}'_N \to \boldsymbol{\theta}' \quad \text{strongly in} \quad L^2(0,T;V(\Omega)) \end{aligned} \quad \text{as} \quad N \to \infty. \tag{3.13}$$

Taking into account (3.13), from (3.3) we obtain

$$\int_0^T B^\Omega(\boldsymbol{w}_\nu(t), \boldsymbol{\theta}_\nu(t))dt - \int_0^T (\boldsymbol{w}'_\nu(t), \boldsymbol{\theta}'_\nu(t))dt = \int_0^T (\boldsymbol{f}(t), \boldsymbol{\theta}_\nu(t))dt$$
$$+ \int_0^T (\boldsymbol{g}^+(t), \boldsymbol{\theta}_\nu(t))_{\mathbf{L}^2(\Gamma^+)} dt + \int_0^T (\boldsymbol{g}^-(t), \boldsymbol{\theta}_\nu(t))_{\mathbf{L}^2(\Gamma^-)} dt.$$

Passing to the limit as $\nu \to \infty$ and applying (3.12), (3.13) from this equality we obtain

$$\int_0^T B^\Omega(\tilde{\boldsymbol{u}}(t), \boldsymbol{\theta}(t))dt - \int_0^T (\tilde{\boldsymbol{u}}'(t), \boldsymbol{\theta}'(t))dt = \int_0^T (\boldsymbol{f}(t), \boldsymbol{\theta}(t))dt$$
$$+ \int_0^T (\boldsymbol{g}^+(t), \boldsymbol{\theta}(t))_{\mathbf{L}^2(\Gamma^+)} dt + \int_0^T (\boldsymbol{g}^-(t), \boldsymbol{\theta}(t))_{\mathbf{L}^2(\Gamma^-)} dt.$$

Hence, $\tilde{\boldsymbol{u}} \in L^\infty(0,T;V(\Omega))$, $\tilde{\boldsymbol{u}}' \in L^\infty(0,T;\mathbf{L}^2(\Omega))$ and satisfies equation (3.1).

Therefore $\tilde{\boldsymbol{u}} \in C^0([0,T];\mathbf{L}^2(\Omega)) \cap L^\infty(0,T;V(\Omega))$, $\tilde{\boldsymbol{u}}' \in C^0([0,T];V'(\Omega)) \cap L^\infty(0,T;\mathbf{L}^2(\Omega))$ ($V'(\Omega)$ denotes the dual space of $V(\Omega)$) and consequently

$\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{u}}'$ are scalarly continuous functions from $[0,T]$ to spaces $V(\Omega)$ and $\mathbf{L}^2(\Omega)$, respectively [25]. Note that $\tilde{\boldsymbol{u}}$ satisfies the energetical identity, from which we obtain the continuity of $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{u}}'$, $\tilde{\boldsymbol{u}} \in C^0([0,T];V(\Omega))$, $\tilde{\boldsymbol{u}}' \in C^0([0,T];\mathbf{L}^2(\Omega))$, and as $\boldsymbol{\varphi}_N \to \boldsymbol{\varphi}$ in V, $\boldsymbol{\psi}_N \to \boldsymbol{\psi}$ in $\mathbf{L}^2(\Omega)$, $\tilde{\boldsymbol{u}}$ satisfies the initial conditions (3.2).

Since problem (3.1), (3.2) has a unique solution $\boldsymbol{u}$, we get that $\tilde{\boldsymbol{u}} = \boldsymbol{u}$ and the whole sequence $\boldsymbol{w}_N$ has property (3.12), i.e. when $N \to \infty$,

$$\begin{aligned} &\boldsymbol{w}_N \to \boldsymbol{u} \ \text{ weakly in } L^2(0,T;V(\Omega)), \ \text{ weakly-}* \text{ in } L^\infty(0,T;V(\Omega)), \\ &\boldsymbol{w}_N' \to \boldsymbol{u}' \ \text{ weakly in } L^2(0,T;\mathbf{L}^2(\Omega)), \ \text{weakly-}* \text{ in } L^\infty(0,T;\mathbf{L}^2(\Omega)). \end{aligned} \tag{3.14}$$

Let us prove that $\{\boldsymbol{w}_N\}$ satisfies the convergence properties formulated in the theorem. Applying the energetical equalities for $\boldsymbol{u}(t)$ and $\boldsymbol{w}_N(t)$, we obtain the following equality for the difference $\boldsymbol{\delta}_N(t) = \boldsymbol{u}(t) - \boldsymbol{w}_N(t)$:

$$\begin{aligned} &\big(\boldsymbol{\delta}_N'(t), \boldsymbol{\delta}_N'(t)\big) + B^\Omega\big(\boldsymbol{\delta}_N(t), \boldsymbol{\delta}_N(t)\big) + 2B^\Omega\big(\boldsymbol{u}(t), \boldsymbol{w}_N(t)\big) \\ &+ 2\big(\boldsymbol{u}'(t), \boldsymbol{w}_N'(t)\big) = \big(\boldsymbol{\delta}_N'(0), \boldsymbol{\delta}_N'(0)\big) + B^\Omega(\boldsymbol{\delta}_N(0), \boldsymbol{\delta}_N(0)) \\ &+ 2\big(\boldsymbol{u}'(0), \boldsymbol{w}_N'(0)\big) + 2B^\Omega(\boldsymbol{u}(0), \boldsymbol{w}_N(0)) + 2\tilde{L}^\Omega(\boldsymbol{\delta}_N)(t) + 4\tilde{L}^\Omega(\boldsymbol{w}_N)(t). \end{aligned} \tag{3.15}$$

Denote

$$\begin{aligned} J_N(t) = \big(\boldsymbol{u}'(0), \boldsymbol{w}_N'(0)\big) &+ B^\Omega\big(\boldsymbol{u}(0), \boldsymbol{w}_N(0)\big) - B^\Omega(\boldsymbol{u}(t), \boldsymbol{w}_N(t)) \\ &- \big(\boldsymbol{u}'(t), \boldsymbol{w}_N'(t)\big) + 2\tilde{L}^\Omega(\boldsymbol{w}_N)(t). \end{aligned}$$

Then from (3.15) we obtain

$$\begin{aligned} \big(\boldsymbol{\delta}_N'(t), \boldsymbol{\delta}_N'(t)\big) &+ B^\Omega\big(\boldsymbol{\delta}_N(t), \boldsymbol{\delta}_N(t)\big) = \big(\boldsymbol{\delta}_N'(0), \boldsymbol{\delta}_N'(0)\big) \\ &+B^\Omega\big(\boldsymbol{\delta}_N(0), \boldsymbol{\delta}_N(0)\big) + 2\tilde{L}^\Omega(\boldsymbol{\delta}_N)(t) + 2J_N(t). \end{aligned} \tag{3.16}$$

From (3.11) it follows that for each $t \in (0,T]$ there exists a sequence $\{\boldsymbol{w}_{\nu_1}(t)\}$ such that

$$\begin{aligned} \boldsymbol{w}_{\nu_1}(t) &\to \boldsymbol{\chi}_1 \ \text{ weakly in } \ V(\Omega), \\ \boldsymbol{w}_{\nu_1}'(t) &\to \boldsymbol{\chi}_2 \ \text{ weakly in } \ \mathbf{L}^2(\Omega) \end{aligned} \qquad \text{as } \ \nu_1 \to \infty. \tag{3.17}$$

Let us take $\zeta \in C^1([0,T])$, $\zeta(0) = 0, \zeta(t) \neq 0$ and consider the above-mentioned vector-functions $\boldsymbol{\theta} = \zeta \boldsymbol{v}, \boldsymbol{\theta}_N = \zeta \boldsymbol{v}_N$, which also have property (3.13). By the integration by parts we have

$$\int_0^t (\boldsymbol{w}_{\nu_1}'(\tau), \boldsymbol{\theta}_{\nu_1}(\tau))d\tau = (\boldsymbol{w}_{\nu_1}(t), \boldsymbol{\theta}_{\nu_1}(t)) - \int_0^t (\boldsymbol{w}_{\nu_1}(\tau), \boldsymbol{\theta}_{\nu_1}'(\tau))d\tau.$$

Applying (3.14), (3.17) and passing to the limit as $\nu_1 \to \infty$, from the latter equality we obtain

$$\int_0^t (\boldsymbol{u}'(\tau), \boldsymbol{\theta}(\tau))d\tau = (\boldsymbol{\chi}_1, \boldsymbol{\theta}(t)) - \int_0^t (\boldsymbol{u}(\tau), \boldsymbol{\theta}'(\tau))d\tau.$$

On the other hand,

$$\int_0^t (\boldsymbol{u}'(\tau), \boldsymbol{\theta}(\tau))d\tau = (\boldsymbol{u}(t), \boldsymbol{\theta}(t)) - \int_0^t (\boldsymbol{u}(\tau), \boldsymbol{\theta}'(\tau))d\tau.$$

From the latter equalities we have $(\boldsymbol{u}(t),\boldsymbol{v})=(\boldsymbol{\chi}_1,\boldsymbol{v}),\ \forall\,\boldsymbol{v}\in V(\Omega)$, and therefore $\boldsymbol{\chi}_1=\boldsymbol{u}(t)$.

Hence since $\boldsymbol{\chi}_1$ is unique and is equal to $\boldsymbol{u}(t)$, then for any fixed t the whole sequence $\{\boldsymbol{w}_N(t)\}$ converges to $\boldsymbol{u}(t)$ weakly in $V(\Omega)$.

Also, since $\boldsymbol{w}_N$ satisfies equation (3.3), we have

$$\begin{aligned}-\int_0^t(\boldsymbol{w}'_{\nu_1}(\tau),\boldsymbol{\theta}'_{\nu_1}(\tau))d\tau+\int_0^t B^\Omega(\boldsymbol{w}_{\nu_1}(\tau),\boldsymbol{\theta}_{\nu_1}(\tau))d\tau&\\=-(\boldsymbol{w}'_{\nu_1}(t),\boldsymbol{\theta}_{\nu_1}(t))+\int_0^t L^\Omega(\boldsymbol{\theta}_{\nu_1}(\tau))d\tau,&\end{aligned}$$

from which, passing to the limit as $\nu_1\to\infty$ and applying (3.14), (3.17), we obtain

$$-\int_0^t(\boldsymbol{u}'(\tau),\boldsymbol{\theta}'(\tau))d\tau+\int_0^t B^\Omega(\boldsymbol{u}(\tau),\boldsymbol{\theta}(\tau))d\tau=-(\boldsymbol{\chi}_2,\boldsymbol{\theta}(t))+\int_0^t L^\Omega(\boldsymbol{\theta}(\tau))d\tau.$$

Since $\boldsymbol{u}$ is a solution of problem (3.1), (3.2), we get

$$-\int_0^t(\boldsymbol{u}'(\tau),\boldsymbol{\theta}'(\tau))d\tau+\int_0^t B^\Omega(\boldsymbol{u}(\tau),\boldsymbol{\theta}(\tau))d\tau=-(\boldsymbol{u}'(t),\boldsymbol{\theta}(t))+\int_0^t L^\Omega(\boldsymbol{\theta}(\tau))d\tau,$$

and therefore

$$(\boldsymbol{u}'(t),\boldsymbol{v})=(\boldsymbol{\chi}_2,\boldsymbol{v}),\quad \forall\,\boldsymbol{v}\in V(\Omega).$$

From the density of $V(\Omega)$ in $\mathbf{L}^2(\Omega)$, it follows that $\boldsymbol{\chi}_2=\boldsymbol{u}'(t)$. As in the case of $\boldsymbol{w}_{\nu_1}(t)$, we obtain that the whole sequence $\boldsymbol{w}'_N(t)$ converges to $\boldsymbol{u}'(t)$ weakly in $\mathbf{L}^2(\Omega)$. Thus, for any $t\in[0,T]$,

$$\begin{matrix}\boldsymbol{w}_N(t)\to\boldsymbol{u}(t) & \text{weakly in} & V(\Omega),\\ \boldsymbol{w}'_N(t)\to\boldsymbol{u}'(t) & \text{weakly in} & \mathbf{L}^2(\Omega)\end{matrix}\qquad \text{as}\quad N\to\infty.$$

Therefore passing to the limit in $J_N(t)$ as $N\to\infty$, by the energetical identity we obtain

$$\begin{aligned}J_N(t)\to\big(\boldsymbol{u}'(0),\boldsymbol{u}'(0)\big)+B^\Omega\big(\boldsymbol{u}(0),\boldsymbol{u}(0)\big)+2\tilde{L}^\Omega\big(\boldsymbol{u}\big)(t)&\\-\big(\boldsymbol{u}'(t),\boldsymbol{u}'(t)\big)-B^\Omega\big(\boldsymbol{u}(t),\boldsymbol{u}(t)\big)=0.&\end{aligned}\tag{3.18}$$

So, from (3.16) we have

$$\begin{aligned}|\boldsymbol{\delta}'_N(t)|^2+\|\boldsymbol{\delta}_N(t)\|^2\le c\Big(2J_N(t)+\big(\boldsymbol{\delta}'_N(0),\boldsymbol{\delta}'_N(0)\big)&\\+B^\Omega\big(\boldsymbol{\delta}_N(0),\boldsymbol{\delta}_N(0)\big)+2\tilde{L}^\Omega(\boldsymbol{\delta}_N)(t)\Big).&\end{aligned}\tag{3.19}$$

From the conditions of the theorem we have that $\boldsymbol{\delta}_N(0)\to 0$ strongly in $V(\Omega)$ and $\boldsymbol{\delta}'_N(0)\to 0$ strongly in $\mathbf{L}^2(\Omega)$. Thus due to (3.14), (3.18) we obtain

$$\big(\boldsymbol{\delta}'_N(0),\boldsymbol{\delta}'_N(0)\big)+B^\Omega(\boldsymbol{\delta}_N(0),\boldsymbol{\delta}_N(0))+2J_N(t)+2\tilde{L}^\Omega(\boldsymbol{\delta}_N)(t)\to 0\quad\text{as}\quad N\to\infty.$$

Therefore from (3.19) we have

$$|\boldsymbol{\delta}'_N(t)|^2+\|\boldsymbol{\delta}_N(t)\|^2\to 0\quad\text{as}\quad N\to\infty.$$

Consequently,

$$\begin{array}{lll} \boldsymbol{w}_N(t) \to \boldsymbol{u}(t) & \text{strongly in} & V(\Omega), \\ \boldsymbol{w}'_N(t) \to \boldsymbol{u}'(t) & \text{strongly in} & \mathbf{L}^2(\Omega) \end{array} \quad \text{as} \quad N \to \infty, \quad \forall\, t \in [0,T].$$

Now we will prove the validity of the estimates of the theorem. The solution $\boldsymbol{u}$ of the three-dimensional problem satisfies equation (3.1) for any $\boldsymbol{v} \in V(\Omega)$ and hence satisfies it for any $\boldsymbol{v}_N \in V_N(\Omega) \subset V(\Omega)$, i.e.

$$\frac{d}{dt}(\boldsymbol{u}', \boldsymbol{v}_N) + B^\Omega(\boldsymbol{u}, \boldsymbol{v}_N) = L^\Omega(\boldsymbol{v}_N), \quad \forall\, \boldsymbol{v}_N \in V_N(\Omega).$$

Thus, if we take into account that $\boldsymbol{w}_N$ corresponds to the solution $\vec{w}_N$ of the reduced two-dimensional problem (3.5), (3.6), then from the latter equation we have

$$\frac{d}{dt}\big((\boldsymbol{u} - \boldsymbol{w}_N)', \boldsymbol{v}_N\big) + B^\Omega(\boldsymbol{u} - \boldsymbol{w}_N, \boldsymbol{v}_N) = 0, \quad \forall\, \boldsymbol{v}_N \in V_N(\Omega).$$

Let $\boldsymbol{u} \in L^2(0,T;\mathbf{H}^{s_0}(\Omega))$, $\boldsymbol{u}' \in L^2(0,T;\mathbf{H}^{s_1}(\Omega))$, $\boldsymbol{u}'' \in L^2(0,T;\mathbf{H}^{s_2}(\Omega))$, $s_0 \geq s_1 \geq s_2 \geq 1$, $s_1 \geq 2$. Let us consider the series expansion of the vector-function $\boldsymbol{u}$ by Legendre polynomials with respect to x_3. Denote by $\boldsymbol{u}_N$ the piece of series consisting of first $N+1$ terms, and by $\boldsymbol{\alpha}_N$, the remainder term, i.e.

$$\boldsymbol{u} = \boldsymbol{u}_N + \boldsymbol{\alpha}_N = \sum_{k=0}^{N} a\Big(k + \frac{1}{2}\Big) \overset{k}{\boldsymbol{u}}\, P_k(ax_3 - b) + \boldsymbol{\alpha}_N, \overset{k}{\boldsymbol{u}} = \int_{h^-}^{h^+} \boldsymbol{u} P_k(ax_3 - b)dx_3,$$

$k = \overline{0,N}$. Let us take initial conditions $\vec{\varphi}_N$, $\vec{\psi}_N$ of the reduced problem (3.5), (3.6) such that $\vec{\varphi}_N = (\overset{0}{\boldsymbol{\varphi}}, \dots, \overset{N}{\boldsymbol{\varphi}})$, $\vec{\psi}_N = (\overset{0}{\boldsymbol{\psi}}, \dots, \overset{N}{\boldsymbol{\psi}})$,

$$\overset{k}{\boldsymbol{\varphi}} = \int_{h^-}^{h^+} \boldsymbol{\varphi} P_k(ax_3 - b)dx_3, \quad \overset{k}{\boldsymbol{\psi}} = \int_{h^-}^{h^+} \boldsymbol{\psi} P_k(ax_3 - b)dx_3, \quad k = \overline{0,N}.$$

Thus the vector-function $\boldsymbol{\Delta}_N = \boldsymbol{u}_N - \boldsymbol{w}_N$ is a solution of the problem

$$\begin{gathered} \frac{d}{dt}(\boldsymbol{\Delta}'_N, \boldsymbol{v}_N) + B^\Omega(\boldsymbol{\Delta}_N, \boldsymbol{v}_N) = \big((\boldsymbol{\alpha}''_N, \boldsymbol{v}_N) + B^\Omega(\boldsymbol{\alpha}_N, \boldsymbol{v}_N)\big)\ \forall \boldsymbol{v}_N \in V_N(\Omega), \\ \boldsymbol{\Delta}_N(0) = \boldsymbol{u}_N(0) - \boldsymbol{\varphi}_N = 0, \boldsymbol{\Delta}'_N(0) = \boldsymbol{u}'_N(0) - \boldsymbol{\psi}_N = 0. \end{gathered} \tag{3.20}$$

Applying Theorem 3.1 to problem (3.20), we obtain

$$\begin{aligned} \big(\boldsymbol{\Delta}'_N(t), \boldsymbol{\Delta}'_N(t)\big) + B^\Omega\big(\boldsymbol{\Delta}_N(t), \boldsymbol{\Delta}_N(t)\big) &= -2\int_0^t \big(\boldsymbol{\alpha}''_N(\tau), \boldsymbol{\Delta}'_N(\tau)\big)d\tau \\ -2B^\Omega\big(\boldsymbol{\alpha}_N(t), \boldsymbol{\Delta}_N(t)\big) &+ 2\int_0^t B^\Omega\big(\boldsymbol{\alpha}'_N(\tau), \boldsymbol{\Delta}_N(\tau)\big)d\tau, \quad \forall\, t \in [0,T]. \end{aligned}$$

From the latter equality we get

$$\begin{aligned} |\boldsymbol{\Delta}'_N(t)|^2 + \|\boldsymbol{\Delta}_N(t)\|^2 \leq c\int_0^t \big(|\boldsymbol{\Delta}'_N(\tau)|^2 + \|\boldsymbol{\Delta}_N(\tau)\|^2\big)d\tau + c\Big(\int_0^t |\boldsymbol{\alpha}''_N(\tau)|^2 d\tau \\ + \frac{1}{2\varepsilon}\|\boldsymbol{\alpha}_N(t)\|^2 + \frac{\varepsilon}{2}\|\boldsymbol{\Delta}_N(t)\|^2 + \int_0^t \|\boldsymbol{\alpha}'_N(\tau)\|^2 d\tau\Big), \end{aligned}$$

where c depends only on λ, μ, Ω and Γ^0.

So by taking ε properly we have

$$|\boldsymbol{\Delta}'_N(t)|^2+\|\boldsymbol{\Delta}_N(t)\|^2\le c_1\bigg(\int_0^t\big(|\boldsymbol{\Delta}'_N(\tau)|^2+\|\boldsymbol{\Delta}_N(\tau)\|^2\big)d\tau+\int_0^t|\boldsymbol{\alpha}''_N(\tau)|^2d\tau+\|\boldsymbol{\alpha}_N(t)\|^2+\int_0^t\|\boldsymbol{\alpha}'_N(\tau)\|^2d\tau\bigg). \tag{3.21}$$

By Gronwall's lemma, from (3.21) we obtain that for any $t\in[0,T]$,

$$|\boldsymbol{\Delta}'_N(t)|^2+\|\boldsymbol{\Delta}_N(t)\|^2\le c_2\bigg(\int_0^t|\boldsymbol{\alpha}''_N(\tau)|^2d\tau+\|\boldsymbol{\alpha}_N(t)\|^2+\int_0^t\|\boldsymbol{\alpha}'_N(\tau)\|^2d\tau\bigg).$$

As in the proof of Theorem 2.2 we can show that

$$\begin{gathered}\int_0^t|\boldsymbol{\alpha}''_N(\tau)|^2d\tau\le\frac{1}{N^{2s_2}}\bar q(h^+,h^-,N),\ \ \|\boldsymbol{\alpha}_N(t)\|^2\le\frac{1}{N^{2s_1-3}}\bar q(h^+,h^-,N),\\ \int_0^t\|\boldsymbol{\alpha}'_N(\tau)\|^2d\tau\le\frac{1}{N^{2s_1-3}}\bar q(h^+,h^-,N),\ \ \bar q(h^+,h^-,N)\to0\ \text{ as }\ N\to\infty.\end{gathered} \tag{3.22}$$

Therefore

$$|\boldsymbol{\Delta}'_N(t)|^2+\|\boldsymbol{\Delta}_N(t)\|^2\le\frac{1}{N^{2s}}\hat q(\Omega,\Gamma^0,N),\quad \hat q(\Omega,\Gamma^0,N)\to0\quad\text{as}\quad N\to\infty,$$

where $s=\min\{s_2,s_1-3/2\}$.

Since

$$|\boldsymbol{\alpha}'_N(t)|^2\le\frac{1}{N^{2s_2}}\tilde q(h^+,h^-,N),\quad \tilde q(h^+,h^-,N)\to0\quad\text{as}\quad N\to\infty,$$

we obtain

$$|\boldsymbol{u}'(t)-\boldsymbol{w}'_N(t)|^2+\|\boldsymbol{u}(t)-\boldsymbol{w}_N(t)\|^2\le\frac{1}{N^{2s}}q(\Omega,\Gamma^0,N),$$

where $q(\Omega,\Gamma^0,N)\to0$ as $N\to\infty$.

If $\|\boldsymbol{u}\|_{L^2(0,T;\mathbf{H}^{s_0}(\Omega))}\le c$, $\|\boldsymbol{u}'\|_{L^2(0,T;\mathbf{H}^{s_1}(\Omega))}\le c$, $\|\boldsymbol{u}''\|_{L^2(0,T;\mathbf{H}^{s_2}(\Omega))}\le c$, where c is independent of $h=\max\limits_{(x_1,x_2)\in\bar\omega}(h^+(x_1,x_2)-h^-(x_1,x_2))$, then instead of (3.22) we have

$$\begin{gathered}\int_0^t|\boldsymbol{\alpha}''_N(\tau)|^2d\tau\le\frac{h^{2s_2}}{N^{2s_2}}\bar q_1(N),\quad \|\boldsymbol{\alpha}_N(t)\|^2\le\frac{h^{2(s_1-1)}}{N^{2s_1-3}}\bar q_1(N),\\ \int_0^t\|\boldsymbol{\alpha}'_N(\tau)\|^2d\tau\le\frac{h^{2(s_1-1)}}{N^{2s_1-3}}\bar q_1(N),\quad \bar q_1(N)\to0\quad\text{as}\quad N\to\infty,\end{gathered}$$

and hence

$$\begin{gathered}|\boldsymbol{\Delta}'_N(t)|^2+\|\boldsymbol{\Delta}_N(t)\|^2\le\hat q_1(\Omega,\Gamma^0)\frac{h^{2\bar s}}{N^{2s}}\hat q_2(N),\\ \hat q_2(N)\to0\quad\text{as}\quad N\to\infty,\end{gathered} \tag{3.23}$$

where $\bar s=\min\{s_2,s_1-1\}$.

Taking into account that

$$|\boldsymbol{\alpha}'_N(t)|^2\le\frac{h^{2s_2}}{N^{2s_2}}\tilde q_1(N),\quad \tilde q_1(N)\to0\quad\text{as}\quad N\to\infty,$$

from (3.23) we obtain the second estimate of the theorem. □

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