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# LOCAL GROWTH OF WEIERSTRASS $\sigma$-FUNCTION AND WHITTAKER-TYPE DERIVATIVE SAMPLING

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*Dedicated to N. N. Leonenko on the occasion of his 50th birthday*

**Abstract.** Two explicit guard functions $K_j = K_j(\delta_z)$, $j = 1, 2$, are obtained, which depend on the distance $\delta_z$ between $z$ and the nearest point of the integer lattice in the complex plane, such that $\delta_z K_1(\delta_z) \le |\sigma(z)|e^{-\pi|z|^2/2} \le \delta_z K_2(\delta_z)$, $z \in \mathbb{C}$, where $\sigma(z)$ stands for the Weierstraß $\sigma$-function. This result is used to improve the circular truncation error upper bound in the $q$-th order Whittaker-type derivative sampling for the Leont'ev functions space $[2, \frac{\pi q}{2})$, $q \ge 1$.

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## 1. Introduction

The Weierstraß $\sigma$-function described as an infinite product is given by

$$\sigma(z) = z \prod_{(m,n)\in\mathbb{Z}^2}{}' \left(1 - \frac{z}{m+ni}\right) \exp\left\{\frac{z}{m+ni} + \frac{z^2}{2(m+ni)^2}\right\},$$

where the dashed product means that the factor with $m = n = 0$ is omitted. The local growth estimation of $\sigma(z)$, interesting for different purposes, has quite a long history. The first result of this kind known by the author is given in [3], where $\ln M_\sigma(r) \sim \pi r^2/2$, $r \to \infty$, is proved ($M_\sigma(r)$ stands for the maximum modulus of $\sigma$ on the circle $|z| = r$), see also [7, Chapter 4, §1, Problem 49], where this result is quoted from the book [3]. After that Hayman proved that there exist absolute constants $\mathbf{K}_1, \mathbf{K}_2$, $\mathbf{K}_1 < \mathbf{K}_2$, for which

$$\mathbf{K}_1 \le \frac{|\sigma(z)|}{\operatorname{dist}(z, \mathbb{Z}^2)}\, e^{-\pi|z|^2/2} \le \mathbf{K}_2, \quad z \in \mathbb{C}. \tag{1}$$

In his article [1] no specific comments were given on the nature of $\mathbf{K}_j$'s. (Here we have to point out that Hayman's proof is not correct; he wrongly deduced that the type of $\sigma(z)$ is $\pi/4$ instead of the true value $\pi/2$, [1].) Seip confirms (1) on the lattice $\Lambda_\alpha = \sqrt{\frac{\pi}{\alpha}}\mathbb{Z}^2$, $\alpha > 0$, but without any closer specification of $\mathbf{K}_j$'s, [8] (these estimates of $\sigma(z)$ were needed as convergence tools).

In this short note we obtain the values for these uniform constants ($\mathbf{K}_1 \approx 0.266$ and $\mathbf{K}_2 = 1$) as a corollary of our main result. Our principal result is to

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establish two positive guard functions $K_j = K_j(\delta_z)$, $j = 1, 2$, depending just on $\delta_z = \text{dist}(z, \mathbb{Z}^2)$ such that (1) holds below; the proposed numerical bounds (see the Proof of the Corollary 1), are the minimal and maximal values of these guard functions respectively (therefore $\mathbf{K}_j$ cannot be improved in our setting).

In the third section of the article we consider the truncation error upper bound appearing in the Whittaker-type derivative sampling restoration formula, i.e. the sampling restoration formula which involves not just the sampled values of the function $f$, but the sampled values of $f^{(j)}$, $j = \overline{0, q-1}$, i.e., of the first $q$ derivatives of the functions sampled at points of the lattice $\mathbb{Z}^2$. This type of sampling reconstruction holds for the Leont'ev type functions space $[2, \frac{\pi q}{2})$ [1], [4]. Our second main goal is to improve the truncation error upper bound in [6], Theorem 1, with the aid of inequalities (2).

## 2. Local Growth Behaviour of $\sigma(z)$-Function

Along with the notation already introduced the following symbols will also be used: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$ denote the sets of natural, integer and complex numbers, respectively; the symbol $\text{Cl}\{A\}$ denotes the closure of the set $A$, $t^\star$ is the complex conjugate of $t \in \mathbb{C}$, the circle $\Gamma_r = \{\zeta | \, |\zeta| = r\}$ contains no point of $\mathbb{Z}^2$.

**Theorem 1.** *For all $z \in \mathbb{C}$ we have*

$$\delta_z K_1(\delta_z) \le |\sigma(z)| \exp\left\{-\frac{\pi}{2}|z|^2\right\} \le \delta_z K_2(\delta_z), \tag{2}$$

*where*

$$K_1(\delta_z) = \left(1 - \frac{\pi^4 \delta_z^4}{90}\right)\left(1 - A_{22}\delta_z^4\right)^2 \exp\left\{-\frac{\pi}{2}\delta_z^2\right\}, \tag{3}$$

$$K_2(\delta_z) = \exp\left\{\left(\frac{\pi^4}{90} + A_{22}\right)\delta_z^4 - \frac{\pi}{2}\delta_z^2\right\}, \tag{4}$$

*and* $A_{22} := \sum_{m,n\in\mathbb{N}} (m^2+n^2)^{-2}$.

*Proof.* Let $P_{\frac{1}{2}} = (-\frac{1}{2}, \frac{1}{2})^2$ be the period cell of the Weierstraß $\sigma$-function. First we remark that for all $z \in \text{Cl}\{P_{\frac{1}{2}}\}$,

$$\delta_z(1 - A_{22}\delta_z^4)^2 \left(1 - \frac{\pi^4\delta_z^4}{90}\right) \le |\sigma(z)| \le \delta_z \exp\left\{\left(\frac{\pi^4}{90} + A_{22}\right)\delta_z^4\right\}. \tag{5}$$

Indeed, if $z = x + iy \in \text{Cl}\{P_{\frac{1}{2}}\}$, then we conclude that $\delta_z = \text{dist}(z, \mathbb{Z}^2) = \text{dist}(z, 0) = |z|$. Put

$$a_{m,n} := \left(1 - \frac{z}{m+ni}\right) \exp\left\{\frac{z}{m+ni} + \frac{z^2}{2(m+ni)^2}\right\}.$$

[1]The infrequently used notation $[\rho, \psi)$, $[\rho, \psi]$ stands for the functions space consisting of all entire functions of order less then or equal to $\rho$, and $[\rho, \psi)$ denotes the case when the function of order $\rho$ has a type less then $\psi$, while $[\rho, \psi]$ is used in the case when the function of order $\rho$ possesses a type less then or equal to $\psi$, [4].

Consequently, we directly get

$$\begin{aligned}|\sigma(z)|^2 &= |z|^2 \prod_{n\in\mathbb{N}} |a_{n,0}a_{-n,0}a_{0,n}a_{0,-n}|^2 \prod_{m,n\in\mathbb{N}} |a_{m,n}a_{m,-n}a_{-m,n}a_{-m,-n}|^2 \\ &= |z|^2 \prod_{n\in\mathbb{N}} \left|\left(1-\frac{z^2}{n^2}\right)\left(1+\frac{z^2}{n^2}\right)\right|^2 \\ &\times \prod_{m,n\in\mathbb{N}} \left|\left(1-\left(\frac{z}{m+ni}\right)^2\right)\left(1-\left(\frac{z}{m-ni}\right)^2\right)\right. \\ &\times \left.\exp\left\{z^2\left(\frac{1}{(m+ni)^2}+\frac{1}{(m-ni)^2}\right)\right\}\right|^2 && (5a) \\ &\le |z|^2 \prod_{n\in\mathbb{N}} \left(1+\frac{|z|^4}{n^4}\right)^2 && (5b) \\ &\times \prod_{m,n\in\mathbb{N}} \left(1-2\frac{(x^2-y^2)(m^2-n^2)+4mnxy}{(m^2+n^2)^2}+\frac{|z|^4}{(m^2+n^2)^2}\right) && (5c) \\ &\times \left(1-2\frac{(x^2-y^2)(m^2-n^2)-4mnxy}{(m^2+n^2)^2}+\frac{|z|^4}{(m^2+n^2)^2}\right) && (5d) \\ &\times \exp\left\{4\frac{(x^2-y^2)(m^2-n^2)}{(m^2+n^2)^2}\right\}.\end{aligned}$$

Now, we use $1+t\le e^t$ to majorize the factors in displays $(5b,c,d)$. So, straightforward calculation results in

$$|\sigma(z)|^2 \le |z|^2 \exp\left\{2\left(\frac{\pi^4}{90}+\sum_{m,n\in\mathbb{N}}\frac{1}{(m^2+n^2)^2}\right)|z|^4\right\}, \tag{6}$$

which is the asserted upper bound in (5).

The lower bound derivation will be realized in a somewhat different way. At first an auxiliary inequality is established. Namely, let $a\in\mathbb{C}$, $|a|<1$. Then there holds

$$|(1-a)e^a| \ge 1-|a|^2. \tag{7}$$

Indeed, after fixing $a=|a|e^{i\phi}$, $|a|$, we get by direct calculation

$$\begin{aligned}h(\phi) &:= |(1-a)e^a| = e^{|a|\cos\phi}\sqrt{1-2|a|\cos\phi+|a|^2} \\ &\ge \min_{\phi\in[0,2\pi)} h(\phi) = h(0) = (1-|a|)e^{|a|}.\end{aligned}$$

Finally, by $e^t\ge 1+t$ we deduce estimate (7).

To continue, assume $\lambda_n\in(0,1), n\in\mathbb{N}$. Then if $\sum_{n\in\mathbb{N}}\lambda_n$ converges, there holds

$$\prod_{n\in\mathbb{N}}(1-\lambda_n) \ge 1-\sum_{n\in\mathbb{N}}\lambda_n. \tag{8}$$

This result is a generalization of the inequality due to Weierstraß. (In fact, (8) is a straightforward consequence of the inequality concerning the finite product/sum case, cf. (1) in [5, **3.2.37.**, p. 207]).

Now we are ready to establish the lower bound for $|\sigma(z)|, z \in \mathrm{Cl}\{P_{\frac{1}{2}}\}$. We have

$$\begin{aligned} |\sigma(z)| &= |z| \left| \prod_{n\in\mathbb{N}} \left(1 - \frac{z^4}{n^4}\right) \right| \\ &\quad \times \left| \prod_{m,n\in\mathbb{N}} \left(1 - \frac{z^2}{(m+ni)^2}\right) e^{\frac{z^2}{(m+ni)^2}} \left(1 - \frac{z^2}{(m-ni)^2}\right) e^{\frac{z^2}{(m-ni)^2}} \right| \qquad (9) \\ &\geq |z| \prod_{n\in\mathbb{N}} \left(1 - \frac{|z|^4}{n^4}\right) \prod_{m,n\in\mathbb{N}} \left(1 - \frac{|z|^4}{(m^2+n^2)^2}\right)^2 \qquad (10) \\ &\geq |z| \left(1 - |z|^4 \sum_{n\in\mathbb{N}} \frac{1}{n^4}\right) \left(1 - |z|^4 \sum_{m,n\in\mathbb{N}} \frac{1}{(m^2+n^2)^2}\right)^2 \\ &= |z| \left(1 - \frac{\pi^4}{90}\delta_z^4\right) \left(1 - A_{22}\delta_z^4\right)^2 . \qquad (11) \end{aligned}$$

Indeed, since for $a = z^2(m \pm in)^{-2}$ we have $|a| \leq \frac{1}{4}$ for all $z \in \mathrm{Cl}\{P_{\frac{1}{2}}\}$, $m, n \in \mathbb{N}$, this allows us to use estimate (7) in the double-indexed product in (9), and then, taking into account that the factors $\lambda_n^{(1)} = |z|^4 n^{-4}$, $\lambda_n^{(2)} = |z|^4(m^2+n^2)^{-2}$ belong to the interval $(0,1)$ in display (10), with the help of the Weierstraß-type inequality (8) we easily deduce (11).

Finally, combining estimates (6) and (11), we finish the derivation of (5).

Let $\tau_{\mathbf{k}} : \mathbb{C} \mapsto \mathrm{Cl}\{P_{\frac{1}{2}}\}$ be the translation which defines a unique $\mathbf{k} = (k_u, k_v) \in \mathbb{Z}^2$ and a unique $w = (u, v) \in \mathrm{Cl}\{P_{\frac{1}{2}}\}$ such that

$$z = \mathbf{k} + w = k_u + ik_v + u + iv.$$

The quasi-periodicity property of the Weierstraß $\sigma$-function implies that

$$\sigma(z) = (-1)^{k_u+k_v+k_uk_v}\sigma(w)\exp\{\pi w\mathbf{k}^\star + \pi|\mathbf{k}|^2/2\}.$$

Since $\mathrm{dist}(z, \mathbb{Z}^2) = \mathrm{dist}(w, 0) = \delta_z$, we get

$$\begin{aligned} |\sigma(z)| &= |\sigma(w)||\exp\{\pi w\mathbf{k}^\star + \pi|\mathbf{k}|^2/2\}| \\ &\leq \delta_z \exp\left\{\left(\frac{\pi^4}{90} + A_{22}\right)\delta_z^4 + \frac{\pi}{2}(|\mathbf{k}|^2 + 2k_u u + 2k_v v + |w|^2)\right\} \exp\left\{-\frac{\pi}{2}|w|^2\right\} \\ &= \delta_z \exp\left\{\left(\frac{\pi^4}{90} + A_{22}\right)\delta_z^4 - \frac{\pi}{2}\delta_z^2\right\} \exp\left\{\frac{\pi}{2}|z|^2\right\}. \end{aligned}$$

This finishes the proof of the upper bound assertion in (2). Finally it remains to derive the lower bound in (2). For this we can argue in a similar way so that by the left-hand estimate in (5) we deduce

$$|\sigma(z)| \;=\; |\sigma(w)||\exp\{\pi w\mathbf{k}^\star + \pi|\mathbf{k}|^2/2\}|$$

$$\geq \ \delta_z \left(1 - \frac{\pi^4\delta_z^4}{90}\right) \left(1 - A_{22}\delta_z^4\right)^2 \exp\left\{-\frac{\pi}{2}\delta_z^2\right\} \exp\left\{\frac{\pi}{2}|z|^2\right\},$$

which is the asserted lower bound. □

*Remark* 1. The *Mathematica* 4.0 gives $A_{22} = \sum_{m,n\in\mathbb{N}}(m^2+n^2)^{-2} \approx 0.42437977$. So we can rearrange (3) and (4) with the approximate value of $A_{22}$.

**Corollary 1.1.** *The following estimates hold:*

$$\delta_z \left(1 - \tfrac{\pi^4}{360}\right) \left(1 - \tfrac{A_{22}}{4}\right)^2 \exp\left\{-\frac{\pi}{4}\right\} \leq |\sigma(z)| \exp\left\{-\frac{\pi}{2}|z|^2\right\} \leq \ \delta_z, \qquad z \in \mathbb{C}.$$

*Proof.* We find $\min K_1(\delta_z)$ and $\max K_2(\delta_z)$ by (3), (4) as $0 \leq \delta_z \leq 1/\sqrt{2}$. So

$$\mathbf{K}_1 \Leftarrow \min_{[0,1/\sqrt{2}]} K_1(\delta_z) = K_1(1/\sqrt{2}) = \left(1 - \tfrac{\pi^4}{360}\right) \left(1 - \tfrac{A_{22}}{4}\right)^2 e^{-\frac{\pi}{4}} \approx 0,26574548,$$

$$\mathbf{K}_2 \Leftarrow \max_{[0,1/\sqrt{2}]} K_2(\delta_z) = K_2(0) = 1$$

are the uniform Hayman's constants. □

*Remark* 2. The sharpness of inequalities (2) is an open question. Namely, starting with (9) and (5$a$), by the Euler's infinite product representation of $\sin \pi\delta_z$ and $\sinh \pi\delta_z$ we get the lower and upper guard functions for $|\sigma(z)|$ as follows:

$$\widetilde{K}_1(\delta_z) = \left(\frac{\sin \pi\delta_z}{\pi\delta_z}\right)^2 (1 - A_{22}\delta_z^4)^2 \exp\left\{-\frac{\pi}{4}\delta_z^2\right\},$$

$$\widetilde{K}_2(\delta_z) = \left(\frac{\sinh \pi\delta_z}{\pi\delta_z}\right)^2 \exp\left\{A_{22}\delta_z^4 - \frac{\pi}{4}\delta_z^2\right\}.$$

Since $\widetilde{K}_1(\delta_z)$ possesses lower minimum than $K_1(\delta_z)$, and $\widetilde{K}_2(\delta_z)$ has larger maximum than $K_2(\delta_z)$ we propose the use of (3) and (4) respectively.

## 3. Whittaker Type Derivative Sampling

In this part of the article we consider the influence of bounds (2) on the convergence rate in the Whittaker-type $q$-th order derivative, uniformly spaced sampling restoration formula for the Leont'ev function space $[2, \frac{\pi q}{2})$.

For $f \in [2, \frac{\pi q}{2}\vartheta], \vartheta \in [0, 1)$ we have

$$f(z) = \sigma^q(z) \sum_{(m,n)\in\mathbb{Z}^2} \sum_{j=0}^{q-1} \sum_{k=0}^{q-1-j} \frac{f^{(q-1-j-k)}(m+ni)R^q_{mnj}}{j!(q-1-j-k)!(z-m-ni)^{k+1}} \tag{12}$$

uniformly on the compact subsets of $\mathbb{C}$; as usual, $f^{(0)}(\cdot) \equiv f(\cdot)$. Here

$$R^q_{mnj} = \lim_{w\to m+ni} \frac{d^j}{dw^j} \left(\frac{w-m-ni}{\sigma(w)}\right)^q;$$

cf. [6], Theorem 1 and Corollary 1 for certain additional details, e.g. truncation error analysis, etc. (In fact, formula (12) for $q = 1$ belongs to Whittaker; the case $q > 1$ appears in [2] as well.)

Let us introduce a subset of $\mathbb{Z}^2$:

$$\mathbf{N}(r) := \{(m,n)|\ |m+ni| < r\}.$$

Under the Whittaker-type derivative sampling series (12) truncated to $\mathbf{N}(r)$ we mean the interpolation formula

$$\mathcal{I}_N(z;f;\sigma;q;r) = \sum_{(m,n)\in\mathbf{N}(r)} \sum_{j=0}^{q-1} \sum_{k=0}^{q-1-j} \frac{\sigma^q(z) f^{(q-1-j-k)}(m+ni) R^q_{mnj}}{j!(q-1-j-k)!(z-m-ni))^{k+1}}; \quad (13)$$

the so-called *circular truncation error* is

$$\epsilon_N(f;z;q;r) := f(z) - \mathcal{I}_N(z;f;\sigma;q;r). \quad (14)$$

Now, we apply estimates (2) to the results in [6], Theorem 1, Corollary 1, where we cannot avoid truncation error bound estimates such that depend on the Hayman constants $\mathbf{K}_1, \mathbf{K}_2$. To prove the principal result in this section, we will need the following evaluation.

**Lemma 1.** *Consider a circle $\Gamma_r = \{\zeta|\ |\zeta| = r\}$ such that contains no point of $\mathbb{Z}^2$. Then there holds*

$$\delta_\zeta \geq \mathrm{dist}(\Gamma_r, \mathbb{Z}^2) \geq \frac{1 - |1 - 2(r^2 - [r^2])|}{4\,r + \sqrt{2}} := H(r), \quad (15)$$

*where $[x]$ is the largest integer less than or equal to $x$.*

*Proof.* Let $A(\mathfrak{a}_1, \mathfrak{a}_2)$ be the closest point in $\mathbb{Z}^2$ to $\Gamma_r$ and denote by $\Delta_A$ the distance between $A$ and the nearest point on $\Gamma_r$. Clearly, $\sqrt{\mathfrak{a}_1^2 + \mathfrak{a}_2^2} = r \pm \Delta_A$, choosing the sign according to where $A$ lies, outside or inside the circle $\Gamma_r$. Thus we can write $\sqrt{\mathfrak{a}_1^2 + \mathfrak{a}_2^2} \leq r + \Delta_A \leq r + 1/\sqrt{2}$. Since $\mathfrak{a}_1^2 + \mathfrak{a}_2^2$ is the positive integer nearest to $r^2$, we have

$$\mathfrak{a}_1^2 + \mathfrak{a}_2^2 = [r^2] \quad \text{if } A \text{ lies inside of } \Gamma_r \quad \big(= [r^2] + 1 \quad \text{if } A \text{ lies outside of } \Gamma_r\big)\,,$$

which gives, that

$$\begin{aligned}\Delta_A = \mathrm{dist}(\Gamma_r, \mathbb{Z}^2) = \Big|r - \sqrt{\mathfrak{a}_1^2 + \mathfrak{a}_2^2}\Big| &= \frac{|r^2 - (\mathfrak{a}_1^2 + \mathfrak{a}_2^2)|}{r + \sqrt{\mathfrak{a}_1^2 + \mathfrak{a}_2^2}} \\ &\geq \frac{\min\{r^2 - [r^2], [r^2] + 1 - r^2\}}{2r + 1/\sqrt{2}}.\end{aligned}$$

According to the definition of a modulus this implies (15). □

**Theorem 2.** *For all $f \in [2, \frac{\pi q}{2}\vartheta]$, $\vartheta \in [0,1)$, and for all*

$$N \geq \left[\frac{|z|}{\sqrt{2(1-\vartheta)}} + \frac{1}{2}\right] + 1, \quad z \in \mathbb{C},$$

*we have*

$$|\epsilon_N(f;z;q;\sqrt{2}(N+1/2))| \leq \frac{A_f(2N+1)(4N+3)^q e^{-2\pi q(1-\vartheta)N}}{\mathbf{K}_1^q(2(1-\sqrt{1-\vartheta})N + 1 + \sqrt{1-\vartheta})}\,, \quad (16)$$

*where* $A_f$ *is the absolute constant which characterizes the* $[2, \frac{\pi q}{2}\vartheta]$*-function* $f$*, i.e. it is given by* $|f(z)| \le A_f \exp\left\{\frac{\pi q \vartheta}{2}\,|z|^2\right\}$, $z \in \mathbb{C}$. *Moreover*

$$\lim_{N\to\infty} \mathcal{I}_N(z; f; \sigma; q; r) = f(z)$$

*uniformly in* $z \in \mathbb{C}$.

*Proof.* First we repeat the procedure for deriving the truncation error upper bound in [6], Theorem 1 with the integration path $\Gamma_r = \{\zeta|\ |\zeta| = r\}$ chosen according to the definition of the index set $\mathbf{N}(r)$ and assume $z \in \mathrm{int}(\Gamma_r)$. We point out that $|f(z)| \le A_f \exp\left\{\frac{\pi q}{2}\vartheta r^2\right\}$ on the circle $\Gamma_r$ with $A_f > 0$. Then using (2), the numerical values of the Hayman constants $\mathbf{K}_1, \mathbf{K}_2$ and (15) in Lemma 1 for estimating $|\sigma(\cdot)|$ in this result in [6], we get

$$\begin{aligned}
|\epsilon_N(f; z; q; r)| &\le \frac{|\sigma(z)|^q}{2\pi} \oint_{\Gamma_r} \frac{|f(\zeta)||d\zeta|}{|\sigma(\zeta)|^q|\zeta - z|} \\
&\le \left(\frac{\delta_z K_2(\delta_z)}{\min_{\zeta\in\Gamma_r} \delta_\zeta K_1(\delta_\zeta)}\right)^q \frac{A_f\, r}{r - |z|}\ \exp\left\{\frac{\pi q}{2}(|z|^2 - (1-\vartheta)r^2)\right\} \\
&\le \left(\frac{\delta_z \mathbf{K}_2}{H(r)\mathbf{K}_1}\right)^q \frac{A_f\, r}{r - |z|}\ \exp\left\{\frac{\pi q}{2}(|z|^2 - (1-\vartheta)r^2)\right\} \\
&\le \frac{A_f\, r(4r + \sqrt{2})^q \exp\left\{\frac{\pi q}{2}(|z|^2 - (1-\vartheta)r^2)\right\}}{(r - |z|)\left(\sqrt{2}\mathbf{K}_1(1 - |1 - 2(r^2 - \lfloor r^2 \rfloor)|)\right)^q}.
\end{aligned} \tag{17}$$

It is not difficult to see that the circle $\Gamma_r$, $r = \sqrt{2}(N + 1/2)$, does not contain any integer point from $\mathbb{Z}^2$, being $r^2 \notin \mathbb{N}$. So $\Gamma_{\sqrt{2}(N+1/2)}$ is a suitable integration contour for which (17) holds. Then, substituting $r = \sqrt{2}(N + 1/2)$ into the bound (17), we deduce

$$\begin{aligned}
&|\epsilon_N(f; z; q; \sqrt{2}(N + 1/2))| \\
&\quad\le \frac{A_f(N + 1/2)(4(N + 1/2) + 1)^q}{\mathbf{K}_1^q((N + 1/2 - \sqrt{1-\vartheta}(N - 1/2))}\ e^{\pi q(1-\vartheta)[(N-1/2)^2 - (N+1/2)^2]} \\
&\quad= \frac{A_f(2N + 1)(4N + 3)^q \exp\{-2\pi q(1 - \vartheta)N\}}{\mathbf{K}_1^q(2(1 - \sqrt{1-\vartheta})N + 1 + \sqrt{1-\vartheta})},
\end{aligned} \tag{18}$$

which is the asserted upper bound (16).

The uniform convergence in $f(z) \approx \mathcal{I}_N(z; f; \sigma; q)$ follows from the truncation error upper bound (16) as $N \to \infty$. Indeed, since the right-hand term in (16) does not depend on $z$ and vanishes with the growth of $N$, the assertion follows. $\square$

*Remark* 3. By fixing the values of $z$ we get the mathematical model

$$\frac{A_f(2N + 1)(4N + 3)^q e^{-2\pi q(1-\vartheta)N}}{\mathbf{K}_1^q(2(1 - \sqrt{1-\vartheta})N + 1 + \sqrt{1-\vartheta})} < \epsilon \tag{19}$$

for a pre-assigned approximation error level $\epsilon > 0$ from (16). Then inequality (19) gives an optimal value of $N$ in finding the minimal size of the approximation sum (13). Moreover, the convergence rate in (12) is

$$|\epsilon_N(f;z)| = \mathcal{O}(N^q e^{-2\pi q(1-\vartheta)N})$$

under the assumptions of Theorem 2.


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