

## LOCAL GROWTH OF WEIERSTRASS $\sigma$ -FUNCTION AND WHITTAKER-TYPE DERIVATIVE SAMPLING

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*Dedicated to N. N. Leonenko on the occasion of his 50th birthday*

**Abstract.** Two explicit guard functions  $K_j = K_j(\delta_z)$ ,  $j = 1, 2$ , are obtained, which depend on the distance  $\delta_z$  between  $z$  and the nearest point of the integer lattice in the complex plane, such that  $\delta_z K_1(\delta_z) \leq |\sigma(z)| e^{-\pi|z|^2/2} \leq \delta_z K_2(\delta_z)$ ,  $z \in \mathbb{C}$ , where  $\sigma(z)$  stands for the Weierstraß  $\sigma$ -function. This result is used to improve the circular truncation error upper bound in the  $q$ -th order Whittaker-type derivative sampling for the Leont'ev functions space  $[2, \frac{\pi q}{2})$ ,  $q \geq 1$ .

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### 1. INTRODUCTION

The Weierstraß  $\sigma$ -function described as an infinite product is given by

$$\sigma(z) = z \prod'_{(m,n) \in \mathbb{Z}^2} \left( 1 - \frac{z}{m + ni} \right) \exp \left\{ \frac{z}{m + ni} + \frac{z^2}{2(m + ni)^2} \right\},$$

where the dashed product means that the factor with  $m = n = 0$  is omitted. The local growth estimation of  $\sigma(z)$ , interesting for different purposes, has quite a long history. The first result of this kind known by the author is given in [3], where  $\ln M_\sigma(r) \sim \pi r^2/2$ ,  $r \rightarrow \infty$ , is proved ( $M_\sigma(r)$  stands for the maximum modulus of  $\sigma$  on the circle  $|z| = r$ ), see also [7, Chapter 4, §1, Problem 49], where this result is quoted from the book [3]. After that Hayman proved that there exist absolute constants  $\mathbf{K}_1, \mathbf{K}_2$ ,  $\mathbf{K}_1 < \mathbf{K}_2$ , for which

$$\mathbf{K}_1 \leq \frac{|\sigma(z)|}{\text{dist}(z, \mathbb{Z}^2)} e^{-\pi|z|^2/2} \leq \mathbf{K}_2, \quad z \in \mathbb{C}. \quad (1)$$

In his article [1] no specific comments were given on the nature of  $\mathbf{K}_j$ 's. (Here we have to point out that Hayman's proof is not correct; he wrongly deduced that the type of  $\sigma(z)$  is  $\pi/4$  instead of the true value  $\pi/2$ , [1].) Seip confirms (1) on the lattice  $\Lambda_\alpha = \sqrt{\frac{\pi}{\alpha}} \mathbb{Z}^2$ ,  $\alpha > 0$ , but without any closer specification of  $\mathbf{K}_j$ 's, [8] (these estimates of  $\sigma(z)$  were needed as convergence tools).

In this short note we obtain the values for these uniform constants ( $\mathbf{K}_1 \approx 0.266$  and  $\mathbf{K}_2 = 1$ ) as a corollary of our main result. Our principal result is to

establish two positive guard functions  $K_j = K_j(\delta_z)$ ,  $j = 1, 2$ , depending just on  $\delta_z = \text{dist}(z, \mathbb{Z}^2)$  such that (1) holds below; the proposed numerical bounds (see the Proof of the Corollary 1), are the minimal and maximal values of these guard functions respectively (therefore  $\mathbf{K}_j$  cannot be improved in our setting).

In the third section of the article we consider the truncation error upper bound appearing in the Whittaker-type derivative sampling restoration formula, i.e. the sampling restoration formula which involves not just the sampled values of the function  $f$ , but the sampled values of  $f^{(j)}$ ,  $j = \overline{0, q-1}$ , i.e., of the first  $q$  derivatives of the functions sampled at points of the lattice  $\mathbb{Z}^2$ . This type of sampling reconstruction holds for the Leont'ev type functions space [2,  $\frac{\pi q}{2}$ ]<sup>1</sup>, [4]. Our second main goal is to improve the truncation error upper bound in [6], Theorem 1, with the aid of inequalities (2).

## 2. LOCAL GROWTH BEHAVIOUR OF $\sigma(z)$ -FUNCTION

Along with the notation already introduced the following symbols will also be used:  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{C}$  denote the sets of natural, integer and complex numbers, respectively; the symbol  $\text{Cl}\{A\}$  denotes the closure of the set  $A$ ,  $t^*$  is the complex conjugate of  $t \in \mathbb{C}$ , the circle  $\Gamma_r = \{\zeta \mid |\zeta| = r\}$  contains no point of  $\mathbb{Z}^2$ .

**Theorem 1.** *For all  $z \in \mathbb{C}$  we have*

$$\delta_z K_1(\delta_z) \leq |\sigma(z)| \exp \left\{ -\frac{\pi}{2} |z|^2 \right\} \leq \delta_z K_2(\delta_z), \quad (2)$$

where

$$K_1(\delta_z) = \left( 1 - \frac{\pi^4 \delta_z^4}{90} \right) (1 - A_{22} \delta_z^4)^2 \exp \left\{ -\frac{\pi}{2} \delta_z^2 \right\}, \quad (3)$$

$$K_2(\delta_z) = \exp \left\{ \left( \frac{\pi^4}{90} + A_{22} \right) \delta_z^4 - \frac{\pi}{2} \delta_z^2 \right\}, \quad (4)$$

and  $A_{22} := \sum_{m,n \in \mathbb{N}} (m^2 + n^2)^{-2}$ .

*Proof.* Let  $P_{\frac{1}{2}} = (-\frac{1}{2}, \frac{1}{2})^2$  be the period cell of the Weierstraß  $\sigma$ -function. First we remark that for all  $z \in \text{Cl}\{P_{\frac{1}{2}}\}$ ,

$$\delta_z (1 - A_{22} \delta_z^4)^2 \left( 1 - \frac{\pi^4 \delta_z^4}{90} \right) \leq |\sigma(z)| \leq \delta_z \exp \left\{ \left( \frac{\pi^4}{90} + A_{22} \right) \delta_z^4 \right\}. \quad (5)$$

Indeed, if  $z = x + iy \in \text{Cl}\{P_{\frac{1}{2}}\}$ , then we conclude that  $\delta_z = \text{dist}(z, \mathbb{Z}^2) = \text{dist}(z, 0) = |z|$ . Put

$$a_{m,n} := \left( 1 - \frac{z}{m + ni} \right) \exp \left\{ \frac{z}{m + ni} + \frac{z^2}{2(m + ni)^2} \right\}.$$

<sup>1</sup>The infrequently used notation  $[\rho, \psi)$ ,  $[\rho, \psi]$  stands for the functions space consisting of all entire functions of order less than or equal to  $\rho$ , and  $[\rho, \psi)$  denotes the case when the function of order  $\rho$  has a type less than  $\psi$ , while  $[\rho, \psi]$  is used in the case when the function of order  $\rho$  possesses a type less than or equal to  $\psi$ , [4].

Consequently, we directly get

$$\begin{aligned}
 |\sigma(z)|^2 &= |z|^2 \prod_{n \in \mathbb{N}} |a_{n,0} a_{-n,0} a_{0,n} a_{0,-n}|^2 \prod_{m,n \in \mathbb{N}} |a_{m,n} a_{m,-n} a_{-m,n} a_{-m,-n}|^2 \\
 &= |z|^2 \prod_{n \in \mathbb{N}} \left| \left(1 - \frac{z^2}{n^2}\right) \left(1 + \frac{z^2}{n^2}\right) \right|^2 \\
 &\times \prod_{m,n \in \mathbb{N}} \left| \left(1 - \left(\frac{z}{m+ni}\right)^2\right) \left(1 - \left(\frac{z}{m-ni}\right)^2\right) \right. \\
 &\times \left. \exp \left\{ z^2 \left( \frac{1}{(m+ni)^2} + \frac{1}{(m-ni)^2} \right) \right\} \right|^2 \tag{5a}
 \end{aligned}$$

$$\leq |z|^2 \prod_{n \in \mathbb{N}} \left(1 + \frac{|z|^4}{n^4}\right)^2 \tag{5b}$$

$$\times \prod_{m,n \in \mathbb{N}} \left(1 - 2 \frac{(x^2 - y^2)(m^2 - n^2) + 4mnxy}{(m^2 + n^2)^2} + \frac{|z|^4}{(m^2 + n^2)^2}\right) \tag{5c}$$

$$\times \left(1 - 2 \frac{(x^2 - y^2)(m^2 - n^2) - 4mnxy}{(m^2 + n^2)^2} + \frac{|z|^4}{(m^2 + n^2)^2}\right) \tag{5d}$$

$$\times \exp \left\{ 4 \frac{(x^2 - y^2)(m^2 - n^2)}{(m^2 + n^2)^2} \right\}.$$

Now, we use  $1 + t \leq e^t$  to majorize the factors in displays (5b, c, d). So, straightforward calculation results in

$$|\sigma(z)|^2 \leq |z|^2 \exp \left\{ 2 \left( \frac{\pi^4}{90} + \sum_{m,n \in \mathbb{N}} \frac{1}{(m^2 + n^2)^2} \right) |z|^4 \right\}, \tag{6}$$

which is the asserted upper bound in (5).

The lower bound derivation will be realized in a somewhat different way. At first an auxiliary inequality is established. Namely, let  $a \in \mathbb{C}$ ,  $|a| < 1$ . Then there holds

$$|(1 - a)e^a| \geq 1 - |a|^2. \tag{7}$$

Indeed, after fixing  $a = |a|e^{i\phi}$ ,  $|a|$ , we get by direct calculation

$$\begin{aligned}
 h(\phi) &:= |(1 - a)e^a| = e^{|a| \cos \phi} \sqrt{1 - 2|a| \cos \phi + |a|^2} \\
 &\geq \min_{\phi \in [0, 2\pi)} h(\phi) = h(0) = (1 - |a|)e^{|a|}.
 \end{aligned}$$

Finally, by  $e^t \geq 1 + t$  we deduce estimate (7).

To continue, assume  $\lambda_n \in (0, 1)$ ,  $n \in \mathbb{N}$ . Then if  $\sum_{n \in \mathbb{N}} \lambda_n$  converges, there holds

$$\prod_{n \in \mathbb{N}} (1 - \lambda_n) \geq 1 - \sum_{n \in \mathbb{N}} \lambda_n. \tag{8}$$

This result is a generalization of the inequality due to Weierstraß. (In fact, (8) is a straightforward consequence of the inequality concerning the finite product/sum case, cf. (1) in [5, **3.2.37.**, p. 207]).

Now we are ready to establish the lower bound for  $|\sigma(z)|$ ,  $z \in \text{Cl}\{P_{\frac{1}{2}}\}$ . We have

$$|\sigma(z)| = |z| \left| \prod_{n \in \mathbb{N}} \left( 1 - \frac{z^4}{n^4} \right) \right| \times \left| \prod_{m, n \in \mathbb{N}} \left( 1 - \frac{z^2}{(m+ni)^2} \right) e^{\frac{z^2}{(m+ni)^2}} \left( 1 - \frac{z^2}{(m-ni)^2} \right) e^{\frac{z^2}{(m-ni)^2}} \right| \quad (9)$$

$$\geq |z| \prod_{n \in \mathbb{N}} \left( 1 - \frac{|z|^4}{n^4} \right) \prod_{m, n \in \mathbb{N}} \left( 1 - \frac{|z|^4}{(m^2 + n^2)^2} \right)^2 \quad (10)$$

$$\geq |z| \left( 1 - |z|^4 \sum_{n \in \mathbb{N}} \frac{1}{n^4} \right) \left( 1 - |z|^4 \sum_{m, n \in \mathbb{N}} \frac{1}{(m^2 + n^2)^2} \right)^2 = |z| \left( 1 - \frac{\pi^4}{90} \delta_z^4 \right) (1 - A_{22} \delta_z^4)^2. \quad (11)$$

Indeed, since for  $a = z^2(m \pm in)^{-2}$  we have  $|a| \leq \frac{1}{4}$  for all  $z \in \text{Cl}\{P_{\frac{1}{2}}\}$ ,  $m, n \in \mathbb{N}$ , this allows us to use estimate (7) in the double-indexed product in (9), and then, taking into account that the factors  $\lambda_n^{(1)} = |z|^4 n^{-4}$ ,  $\lambda_n^{(2)} = |z|^4 (m^2 + n^2)^{-2}$  belong to the interval  $(0, 1)$  in display (10), with the help of the Weierstraß-type inequality (8) we easily deduce (11).

Finally, combining estimates (6) and (11), we finish the derivation of (5).

Let  $\tau_{\mathbf{k}} : \mathbb{C} \mapsto \text{Cl}\{P_{\frac{1}{2}}\}$  be the translation which defines a unique  $\mathbf{k} = (k_u, k_v) \in \mathbb{Z}^2$  and a unique  $w = (u, v) \in \text{Cl}\{P_{\frac{1}{2}}\}$  such that

$$z = \mathbf{k} + w = k_u + ik_v + u + iv.$$

The quasi-periodicity property of the Weierstraß  $\sigma$ -function implies that

$$\sigma(z) = (-1)^{k_u + k_v + k_u k_v} \sigma(w) \exp\{\pi w \mathbf{k}^* + \pi |\mathbf{k}|^2 / 2\}.$$

Since  $\text{dist}(z, \mathbb{Z}^2) = \text{dist}(w, 0) = \delta_z$ , we get

$$\begin{aligned} |\sigma(z)| &= |\sigma(w)| \exp\{\pi w \mathbf{k}^* + \pi |\mathbf{k}|^2 / 2\} \\ &\leq \delta_z \exp \left\{ \left( \frac{\pi^4}{90} + A_{22} \right) \delta_z^4 + \frac{\pi}{2} (|\mathbf{k}|^2 + 2k_u u + 2k_v v + |w|^2) \right\} \exp \left\{ -\frac{\pi}{2} |w|^2 \right\} \\ &= \delta_z \exp \left\{ \left( \frac{\pi^4}{90} + A_{22} \right) \delta_z^4 - \frac{\pi}{2} \delta_z^2 \right\} \exp \left\{ \frac{\pi}{2} |z|^2 \right\}. \end{aligned}$$

This finishes the proof of the upper bound assertion in (2). Finally it remains to derive the lower bound in (2). For this we can argue in a similar way so that by the left-hand estimate in (5) we deduce

$$|\sigma(z)| = |\sigma(w)| \exp\{\pi w \mathbf{k}^* + \pi |\mathbf{k}|^2 / 2\}$$

$$\geq \delta_z \left(1 - \frac{\pi^4 \delta_z^4}{90}\right) (1 - A_{22} \delta_z^4)^2 \exp \left\{-\frac{\pi}{2} \delta_z^2\right\} \exp \left\{\frac{\pi}{2} |z|^2\right\},$$

which is the asserted lower bound. □

*Remark 1.* The *Mathematica* 4.0 gives  $A_{22} = \sum_{m,n \in \mathbb{N}} (m^2 + n^2)^{-2} \approx 0.42437977$ . So we can rearrange (3) and (4) with the approximate value of  $A_{22}$ .

**Corollary 1.1.** *The following estimates hold:*

$$\delta_z \left(1 - \frac{\pi^4}{360}\right) \left(1 - \frac{A_{22}}{4}\right)^2 \exp \left\{-\frac{\pi}{4}\right\} \leq |\sigma(z)| \exp \left\{-\frac{\pi}{2} |z|^2\right\} \leq \delta_z, \quad z \in \mathbb{C}.$$

*Proof.* We find  $\min K_1(\delta_z)$  and  $\max K_2(\delta_z)$  by (3), (4) as  $0 \leq \delta_z \leq 1/\sqrt{2}$ . So

$$\mathbf{K}_1 \Leftarrow \min_{[0, 1/\sqrt{2}]} K_1(\delta_z) = K_1(1/\sqrt{2}) = \left(1 - \frac{\pi^4}{360}\right) \left(1 - \frac{A_{22}}{4}\right)^2 e^{-\frac{\pi}{4}} \approx 0,26574548,$$

$$\mathbf{K}_2 \Leftarrow \max_{[0, 1/\sqrt{2}]} K_2(\delta_z) = K_2(0) = 1$$

are the uniform Hayman’s constants. □

*Remark 2.* The sharpness of inequalities (2) is an open question. Namely, starting with (9) and (5a), by the Euler’s infinite product representation of  $\sin \pi \delta_z$  and  $\sinh \pi \delta_z$  we get the lower and upper guard functions for  $|\sigma(z)|$  as follows:

$$\tilde{K}_1(\delta_z) = \left(\frac{\sin \pi \delta_z}{\pi \delta_z}\right)^2 (1 - A_{22} \delta_z^4)^2 \exp \left\{-\frac{\pi}{4} \delta_z^2\right\},$$

$$\tilde{K}_2(\delta_z) = \left(\frac{\sinh \pi \delta_z}{\pi \delta_z}\right)^2 \exp \left\{A_{22} \delta_z^4 - \frac{\pi}{4} \delta_z^2\right\}.$$

Since  $\tilde{K}_1(\delta_z)$  possesses lower minimum than  $K_1(\delta_z)$ , and  $\tilde{K}_2(\delta_z)$  has larger maximum than  $K_2(\delta_z)$  we propose the use of (3) and (4) respectively.

### 3. WHITTAKER TYPE DERIVATIVE SAMPLING

In this part of the article we consider the influence of bounds (2) on the convergence rate in the Whittaker-type  $q$ -th order derivative, uniformly spaced sampling restoration formula for the Leont’ev function space  $[2, \frac{\pi q}{2})$ .

For  $f \in [2, \frac{\pi q}{2} \vartheta]$ ,  $\vartheta \in [0, 1)$  we have

$$f(z) = \sigma^q(z) \sum_{(m,n) \in \mathbb{Z}^2} \sum_{j=0}^{q-1} \sum_{k=0}^{q-1-j} \frac{f^{(q-1-j-k)}(m+ni) R_{mnj}^q}{j!(q-1-j-k)!(z-m-ni)^{k+1}} \tag{12}$$

uniformly on the compact subsets of  $\mathbb{C}$ ; as usual,  $f^{(0)}(\cdot) \equiv f(\cdot)$ . Here

$$R_{mnj}^q = \lim_{w \rightarrow m+ni} \frac{d^j}{dw^j} \left(\frac{w-m-ni}{\sigma(w)}\right)^q;$$

cf. [6], Theorem 1 and Corollary 1 for certain additional details, e.g. truncation error analysis, etc. (In fact, formula (12) for  $q = 1$  belongs to Whittaker; the case  $q > 1$  appears in [2] as well.)

Let us introduce a subset of  $\mathbb{Z}^2$ :

$$\mathbf{N}(r) := \{(m, n) \mid |m + ni| < r\}.$$

Under the Whittaker-type derivative sampling series (12) truncated to  $\mathbf{N}(r)$  we mean the interpolation formula

$$\mathcal{I}_N(z; f; \sigma; q; r) = \sum_{(m,n) \in \mathbf{N}(r)} \sum_{j=0}^{q-1} \sum_{k=0}^{q-1-j} \frac{\sigma^q(z) f^{(q-1-j-k)}(m+ni) R_{mnj}^q}{j!(q-1-j-k)!(z-m-ni)^{k+1}}; \quad (13)$$

the so-called *circular truncation error* is

$$\epsilon_N(f; z; q; r) := f(z) - \mathcal{I}_N(z; f; \sigma; q; r). \quad (14)$$

Now, we apply estimates (2) to the results in [6], Theorem 1, Corollary 1, where we cannot avoid truncation error bound estimates such that depend on the Hayman constants  $\mathbf{K}_1, \mathbf{K}_2$ . To prove the principal result in this section, we will need the following evaluation.

**Lemma 1.** *Consider a circle  $\Gamma_r = \{\zeta \mid |\zeta| = r\}$  such that contains no point of  $\mathbb{Z}^2$ . Then there holds*

$$\delta_\zeta \geq \text{dist}(\Gamma_r, \mathbb{Z}^2) \geq \frac{1 - |1 - 2(r^2 - [r^2])|}{4r + \sqrt{2}} := H(r), \quad (15)$$

where  $[x]$  is the largest integer less than or equal to  $x$ .

*Proof.* Let  $A(\mathbf{a}_1, \mathbf{a}_2)$  be the closest point in  $\mathbb{Z}^2$  to  $\Gamma_r$  and denote by  $\Delta_A$  the distance between  $A$  and the nearest point on  $\Gamma_r$ . Clearly,  $\sqrt{\mathbf{a}_1^2 + \mathbf{a}_2^2} = r \pm \Delta_A$ , choosing the sign according to where  $A$  lies, outside or inside the circle  $\Gamma_r$ . Thus we can write  $\sqrt{\mathbf{a}_1^2 + \mathbf{a}_2^2} \leq r + \Delta_A \leq r + 1/\sqrt{2}$ . Since  $\mathbf{a}_1^2 + \mathbf{a}_2^2$  is the positive integer nearest to  $r^2$ , we have

$$\mathbf{a}_1^2 + \mathbf{a}_2^2 = [r^2] \quad \text{if } A \text{ lies inside of } \Gamma_r \quad (= [r^2] + 1 \quad \text{if } A \text{ lies outside of } \Gamma_r),$$

which gives, that

$$\begin{aligned} \Delta_A = \text{dist}(\Gamma_r, \mathbb{Z}^2) &= \left| r - \sqrt{\mathbf{a}_1^2 + \mathbf{a}_2^2} \right| = \frac{|r^2 - (\mathbf{a}_1^2 + \mathbf{a}_2^2)|}{r + \sqrt{\mathbf{a}_1^2 + \mathbf{a}_2^2}} \\ &\geq \frac{\min\{r^2 - [r^2], [r^2] + 1 - r^2\}}{2r + 1/\sqrt{2}}. \end{aligned}$$

According to the definition of a modulus this implies (15).  $\square$

**Theorem 2.** *For all  $f \in [2, \frac{\pi q}{2}\vartheta]$ ,  $\vartheta \in [0, 1)$ , and for all*

$$N \geq \left\lceil \frac{|z|}{\sqrt{2(1-\vartheta)}} + \frac{1}{2} \right\rceil + 1, \quad z \in \mathbb{C},$$

we have

$$|\epsilon_N(f; z; q; \sqrt{2}(N + 1/2))| \leq \frac{A_f(2N + 1)(4N + 3)^q e^{-2\pi q(1-\vartheta)N}}{\mathbf{K}_1^q(2(1 - \sqrt{1-\vartheta})N + 1 + \sqrt{1-\vartheta})}, \quad (16)$$

where  $A_f$  is the absolute constant which characterizes the  $[2, \frac{\pi q}{2}\vartheta]$ -function  $f$ , i.e. it is given by  $|f(z)| \leq A_f \exp\{\frac{\pi q \vartheta}{2} |z|^2\}$ ,  $z \in \mathbb{C}$ . Moreover

$$\lim_{N \rightarrow \infty} \mathcal{I}_N(z; f; \sigma; q; r) = f(z)$$

uniformly in  $z \in \mathbb{C}$ .

*Proof.* First we repeat the procedure for deriving the truncation error upper bound in [6], Theorem 1 with the integration path  $\Gamma_r = \{\zeta \mid |\zeta| = r\}$  chosen according to the definition of the index set  $\mathbf{N}(r)$  and assume  $z \in \text{int}(\Gamma_r)$ . We point out that  $|f(z)| \leq A_f \exp\{\frac{\pi q}{2} \vartheta r^2\}$  on the circle  $\Gamma_r$  with  $A_f > 0$ . Then using (2), the numerical values of the Hayman constants  $\mathbf{K}_1, \mathbf{K}_2$  and (15) in Lemma 1 for estimating  $|\sigma(\cdot)|$  in this result in [6], we get

$$\begin{aligned} |\epsilon_N(f; z; q; r)| &\leq \frac{|\sigma(z)|^q}{2\pi} \oint_{\Gamma_r} \frac{|f(\zeta)| |d\zeta|}{|\sigma(\zeta)|^q |\zeta - z|} \\ &\leq \left( \frac{\delta_z K_2(\delta_z)}{\min_{\zeta \in \Gamma_r} \delta_\zeta K_1(\delta_\zeta)} \right)^q \frac{A_f r}{r - |z|} \exp\left\{ \frac{\pi q}{2} (|z|^2 - (1 - \vartheta)r^2) \right\} \\ &\leq \left( \frac{\delta_z \mathbf{K}_2}{H(r) \mathbf{K}_1} \right)^q \frac{A_f r}{r - |z|} \exp\left\{ \frac{\pi q}{2} (|z|^2 - (1 - \vartheta)r^2) \right\} \\ &\leq \frac{A_f r (4r + \sqrt{2})^q \exp\left\{ \frac{\pi q}{2} (|z|^2 - (1 - \vartheta)r^2) \right\}}{(r - |z|) (\sqrt{2} \mathbf{K}_1 (1 - |1 - 2(r^2 - \lfloor r^2 \rfloor)|))^q}. \end{aligned} \tag{17}$$

It is not difficult to see that the circle  $\Gamma_r$ ,  $r = \sqrt{2}(N + 1/2)$ , does not contain any integer point from  $\mathbb{Z}^2$ , being  $r^2 \notin \mathbb{N}$ . So  $\Gamma_{\sqrt{2}(N+1/2)}$  is a suitable integration contour for which (17) holds. Then, substituting  $r = \sqrt{2}(N + 1/2)$  into the bound (17), we deduce

$$\begin{aligned} &|\epsilon_N(f; z; q; \sqrt{2}(N + 1/2))| \\ &\leq \frac{A_f(N + 1/2)(4(N + 1/2) + 1)^q}{\mathbf{K}_1^q((N + 1/2 - \sqrt{1 - \vartheta}(N - 1/2)))} e^{\pi q(1 - \vartheta)[(N - 1/2)^2 - (N + 1/2)^2]} \\ &= \frac{A_f(2N + 1)(4N + 3)^q \exp\{-2\pi q(1 - \vartheta)N\}}{\mathbf{K}_1^q(2(1 - \sqrt{1 - \vartheta})N + 1 + \sqrt{1 - \vartheta})}, \end{aligned} \tag{18}$$

which is the asserted upper bound (16).

The uniform convergence in  $f(z) \approx \mathcal{I}_N(z; f; \sigma; q)$  follows from the truncation error upper bound (16) as  $N \rightarrow \infty$ . Indeed, since the right-hand term in (16) does not depend on  $z$  and vanishes with the growth of  $N$ , the assertion follows.  $\square$

*Remark 3.* By fixing the values of  $z$  we get the mathematical model

$$\frac{A_f(2N + 1)(4N + 3)^q e^{-2\pi q(1 - \vartheta)N}}{\mathbf{K}_1^q(2(1 - \sqrt{1 - \vartheta})N + 1 + \sqrt{1 - \vartheta})} < \epsilon \tag{19}$$

for a pre-assigned approximation error level  $\epsilon > 0$  from (16). Then inequality (19) gives an optimal value of  $N$  in finding the minimal size of the approximation

sum (13). Moreover, the convergence rate in (12) is

$$|\epsilon_N(f; z)| = \mathcal{O}(N^q e^{-2\pi q(1-\vartheta)N})$$

under the assumptions of Theorem 2.

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