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SINGULAR INTEGRALS IN WEIGHTED LEBESGUE SPACES WITH VARIABLE EXPONENT

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Abstract. In the weighted Lebesgue space with variable exponent the boundedness of the Calderón–Zygmund operator is established. The variable exponent $p(x)$ is assumed to satisfy the logarithmic Dini condition and the exponent β of the power weight $\rho(x) = |x - x_0|^\beta$ is related only to the value $p(x_0)$. The mapping properties of Cauchy singular integrals defined on the Lyapunov curve and on curves of bounded rotation are also investigated within the framework of the above-mentioned weighted space.

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1. Introduction

The generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and the related Sobolev type spaces $W^{m,p(x)}(\Omega)$ with variable exponent have proved to be an appropriate tool in studying models with non-standard local growth (in elasticity theory, fluid mechanics, differential equations, see for example Ružička [18], [6] and the references therein).

These applications stimulate a quick progress of the theory of the spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. We mention the papers by Sharapudinov [23] (1979), [24] (1996), Kováčik and Rákosník [17] (1991), Edmunds and Rákosník [11] (1992), Samko [19]–[20] (1998), [21] (1999), Edmunds, Lang, and Nekvinda [8] (1999), Cruz-Uribe, Fiorenza, and Neugebauer [3] (2002), Diening [4]–[5] (2002), Diening and Ružička [6] (2002), Edmunds abd Nekvinda [10] (2002), Edmunds and Meskhi [9], Fiorenza [12](2002), Kokilashvili abd Samko [14]–[16] (2002), see also the references therein.

Although the spaces $L^{p(\cdot)}(\Omega)$ possess some undesirable properties (functions from these spaces are not $p(x)$-mean continuous, the space $L^{p(\cdot)}(\Omega)$ is not translation invariant, convolution operators in general do not behave well and so on), there is an evident progress in their study, stimulated by applications, first of all for continuous exponents $p(x)$ satisfying the logarithmic Dini condition. We mention in particular the result on the denseness of C_0^∞-functions in the Sobolev space $W^{m,p(x)}(\Omega)$, see [21], and the breakthrough connected with the study of maximal operators in [4], [5].

Because of applications, the reconsideration of the main theorems of harmonic analysis is topical in order to find out what theorems remain valid for variable exponents or to find their substituting analogs. Among the challenging

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problems there were: a Sobolev type theorem for the Riesz potential operator I^α and the boundedness of singular integral operators. A Sobolev type theorem for bounded domains is proved in [19] under the condition that the maximal operator is bounded in the spaces $L^{p(\cdot)}$. This condition is fulfilled due to the result on maximal operators obtained in [4]–[5] (we refer also to [3] for maximal operators on unbounded domains).

Singular operators were treated in [6] and [16] within the framework of the spaces with variable exponents.

The main goal of the present paper is to establish the boundedness of Calderón–Zygmund singular operators in weighted spaces $L^{p(\cdot)}_\rho$. In particular, we obtain a weighted mapping theorem for finite Hilbert transform and apply this result to the boundedness of Cauchy singular operators on curves in the complex plane.

2. Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^n$ and $p(x)$ a measurable function on Ω such that

$$1 < \underline{p} \leq p(x) \leq \overline{p} < \infty, \quad x \in \Omega \tag{2.1}$$

and

$$|p(x) - p(y)| \leq \frac{A}{\ln \frac{1}{|x-y|}}, \quad |x-y| \leq \frac{1}{2}, \quad x, y \in \Omega. \tag{2.2}$$

We denote by $\mathcal{P} = \mathcal{P}(\Omega)$ the set of functions $p(x)$ satisfying conditions (2.1)–(2.2). We refer to Appendix A for examples of non-Hölderian functions satisfying condition (2.2). By $L^{p(\cdot)}$ we denote the space of functions $f(x)$ on Ω such that

$$A_p(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty.$$

This is a Banach function space with respect to the norm

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : A_p\left(\frac{f}{\lambda}\right) \leq 1 \right\} \tag{2.3}$$

(see, e.g., [5]). We denote

$$\frac{1}{q(x)} = 1 - \frac{1}{p(x)}.$$

Under condition (2.1) the space $L^{p(\cdot)}$ coincides with the space

$$\left\{ f(x) : \left| \int_\Omega f(x)\varphi(x)\, dx \right| < \infty \quad \text{for all} \quad \varphi(x) \in L^{q(\cdot)}(\Omega) \right\} \tag{2.4}$$

up to the equivalence of the norms

$$\|f\|_{L^{p(\cdot)}} \sim \sup_{\|\varphi\|_{L^{q(\cdot)}} \leq 1} \left| \int_\Omega f(x)\varphi(x)\, dx \right| \sim \sup_{A_q(\varphi) \leq 1} \left| \int_\Omega f(x)\varphi(x)\, dx \right|, \tag{2.5}$$

see [17], Theorem 2.3 or [20], Theorem 3.5.

Let ρ be a measurable almost everywhere positive integrable function. Such functions are usually called weights. The weighted Lebesgue space $L^{p(\cdot)}_\rho$ is defined as the set of all measurable functions for which

$$\|f\|_{L^{p(\cdot)}_\rho} = \|\rho f\|_{L^{p(\cdot)}} < \infty.$$

The space $L^{p(\cdot)}_\rho$ is a Banach space.

We deal with the following integral operators:
the Calderón–Zygmund singular operator

$$Tf(x) = \int_\Omega K(x,y)f(y)\,dy, \tag{2.6}$$

(as treated in [6]), the maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,dy, \tag{2.7}$$

the Riesz potential operator

$$I_\alpha f(x) = \int_\Omega \frac{f(y)}{|x-y|^{n-\alpha}}\,dy, \quad \alpha > 0,$$

and the Cauchy singular operator

$$S_\Gamma f(t) = \int_\Gamma \frac{f(\tau)d\tau}{\tau - t}, \qquad t = t(s), \quad 0 \le s \le \ell, \tag{2.8}$$

along a finite rectifiable Jordan curve Γ of the complex plane on which the arc-length is chosen as a parameter starting from any fixed point.

In the definition of the maximal function we assume that $f(x) = 0$ when $x \notin \Omega$.

In [5] the boundedness of the maximal operator in the space $L^{p(\cdot)}$ was proved. Later in [6] an analogous result for Calderón–Zygmund operator (2.6) was obtained.

The boundedness of the maximal operator M in the weighted Lebesgue space $L^{p(\cdot)}_\rho$ with the power weight $\rho(x) = |x - x_0|^\beta$ was established by the authors in [15], see also [14]. The main point of the result in [14], [15] is that the exponent β is related to the value of $p(x)$ at the point x_0. Recently, we have also established the boundedness of various integral operators, in particular Calderón–Zygmund operators, in weighted Lorentz type spaces with variable exponent [15], see also [16]. However the result of [15], [16] does not imply the boundedness of singular operators in the Lebesgue spaces with variable exponent.

3. Statements of the Main Results

Let

$$T^* f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|$$

be a maximal singular operator, where $T_\varepsilon f(x)$ is the usual truncation

$$T_\varepsilon f(x) = \int\limits_{|x-y|\geq\varepsilon} K(x,y)f(y)\,dy$$

and we assume that $f(x)=0$ outside Ω. In what follows,

$$\rho(x) = \prod_{k=1}^{m} |x-a_k|^{\beta_k}, \tag{3.1}$$

where $a_k \in \overline{\Omega}, \quad k=1,\cdots,m$.

Theorem 1. *Let $p(x)\in\mathcal{P}(\Omega)$ and $\rho(x)$ be weight function* (3.1)*. Then the operators T and T^* are bounded in the space $L^{p(\cdot)}_\rho(\Omega)$ if*

$$-\frac{n}{p(a_k)} < \beta_k < \frac{n}{q(a_k)}, \qquad k=1,\ldots,m. \tag{3.2}$$

Besides operator (2.8), we also consider the corresponding maximal singular operator

$$S^*f(t) = \sup_{\varepsilon>0}\left|\int\limits_{|s-\sigma|>\varepsilon} \frac{f[\tau(\sigma)]\tau'(\sigma)}{\tau(\sigma)-\tau(s)}\,d\sigma\right| \tag{3.3}$$

where it is supposed that $f[t(\sigma)]=0$ when $s\notin[0,\ell]$.

We remind that Γ is called the Lyapunov curve if $t'(s)\in Lip\,\gamma, 0<\gamma\leq 1$ and that in this case

$$\frac{t'(s)}{t(\sigma)-t(s)} = \frac{1}{\sigma-s} + h(s,\sigma), \quad \text{with} \quad |h(s,\sigma)| \leq \frac{c}{|\sigma-s|^{1-\gamma}} \tag{3.4}$$

see [13]; observe that

$$h(s,\sigma) = \frac{1}{t(\sigma)-t(s)}\int_0^1 [t'(s)-t'(s+\xi(\sigma-s))]\,d\xi \tag{3.5}$$

from which the estimate for $h(s,\sigma)$ follows.

If $t'(s)$ is a function of bounded variation, Γ is called a curve of bounded rotation. When Γ is a curve of bounded rotation without cusps, Γ satisfies the chord-arc condition

$$\left|\frac{t(s)-t(\sigma)}{s-\sigma}\right| \geq m > 0. \tag{3.6}$$

Thus we have $|t(s)-t(s_0)| \approx |s-s_0|$.

When dealing with the operators S and S^*, we assume the functions $p(s)$ and $\rho(s)\geq 0$ to be defined on $[0,\ell]$ and put

$$L^{p(\cdot)}_\rho = \left\{f : \|f[t(s)]\rho(s)\|_{L^{p(s)}} < \infty\right\}.$$

In the next theorem we take

$$\rho(s) = \prod_{k=1}^{m} |t(s)-t(c_k)|^{\beta_k} \approx \prod_{k=1}^{m} |s-c_k|^{\beta_k}, \tag{3.7}$$

where $c_k\in[0,\ell],\ k=1,2,\ldots,m$.

Theorem 2. *Let Γ be a Lyapunov curve or a curve of bounded rotation without cusps and let $p(s) \in \mathcal{P}$. The operators S_Γ and S^* are bounded in the space $L^{p(\cdot)}_\rho(\Gamma)$ with the weight function* (3.7) *if and only if*

$$-\frac{1}{p(c_k)} < \beta_k < \frac{1}{q(c_k)}, \qquad k = 1, 2, \dots, m. \tag{3.8}$$

4. Auxiliary Results

In this section we present some basic results which we need to prove our main statements. Let

$$M^\beta f(x) = \sup_{r>0} \frac{|x - x_0|^\beta}{|B(x,r)|} \int\limits_{B(x,r)} \frac{|f(y)|\,dy}{|y - x_0|^\beta}, \tag{4.1}$$

where $x_0 \in \overline{\Omega}$.

Theorem A ([15])**.** *Let $p(x) \in \mathcal{P}$. The operator M^β with $x_0 \in \Omega$ is bounded in the space $L^{p(\cdot)}(\Omega)$ if and only if*

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}. \tag{4.2}$$

When $x_0 \in \partial\Omega$, condition (4.2) *is sufficient in the case of any point x_0 and necessary if the point x_0 satisfies the condition $|\Omega_r(x_0)| \sim r^n$, where $\Omega_r(x_0) = \{y \in \Omega : r < |y - x_0| < 2r\}$.*

Theorem B ([15])**.** *Let $p(x) \in \mathcal{P}$. The Riesz potential operator I_α acts boundedly from the space $L^{p(\cdot)}_\rho(\Omega)$ with the weight $\rho(x) = |x - x_0|^\beta$, $x_0 \in \Omega$, into itself if condition* (4.2) *is satisfied.*

Let $F \in L_{loc}(\mathbb{R}^1)$ and

$$F^\#(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |F(y) - F_{B(x,r)}|\,dy, \tag{4.3}$$

where

$$F_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(z)\,dz.$$

Proposition A ([1])**.** *Let T be a Calderón–Zygmund operator. Then for arbitrary s, $0 < s < 1$, there exists a constant $c_s > 0$ such that*

$$\left[(|Tf|^s)^\#(x)\right]^{\frac{1}{s}} \le c_s Mf(x)$$

for all $f \in C_0^\infty(\mathbb{R}^n)$, $x \in \mathbb{R}^n$.

The following statement holds (see [4], Lemma 3.5).

Proposition B. *Let $p(x) \in \mathcal{P}$. Then for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$ there holds*

$$\left|\int_\Omega f(x)g(x)\,dx\right| \le c \int_\Omega f^\#(x) Mg(x)\,dx$$

with a constant $c > 0$ not depending on f.

Lemma 4.1. *Let $p(x) \in \mathcal{P}$, $w(x) = |x - x_0|^\gamma, x_0 \in \overline{\Omega}$, $-\frac{n}{p(x_0)} < \gamma < \frac{n}{q(x_0)}$. Then*

$$\|fw\|_{L^{p(\cdot)}} \leq c\|f^{\#}w\|_{L^{p(\cdot)}}$$

with a constant $c > 0$ not depending on f.

Proof. By (2.5) we have

$$\|fw\|_{L^{p(\cdot)}} \leq c \sup_{\|g\|_{L^{q(\cdot)}} \leq 1} \left| \int_\Omega f(x)g(x)w(x)\, dx \right|.$$

According to Proposition B,

$$\|fw\|_{L^{p(\cdot)}} \leq c \sup_{\|g\|_{L^{q(\cdot)}} \leq 1} \left| \int_\Omega f^{\#}(x)w(x)[w(x)]^{-1}M(gw)\, dx \right|.$$

Making use of the Hölder inequality for $L^{p(\cdot)}$, we derive

$$\|fw\|_{L^{p(\cdot)}} \leq c \sup_{\|g\|_{L^{q(\cdot)}} \leq 1} \|f^{\#}w\|_{L^{p(\cdot)}} \|w^{-1}M(gw)\|_{L^{q(\cdot)}}.$$

We observe that $-\frac{1}{q(x_0)} < -\gamma < \frac{1}{p(x_0)}$. Therefore we may apply Theorem A for the space $L^{q(\cdot)}$ with $\beta = \gamma$ and conclude that

$$\|fw\|_{L^{p(\cdot)}} \leq c \sup_{\|g\|_{L^{q(\cdot)}} \leq 1} \|f^{\#}w\|_{L^{p(\cdot)}} \|g\|_{L^{q(\cdot)}} \leq \|f^{\#}w\|_{L^{p(\cdot)}}.$$

□

Lemma 4.1 in the case of the constant $p(\cdot)$ and $w \equiv 1$ is well known [25]. For variable exponent and $w \equiv 1$ it was proved in [6].

Theorem 4.1. *Let $p(x)$ be a measurable function on $\mathbb{R}^n$ such that $1 \leq \underline{p} \leq p(x) < \overline{p} < \infty$, $\rho(x) \geq 0$ and $|\{x \in \mathbb{R}^n : \rho(x) = 0\}| = 0$ and*

$$w(x) = [\rho(x)]^{p(x)} \in L^1_{loc}(\mathbb{R}^n). \tag{4.4}$$

Then $C_0^\infty(\mathbb{R}^n)$ is dense in the space $L^{p(\cdot)}_\rho(\mathbb{R}^n)$.

Proof. **I.** First we prove that the class $C_0(\mathbb{R}^n)$ of continuous functions with a compact support is dense in the space $L^{p(\cdot)}_\rho(\mathbb{R}^n)$.

Let $f \in L^{p(\cdot)}_\rho(\mathbb{R}^n)$. Since $|\{x \in \mathbb{R}^n : \rho(x) = 0\}| = 0$, the function $f(x)$ is a.e. finite.

1st step. The functions $f_N(x) = \begin{cases} f(x), & |x| < N \\ 0, & |x| > N \end{cases}$ approximate f in $L^{p(\cdot)}_\rho$ since $A_p(\rho|f - f_N|) \to 0$ as $N \to \infty$. Therefore there exists a function $g \in L^{p(\cdot)}_\rho$ with a compact support such that $\|f - g\|_{L^{p(\cdot)}_\rho} < \varepsilon$.

2nd step. The function g can be approximated in $L^{p(\cdot)}_\rho$ by the bounded functions with a compact support $\tilde{g}_N(x) = \begin{cases} g(x), & |g(x)| < N \\ 0, & |g(x)| > N \end{cases}$. Indeed, the passage to the limit

$$A_p(\rho|g - \tilde{g}_N|) = \int_{\mathbb{R}^n} w(x)|g(x) - \tilde{g}_N(x)|^{p(x)}\, dx \ \to 0 \quad \text{as} \quad n \to \infty \tag{4.5}$$

is justified by the Lebesgue dominated convergence theorem since $w(x)|g(x) - g_N(x)|^{p(x)} \le 2w(x)|g(x)|^{p(x)}$ and the integrand tends to zero a.e. (at any point x at which $f(x)$ is finite). So we choose $\tilde{g}_N$ such that $\|g - \tilde{g}_N\|_{L^{p(\cdot)}_\rho} < \varepsilon$.

3rd step. To approximate the function $\tilde{g}_N$ by continuous bounded functions, we choose $\delta > 0$ so that

$$\int_E w(x)\,dx < \varepsilon_1 = \frac{\varepsilon}{(2N)^{\overline{p}}} \tag{4.6}$$

for any measurable set $E \subset \operatorname{supp} g$ with $|E| < \delta$, which is possible by (4.4). Then we choose a function $\varphi(x) \in C(\mathbb{R}^n)$ with $\|\varphi\|_C \le N$ which coincides with $\tilde{g}_N(x)$ everywhere except possibly for the set $A \subset \operatorname{supp} g$ with $|A| < \delta$, by the Luzin theorem. Then

$$A_p(\rho|\tilde{g}_N - \varphi|) = \int_A w(x)|\tilde{g}_N(x) - \varphi(x)|^{p(x)}\,dx \le (2N)^{\overline{p}} \int_A w(x)\,dx < \varepsilon\,.$$

4th step. It remains to approximate in $L^{p(\cdot)}_\rho$ the function φ by a continuous function with a compact support, which is done in the standard way by means of smooth truncation.

II. Approximation in $L^{p(\cdot)}_\rho$ of a continuous function with a compact support by C_0^∞-functions can already be realized via the identity approximation

$$K_t\varphi = \frac{1}{t^n}\int_{\mathbb{R}^n} a\left(\frac{y}{t}\right)\varphi(x-y)\,dy, \quad t > 0,$$

with $a(x) \in C_0^\infty$ and $\int_{\mathbb{R}^n} a(x)\,dx = 1$. Obviously, $K_t\varphi \in C_0^\infty$ for $\varphi \in C_0(\mathbb{R}^n)$ and $|K_t\varphi - \varphi| < \varepsilon$ as $t \to 0$ uniformly on any given compact set. Therefore,

$$A_p(\rho|K_t\varphi - \varphi|) = \int_B w(x)|K_t\varphi - \varphi(x)|^{p(x)}\,dx \le \varepsilon^{\underline{p}} \int_B w(x)\,dx$$

for t small enough, B being a sufficiently large ball. □

Corollary. *Let $p(x)$ and $\rho(x)$ satisfy the assumptions of Theorem* 4.1 *in* Ω. *Then the set $C^\infty(\overline{\Omega})$ is dense in the space $L^{p(\cdot)}_\rho(\Omega)$.*

Indeed, it suffices to observe that functions from $C^\infty(\overline{\Omega})$ can be continued outside Ω as $C_0^\infty(\mathbb{R}^n)$-functions; likewise, $p(x)$ and $\rho(x)$ can be continued with preservation of their properties.

The statement of Theorem 4.1 is well known in the case of constant exponent, while in the case of variable exponent and $\rho(x) \equiv 1$ it was proved in [17]. The denseness of $C_0^\infty(\mathbb{R}^n)$ in non-weighted Sobolev spaces $W^{m,p(\cdot)}$ was proved in [21]–[22].

5. Proofs of the Main Results

It suffices to deal with a weight of the form $\rho(x) = |x - x_0|^\beta$ because the general case (3.1) is easily reduced to this special case by separation of the points a_k by means of the partition of unity, which provides the representation

$$\frac{\prod_{k=1}^{m}|x-a_k|^{\beta_k}}{\prod_{k=1}^{m}|y-a_k|^{\beta_k}}=\sum_{k=1}^{m}c_k(x,y)\frac{|x-a_k|^{\beta_k}}{|y-a_k|^{\beta_k}}$$

with bounded "coefficients" $c_k(x,y)$.

Proof of Theorem 1. Let us consider the operator T. Let $f\in C^\infty(\overline{\Omega})$ and $0<s<1$. Obviously,

$$\|\rho Tf\|_{L^{p(\cdot)}}=\|\rho^s|Tf|^s\|^{\frac{1}{s}}_{L^{\frac{p(\cdot)}{s}}}. \tag{5.1}$$

Applying Lemma 4.1 with $w(x)=[\rho(x)]^s$ and $p(\cdot)$ replaced by $\frac{p(\cdot)}{s}$, we obtain

$$\|\rho Tf\|_{L^{p(\cdot)}}\le c\left\|\rho^s(|Tf|^s)^{\#}\right\|^{\frac{1}{s}}_{L^{\frac{p(\cdot)}{s}}},$$

which is possible since the condition

$$-\frac{n}{\frac{p(x_0)}{s}}<s\beta<\frac{n}{\left(\frac{p(x_0)}{s}\right)'}$$

is satisfied. Thus we have

$$\|\rho Tf\|_{L^{p(\cdot)}}\le c\left\|\rho\left[(|Tf|^s)^{\#}\right]^{\frac{1}{s}}\right\|_{L^{p(\cdot)}} \tag{5.2}$$

because $\|f\|^{\frac{1}{s}}_{L^{\frac{p(\cdot)}{s}}}=\||f|^{\frac{1}{s}}\|_{L^{p(\cdot)}}$. Since a function $f\in C^\infty(\overline{\Omega})$ can be continued outside Ω as a $C_0^\infty(\mathbb{R}^n)$-function, Proposition A is applicable. Therefore, by Proposition A, from estimate (5.2) we get

$$\|\rho Tf\|_{L^{p(\cdot)}}\le c\,\|\rho(Mf)\|_{L^{p(\cdot)}}.$$

Now we apply Theorem A and conclude that

$$\|\rho Tf\|_{L^{p(\cdot)}}\le c\,\|\rho f\|_{L^{p(\cdot)}}$$

for all $f\in C_0^\infty(\mathbb{R}^n)$. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}_\rho$ by Theorem 4.1, we complete the proof of Theorem 1. □

The boundedness of the operator T^* follows from the known estimate

$$T^*f(x)\le c[M(Tf)(x)+Mf(x)],$$

from Theorem A and Theorem 1.

Corollary 1. *Let $\Omega=[a,b]$, $\rho(x)=\prod_{k=1}^{m}|x-a_k|^{\beta_k}$, $a_k\in[a,b]$, $k=1,\dots,m$, and $p(x)\in\mathcal{P}$. Then the finite Hilbert transform and its maximal version*

$$H_{[a,b]}f=\int_a^b\frac{f(y)\,dy}{y-x}\quad\textit{and}\quad H^*_{[a,b]}=\sup_{\varepsilon>0}\left|\int\limits_{|y-x|>\varepsilon}\frac{f(y)\,dy}{y-x}\right| \tag{5.3}$$

are bounded in the space $L^{p(\cdot)}_\rho(a,b)$, if $-\frac{1}{p(a_k)}<\beta_k<\frac{1}{q(a_k)}$, $k=1,2,\dots,m$.

Proof of Theorem 2. We assume the function $p(s)$ to be defined on $[0,l]$. The function $f(t(\sigma))$ will be denoted by $f_0(\sigma)$. In the case of a Lyapunov curve we use equality (3.4) and apply Corollary 1 and Theorem B, which immediately gives the statement of Theorem 2.

Let Γ be a curve of bounded rotation without cusps and let V be the total variation of $t'(s)$ on $[0,l]$. In this case the function $h(s,\sigma)$ can be estimated as

$$|h(s,\sigma)| \leq c\frac{V(s)-V(\sigma)}{s-\sigma} \tag{5.4}$$

according to (3.5) and (3.6) (see [13], Chapter II, Subsection 2.3). Then we have

$$\begin{aligned}
&\left|\int_{|s-\sigma|>\varepsilon} \frac{f_0(\sigma)\,d\sigma}{t(\sigma)-t(s)}\right| \\
&\quad \leq c\left|\int_{|s-\sigma|>\varepsilon} \frac{f_0(\sigma)\,d\sigma}{\sigma-s}\right| + c\int_{|s-\sigma|>\varepsilon} \frac{|f_0(\sigma)|\,(V(\sigma)-V(s))}{\sigma-s}d\sigma \\
&\quad \leq c\left|\int_{|s-\sigma|>\varepsilon} \frac{f_0(\sigma)\,d\sigma}{\sigma-s}\right| + cV(s)\left|\int_{|s-\sigma|>\varepsilon} \frac{|f_0(\sigma)|\,d\sigma}{\sigma-s}\right| \\
&\qquad + c\left|\int_{|s-\sigma|>\varepsilon} \frac{|f_0(\sigma)|\,V(\sigma)}{\sigma-s}\,d\sigma\right|.
\end{aligned}$$

From here by Corollary 1 and the boundedness of the function $V(s)$ we conclude that the operator S^*_Γ is bounded in $L^{p(\cdot)}_\rho$.

Let us prove the necessity part. From the boundedness of S_Γ in $L^{p(s)}_\rho$ it follows that $S_\Gamma f(t)$ exists almost everywhere for arbitrary $f \in L^{p(s)}_\rho$. Thus ρ should be such that $f \in L^1(\Gamma)$ for arbitrary $f \in L^{p(s)}_\rho$. The function $f = f\rho\rho^{-1}$ belongs to $L^1(\Gamma)$ for arbitrary $f \in L^{p(s)}_\rho$ if and only if $\rho^{-1} \in L^{q(s)}$, which follows from equivalence (2.5). Then function $\rho^{-1}(s) = |s-s_0|^{-\beta}$, $s_0 \in [0,l]$, belongs to $L^{q(s)}[0,l]$ if and only if $\beta < \frac{1}{q(s_0)}$. Indeed, we have

$$|s-s_0|^{-\beta q(s)} = m(s)|s-s_0|^{-\beta q(s_0)},$$

where the function $m(s) = |s-s_0|^{-\beta(q(s)-q(s_0))}$ satisfies the condition

$$0 < c \leq m(s) \leq C < \infty$$

in view of (2.2). On the other hand, from $|s-s_0|^{-\beta q(s_0)} \in L^{q(s)}$ we have $\beta < \frac{1}{q(s_0)}$.

The necessity of the condition $-\frac{1}{p(s_0)} < \beta$ follows from the duality argument. □

6. Appendix

The following is an example of the function which satisfies condition (2.2) but is not a Hölder function:

$$p(x) = a(x) + \frac{b(x)}{\left(\ln \frac{A}{|x|}\right)^\gamma}, \qquad x \in \Omega, \tag{6.1}$$

where $a(x)$ and $b(x)$ are Hölder functions, $a(x) \geq 1,\ b(x) \geq 0,\ A > \sup\limits_{x\in\Omega} |x|$ and $\gamma \geq 1$. One may write a little bit more complicated example:

$$p(x) = a(x) + \frac{b(x)}{\left(\ln \frac{A}{|x|}\right)^{\gamma}} \left(\ln \ln \ln \cdots \ln \frac{C}{|x|}\right)^{\mu} \tag{6.2}$$

with arbitrary sufficiently large $C > 1$ and $\mu > 0$, and the same assumptions on $a(x), b(x)$, and A, but $\gamma > 1$. It is also possible to take different powers of different logarithms as factors or superpositions in (6.2).

To prove condition (2.2) for functions (6.1) or (6.2) or functions of a similar type, we do not need to check condition (2.2) directly. For this purpose we can use properties of continuity moduli. It suffices to deal with the case where $a(x) \equiv 0$ and $b(x) = 1$ since we consider differences $p(x) - p(y)$.

We remind that a non-negative function $f(t)$ on $[0, \ell]$ is called a continuity modulus if

$$\omega(f, h) \sim f(h),$$

where $\omega(f, h) = \sup\limits_{\substack{|t_1 - t_2| \leq h \\ t_1, t_2 \in [0,\ell]}} |f(t_1) - f(t_2)|$. The sufficient conditions for a function $f(x)$ to be a continuity modulus are known, see, for example [2] or [7]:

1) $f(x)$ is continuous on $[0, \ell]$,

2) $f(0) = 0$ and $f(x) > 0$ for $x > 0$,

3) $f(x)$ is non-decreasing and $\frac{f(x)}{x}$ is non-increasing in a neighborhood of the point $x = 0$. It is easy to check that functions (6.1)–(6.2) satisfy the above conditions 1)-3).

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