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INVERSION OF THE CAUCHY INTEGRAL TAKEN OVER THE DOUBLE PERIODIC LINE

N. KHATIASHVILI

Dedicated to my school-teacher Grigol Tsulaia

Abstract. The Cauchy type singular integral equation is investigated when the line of integration is the union of a countable number of disconnected segments. The equation is equivalently reduced to an equation with double periodic kernel. Effective solutions having integrable singularities at the end-points of the line of integration are obtained.

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We consider the Cauchy type singular integral equation when the line of integration is doubly-periodical and disconnected. This equation arises in periodical problems of mathematical physics.

The theory of integral equations when the line of integration is the union of a finite number of disconnected arcs was thoroughly studied by N. I. Muskhelishvili [1].

The case, where the line of integration is periodical and disconnected with one period was investigated by N. I. Akhiezer [2], A. V. Bitsadze [3], S. A. Freidkin [4].

Let L be a disconnected double periodic line in the complex z-plane, $z = x + iy$, i.e., L be the union of a countable number of disconnected segments with positive direction from $-a$ to a and from $-b + ih$ to $b + ih$:

$$L = \bigcup_{m,n} \left(L^1_{mn} \bigcup L^2_{mn} \right), \quad m, n = 0, \pm 1, \pm 2, \dots ,$$
$$L^1_{mn} = \{-a + 2m\omega \le x \le a + 2m\omega,\ y = 2nh\},$$
$$L^2_{mn} = \{-b + 2m\omega \le x \le b + 2m\omega,\ y = (2n + 1)h\},$$

where a, b, ω and h are constants, $0 < a, b < \omega;\ h > 0$.

It is required to solve the singular integral equation

$$\frac{1}{\pi} \int_L \left[\frac{u(t)}{t - z} - \frac{u(t)}{t} \right] dt = f(z), \quad z \in L, \tag{1}$$

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where $f(z)$ is a given real function which belongs to the Hölder's class on every $L^1_{mn}\bigcup L^2_{mn}$, $m,n=0,\pm1,\pm2,\dots$,and satisfies the following conditions:

$$\begin{aligned}
&f(x+2m\omega+2nih)=f(x), \quad x\in[-a,a],\\
&f(x+2m\omega+(2n+1)ih)=f(x+ih), \quad x\in[-b,b],\\
&m,n=0,\pm1,\pm2,\dots,\\
&f(0)=0,
\end{aligned}$$

$u(z)$ is the unknown function satisfying the following conditions:

$$\begin{aligned}
&u(x+2m\omega+2nih)=u(x), \quad x\in[-a,a],\\
&u(x+2m\omega+(2n+1)ih)=u(x+ih), \quad x\in[-b,b],\\
&m,n=0,\pm1,\pm2,\dots,\\
&\int\limits_{L^1_{00}\bigcup L^2_{00}} u(t)dt=0.
\end{aligned} \tag{2}$$

Also, $u(t)$ is assumed to belong to the Muskhelishvili class, the class H^* on $L^1_{mn}\bigcup L^2_{mn}$, $m,n=0,\pm1,\pm2,\dots$ [1].

Definition. If the function $u(t)$, given on $L^1_{mn}\bigcup L^2_{mn}$, $m,n=0,\pm1,\pm2,\dots$, satisfies the Hölder's condition H, on every closed part of $L^1_{mn}\bigcup L^2_{mn}$, $m,n=0,\pm1,\pm2,\dots$, not containing ends, and if near any end c it is of the form

$$u(t)=\frac{u^*(t)}{(t-c)^\alpha}, \quad 0\le\alpha<1,$$

where $u^*(t)$ belongs to the class H, then $u(t)$ is said to belong to the class H^* on $L^1_{mn}\bigcup L^2_{mn}$, $m,n=0,\pm1,\pm2,\dots$.

The integral in the left-hand side of (1) is understood as the series

$$\int\limits_L\left[\frac{u(t)}{t-z}-\frac{u(t)}{t}\right]dt=\sum_{m,n=-\infty}^{+\infty}\int\limits_{L^1_{mn}\bigcup L^2_{mn}}\left[\frac{u(t)}{t-z}-\frac{u(t)}{t}\right]dt,$$

where

$$\int\limits_{L^1_{mn}\bigcup L^2_{mn}}\frac{u(t)}{t-z}dt=\int\limits_{L^1_{00}\bigcup L^2_{00}}\frac{u(t)}{t+2n\omega+2mih-z}dt.$$

Condition (2) is necessary and sufficient for this series, to converge.

Equation (1) must hold for all z on L, with a possible exception for the ends of the line of integration.

We will find a solution of equation (1) in the following classes of functions:

1) $\mathcal{H}(-a,a)$, the class of functions bounded at the ends of the line L^1_{00},

2) $\mathcal{H}(-b+ih,b+ih)$, the class of functions bounded at the ends of the line L^2_{00},

3) $\mathcal{H}(-a,-b+ih)$, the class of functions bounded at the ends $-a,-b+ih$,

4) $\mathcal{H}(a,b+ih)$, the class of functions bounded at the ends $a,b+ih$,

5) $\mathcal{H}(-a,a,-b+ih)$, the class of functions bounded at the ends $-a,a,-b+ih$,

6) $\mathcal{H}(b+ih)$, the class of functions bounded at the end $b+ih$,

7) $\mathcal{H}_0$, the general class of functions having integrable singularities at all end-points,

8) $\mathcal{H}$, the class of functions bounded at all end-points.

Taking into account the representation of Weierstrass function $\zeta(z)$ [5, Part 2, Ch. 1, §14]

$$\zeta(t-z) = \frac{1}{t-z} + \sum_{\substack{m,n=-\infty \\ |m|+|n|\neq 0}}^{+\infty} \left\{ \frac{1}{t-z-2n\omega-2mih} + \frac{1}{2n\omega+2mih} + \frac{t-z}{(2n\omega+2mih)^2} \right\},$$

we can transform equation (1) to the singular integral equation

$$\frac{1}{\pi} \int_{L_{00}^1 \bigcup L_{00}^2} u(t)[\zeta(t-z)-\zeta(t)]dt = f(z), \quad z \in L_{00}^1 \bigcup L_{00}^2, \tag{3}$$

with condition (2).

In the sequel we will use Villa's formula for a strip which represents an analytic periodical function $\Phi(z) = u_0 + iv_0$ with period 2ω in a strip $0 < y < h$ of the complex plane $z = x+iy$ by its real part given on the boundary [6]:

$$\Phi(z) = \frac{1}{\pi i} \int_0^{2\omega} u_0^1(t)[\zeta(t-z)-\zeta(t)]dt - \frac{1}{\pi i} \int_{ih}^{2\omega+ih} u_0^2(t)[\zeta(t-z)-\zeta(t)]dt + iC_0, \tag{4}$$

where C_0 is the real constant, $u_0^1(t)$ and $u_0^2(t)$ are the given functions, satisfying the conditions

$$u_0^1(t) = u_0^+(t), \quad t \in [0, 2\omega],$$
$$u_0^2(t) = u_0^+(t), t \in \{x \in [0,2\omega], y = h\},$$
$$\int_0^{2\omega} u_0^1(t)dt = \int_{ih}^{2\omega+ih} u_0^2(t)dt.$$

We also use the following representation of any elliptic function $g(z)$ of n-th order, with zeros $\alpha_1, \alpha_2, \dots, \alpha_n$ and poles $\beta_1, \beta_2, \dots, \beta_n$ in the period rectangle:

$$g(z) = C \frac{\sigma(z-\alpha_1^*)\sigma(z-\alpha_2)\cdots\sigma(z-\alpha_n)}{\sigma(z-\beta_1)\sigma(z-\beta_2)\cdots\sigma(z-\beta_n)}, \tag{5}$$

where C is an arbitrary constant,

$$\alpha_1^* = (\beta_1+\beta_2+\cdots+\beta_n) - (\alpha_2+\alpha_3+\cdots+\alpha_n)$$

and $\sigma(z)$ is Weierstrass function for the periods 2ω and $2ih$ (see [5]).

The following theorem consisting of eight parts is proved

Theorem.

1. *There exists a unique solution of equation* (1) *of the class* $\mathcal{H}(-a,a)$ *if and only if* $f(z)$ *satisfies the condition*

$$\int\limits_{L^1_{00}\bigcup L^2_{00}} \frac{f(t)-f(ih)}{\sqrt{X_1(t)}}\,dt=0, \tag{6}$$

where

$$X_1(z)=\frac{\sigma(z-a)\sigma(z+a)\sigma^2(z-ih)}{\sigma^2(z)\sigma(z-b-ih)\sigma(z+b-ih)}\ , \tag{7}$$

and this solution is given by

$$u(z)=-\frac{\sqrt{X_1(z)}}{\pi}\int\limits_{L^1_{00}\bigcup L^2_{00}} \frac{f(t)-f(ih)}{\sqrt{X_1(t)}}[\zeta(t-z)-\zeta(t)]dt,\ \ z\in L^1_{00}\bigcup L^2_{00}. \tag{8}$$

2. *A unique solution of the class* $\mathcal{H}(-b+ih,b+ih)$ *exists if and only if* $f(z)$ *satisfies the condition*

$$\int\limits_{L^1_{00}\bigcup L^2_{00}} f(t)\sqrt{X_1(t)}dt=0,$$

and is given by

$$u(z)=-\frac{1}{\pi\sqrt{X_1(z)}}\int\limits_{L^1_{00}\bigcup L^2_{00}} f(t)\sqrt{X_1(t)}[\zeta(t-z)-\zeta(t-ih)]dt,\quad z\in L^1_{00}\bigcup L^2_{00}.$$

3. *There exists a unique solution of equation* (1) *of the class* $\mathcal{H}(-a,-b+ih)$ *if and only if* $f(z)$ *satisfies the condition*

$$\int\limits_{L^1_{00}\bigcup L^2_{00}} \frac{f(t)}{\sqrt{X_2(t)}}\,dt=0,$$

where

$$X_2(z)=e^{2\delta' b}\,\frac{\sigma(z+a)\sigma(z+b-ih)\sigma^2(z-a-b)}{\sigma(z-a)\sigma(z-b-ih)\sigma^2(z)}\ ,$$

δ' *is the constant* [5], *and this solution is given by*

$$u(z)=-\frac{\sqrt{X_2(z)}}{\pi}\int\limits_{L^1_{00}\bigcup L^2_{00}} \frac{f(t)}{\sqrt{X_2(t)}}[\zeta(t-z)-\zeta(t)]dt,\quad z\in L^1_{00}\bigcup L^2_{00}.$$

4. *A unique solution of the class* $\mathcal{H}(a,b+ih)$ *exists if and only if* $f(z)$ *satisfies the condition*

$$\int\limits_{L^1_{00}\bigcup L^2_{00}} f(t)\sqrt{X_2(t)}dt=0,$$

and is given by

$$u(z) = -\frac{1}{\pi\sqrt{X_2(z)}} \int\limits_{L^1_{00}\bigcup L^2_{00}} f(t)\sqrt{X_2(t)}[\zeta(t-z)-\zeta(t-a-b)]dt, \quad z \in L^1_{00}\bigcup L^2_{00}.$$

5. *A unique solution of the class* $\mathcal{H}(-a, a, -b+ih)$ *exists if and only if* $f(z)$ *satisfies the conditions*

$$\int\limits_{L^1_{00}\bigcup L^2_{00}} \frac{f(t)}{\sqrt{X_3(t)}}\, dt = 0,$$

$$\int\limits_{L^1_{00}\bigcup L^2_{00}} \frac{f(t)}{\sqrt{X_3(t)}}\, [\zeta(t+b)-\zeta(t)]dt = 0,$$

where

$$X_3(z) = e^{-2\delta' b}\, \frac{\sigma(z-a)\sigma(z+a)\sigma(z+b-ih)}{\sigma(z-b-ih)\sigma^2(z+b)}\ ,$$

and this solution is given by

$$u(z) = -\frac{\sqrt{X_3(z)}}{\pi} \int\limits_{L^1_{00}\bigcup L^2_{00}} \frac{f(t)}{\sqrt{X_3(t)}}[\zeta(t-z)-\zeta(t+b)]dt, \quad z \in L^1_{00}\bigcup L^2_{00}.$$

6. *A solution of the class* $\mathcal{H}(-b+ih)$ *exists for* $a < b$ *if and only if* $f(z)$ *satisfies the condition*

$$\int\limits_{L^1_{00}\bigcup L^2_{00}} f(t)\sqrt{X_3(t)}dt = 0,$$

and is given by

$$u(z) = -\frac{1}{\pi\sqrt{X_3(z)}} \int\limits_{L^1_{00}\bigcup L^2_{00}} f(t)\sqrt{X_3(t)}[\zeta(t-z)-\zeta(t)]dt, \quad z \in L^1_{00}\bigcup L^2_{00}.$$

For $b < a$ *a solution of the class* $\mathcal{H}(-b+ih)$ *exists if and only if* $f(z)$ *satisfies the condition*

$$\int\limits_{L^1_{00}\bigcup L^2_{00}} (f(t)-f(-b))\sqrt{X_3(t)}dt = 0,$$

and is given by

$$u(z) = -\frac{1}{\pi\sqrt{X_3(z)}} \int\limits_{L^1_{00}\bigcup L^2_{00}} [f(t)-f(-b)]\sqrt{X_3(t)}[\zeta(t-z)-\zeta(t)]dt, \ z \in L^1_{00}\bigcup L^2_{00}.$$

7. *A solution of the class* $\mathcal{H}_0$ *exists if and only if*

$$\int\limits_{L^1_{00}\bigcup L^2_{00}} \left[\frac{f(t)}{\sqrt{X_0(t)}} - C_*\sqrt{X_4(t)}\right] dt = 0, \tag{9}$$

and is given by

$$u(z)=\frac{\sqrt{X_0(z)}}{\pi i}\int\limits_{L^1_{00}\bigcup L^2_{00}}\left[\frac{f(t)}{i\sqrt{X_0(t)}}-C_*\sqrt{X_4(t)}\right][\zeta(t-z)-\zeta(t)]dt$$
$$+\,iK\sqrt{X_0(z)},\quad z\in L^1_{00}\bigcup L^2_{00}, \tag{10}$$

where

$$\begin{aligned}X_0(z)&=\frac{\sigma^2(z-ih)\sigma^2(z)}{\sigma(z-a)\sigma(z+a)\sigma(z-b-ih)\sigma(z+b-ih)}\,,\\ X_4(z)&=\frac{\sigma(z-2ih)\sigma(z)}{\sigma^2(z-ih)}\,,\\ C_*&=f(ih)\,\frac{\sigma(b)|\sigma(a-ih)|}{\sigma^2(ih)}\,,\end{aligned}\tag{11}$$

K is an arbitrary constant.

8. *A solution of the class bounded at the all end-points (class $\mathcal{H}$) exists if and only if $f(z)$ satisfies the conditions*

$$\int\limits_{L^1_{00}\bigcup L^2_{00}}f(t)\sqrt{X_0(t)}dt=0,$$

$$\int\limits_{L^1_{00}\bigcup L^2_{00}}f(t)\sqrt{X_0(t)}[\zeta(t-ih)-\zeta(t)]dt=0,$$

and is given by

$$u(z)=-\frac{1}{\pi\sqrt{X_0(z)}}\int\limits_{L^1_{00}\bigcup L^2_{00}}f(t)\sqrt{X_0(t)}[\zeta(t-z)-\zeta(t)]dt,\quad z\in L^1_{00}\bigcup L^2_{00},$$

where $\zeta(z)$ and $\sigma(z)$ are Weierstrass functions for the periods 2ω and $2ih$ [5]. *$\sqrt{X_j(z)}$, $j=*,0,1,2,3$, is the branch for which $\sqrt{X_j(z)}\geq 0$ when*

$$z\in\{x\in(-\infty,+\infty),\ y=0\}\cup\{x\in(-\infty,+\infty),\ y=h\}\backslash L\equiv L^*.$$

Remark 1. Conditions (6) and (9) are fulfilled for all $f(z)$ satisfying the condition

$$f(z)=-f(-\bar z).$$

Remark 2. The root $\sqrt{X_j(z)}$, $j=*,0,1,2,3$, always indicates the branch which is holomorphic in the z-plane cut along L. The boundary value taken by the root on L from the left is denoted by $[\sqrt{X_j(z)}]^+=\sqrt{X_j(z)}$.

Proof of Theorem. Let us prove the first part. Suppose that a solution of equation (1) exists. In the strip $0<y<h$ let us consider the function

$$\Psi(z)=\frac{1}{\pi i}\int\limits_{L^1_{00}\bigcup L^2_{00}}u(t)[\zeta(t-z)-\zeta(t)]dt. \tag{12}$$

By Villa's formula (4) and conditions (2), (3) this function is periodic and analytic in the strip $0 < y < h$ and satisfies the following mixed boundary conditions:

$$\begin{aligned} &\operatorname{Im}\Psi^+(t) = -f(t), \quad t \in L, \\ &\operatorname{Re}\Psi^+(t) = 0, \quad t \in L^*, \\ &\Psi(0) = 0. \end{aligned} \tag{13}$$

So if we solve the boundary value problem (13) for the function $\Psi(z)$ and take into consideration (3), then a solution of equation (1) is given by

$$\begin{aligned} &\operatorname{Re}\Psi^+(t) = u(t), \quad t \in L_{00}^1, \\ &\operatorname{Re}\Psi^+(t) = -u(t), \quad t \in L_{00}^2. \end{aligned}$$

The function $\sqrt{X_1(z)}$, where $X_1(z)$ is given by (7), satisfies, in the strip $0 < y < h$, the boundary conditions

$$\begin{aligned} &\operatorname{Re}\sqrt{X_1(t)}^+ = 0, \quad t \in L, \\ &\operatorname{Im}\sqrt{X_1(t)}^+ = 0, \quad t \in L^*. \end{aligned} \tag{14}$$

Let us consider the function $\frac{\Psi(z)}{\sqrt{X_1(z)}} - \frac{iC}{\sqrt{X_1(z)}}$, where the real constant C is chosen appropriately. According to (13), (14) this function satisfies the following boundary conditions

$$\begin{aligned} &\operatorname{Re}\left[\frac{\Psi(t)}{\sqrt{X_1(t)}} - \frac{iC}{\sqrt{X_1(t)}}\right]^+ = \frac{f(t)+C}{i\sqrt{X_1(t)}}, \quad t \in L_{00}^1, \\ &\operatorname{Re}\left[\frac{\Psi(t)}{\sqrt{X_1(t)}} - \frac{iC}{\sqrt{X_1(t)}}\right]^+ = -\frac{f(t)+C}{i\sqrt{X_1(t)}}, \quad t \in L_{00}^2, \\ &\operatorname{Re}\left[\frac{\Psi(t)}{\sqrt{X_1(t)}} - \frac{iC}{\sqrt{X_1(t)}}\right]^+ = 0, \quad t \in L^*. \end{aligned} \tag{15}$$

Taking into account Villa's formula (4) and conditions (15), we get

$$\frac{\Psi(z)}{\sqrt{X_1(z)}} - \frac{iC}{\sqrt{X_1(z)}} = \frac{1}{\pi i} \int\limits_{L_{00}^1 \bigcup L_{00}^2} \frac{f(t)+C}{i\sqrt{X_1(t)}}\,[\zeta(t-z) - \zeta(t)]dt + iK_0,$$
$$z \in \{x \in (-\infty, +\infty), \;\; 0 < y < h\},$$

K_0 is a real constant. Since $\frac{1}{\sqrt{X_1(0)}} = 0$, we have $K_0 = 0$, and since $\sqrt{X_1(ih)} = 0$, for the convergence of the integral we have $C = -f(ih)$. Therefore by condition (6) the function Ψ given by (12) is of the form

$$\Psi(z) = -\frac{\sqrt{X_1(z)}}{\pi} \int\limits_{L_{00}^1 \bigcup L_{00}^2} \frac{f(t) - f(ih)}{\sqrt{X_1(t)}}\,[\zeta(t-z) - \zeta(t)]dt + iC, \tag{16}$$
$$z \in \{x \in (-\infty, +\infty), \;\; 0 < y < h\}.$$

Using the Plemelj formula [1], from (16) we get (8).

From condition (6) it follows that the function $\Psi(t)$ is periodic and therefore condition (2) is automatically fulfilled.

The behavior of the singular integral near the end-points of the line of integration implies $u(t) \in \mathcal{H}(-a,a)$ [1].

Now we will prove the uniqueness of the solution of equation (1).

Let us consider the corresponding homogeneous equation

$$\int\limits_{L_{00}^1 \bigcup L_{00}^2} u_0(t)[\zeta(t-z)-\zeta(t)]dt = 0, \quad z \in L_{00}^1 \bigcup L_{00}^2, \tag{17}$$

The corresponding problem for the analytic function $\Psi(z)$ given by (12) is as follows.

Problem. In the strip $0 < y < h$ define the analytic periodic function $\Psi(z)$ satisfying the boundary conditions

$$\begin{aligned} \operatorname{Im} \Psi^+(t) &= 0, \quad t \in L, \\ \operatorname{Re} \Psi^+(t) &= 0, \quad t \in L^*. \end{aligned} \tag{18}$$

As $u_0(t) \in \mathcal{H}(-a,a)$, according to conditions (18) the function $\Psi(z)$ may have poles at the points $-b+ih, b+ih$ and zeros at the points $-a, a$; also, $\Psi(z)$ satisfies the additional condition

$$\Psi(0) = 0. \tag{19}$$

By (18) we conclude that the function $\Psi^2(z)$ has poles at the points $-b+ih, b+ih$ and zeros at the points $-a, a$ and satisfies, in the strip, the boundary conditions

$$\begin{aligned} \operatorname{Im} \Psi^2(t) &= 0, \quad t \in \{x \in (-\infty,+\infty),\ y=0\}, \\ \operatorname{Im} \Psi^2(t) &= 0, \quad t \in \{x \in (-\infty,+\infty),\ y=h\}. \end{aligned} \tag{20}$$

Thus we can continue this function double-periodically in the whole z-plane. Taking into account the properties of double-periodic functions, formula (5) and conditions (18), (19) and (20), we get

$$\Psi^2(z) = C^* \, \frac{\sigma(z-a)\sigma(z+a)\sigma^2(z)}{\sigma^2(z+ih)\sigma(z-b-ih)\sigma(z+b-ih)}\,,$$

where C^* is an arbitrary real constant.

From the behavior of the double-periodic functions it follows that the function $\Psi^2(z)$ changes the sign only at the points $-a, a, -b+ih, b+ih$ and has a pole of second order at the point $z=-ih$ so that $C^*=0$. Thus we conclude that the homogeneous equation (17) has only a trivial solution. Thus the first part of theorem is proved.

The parts 2–6 of the theorem are proved in same way.

Let us prove the seventh part, the eighth part is proved similarly.

We consider the function $\Psi(z)$ given by (12) and the functions $\sqrt{X_0(z)}$, $\sqrt{X_4(z)}$ given by formula (11). In the strip $0 < y < h$ the functions $\sqrt{X_0(z)}$,

$\sqrt{X_4(z)}$ satisfy the following boundary conditions:

$$\begin{aligned}
&\operatorname{Re}\sqrt{X_0(t)}^{+} = 0, \quad t \in L,\\
&\operatorname{Im}\sqrt{X_0(t)}^{+} = 0, \quad t \in L^*,\\
&\operatorname{Im}\sqrt{X}_4(t)^{+} = 0, \quad t \in \{x \in (-\infty, +\infty),\ y = 0\},\\
&\operatorname{Im}\sqrt{X}_4(t)^{+} = 0, \quad t \in \{x \in (-\infty, +\infty),\ y = h\}.
\end{aligned}$$

The function $[\frac{\Psi(z)}{\sqrt{X_0(z)}} - C_*\sqrt{X_4(z)}]$, where the constant C_* is chosen appropriately, satisfies the following boundary conditions:

$$\begin{aligned}
&\operatorname{Re}\left[\frac{\Psi(t)}{\sqrt{X_0(t)}} - C_*\sqrt{X_4(t)}\right]^{+} = \frac{f(t)}{i\sqrt{X_0(t)}} - C_* \operatorname{Re}\sqrt{X_4(t)}, \quad t \in L_{00}^1,\\
&\operatorname{Re}\left[\frac{\Psi(t)}{\sqrt{X_0(t)}} - C_*\sqrt{X_4(t)}\right]^{+} = -\frac{f(t)}{i\sqrt{X_0(t)}} + C_* \operatorname{Re}\sqrt{X_4(t)}, \quad t \in L_{00}^2,\\
&\operatorname{Re}\left[\frac{\Psi(t)}{\sqrt{X_0(t)}} - C_*\sqrt{X_4(t)}\right]^{+} = C_*\sqrt{X_4(t)}, \quad t \in L^*.
\end{aligned}$$

And, similarly to part 1 we obtain

$$\begin{aligned}
\Psi(z) = {}& \frac{\sqrt{X_0(z)}}{\pi i} \int\limits_{L_{00}^1 \bigcup L_{00}^2} \left[\frac{f(t)}{i\sqrt{X_0(t)}} - C_*\sqrt{X_4(t)}\right] [\zeta(t-z) - \zeta(t)]dt\\
&+ C_* \frac{\sqrt{X_0(z)}}{\pi i} \int\limits_{L_{**}} \sqrt{X_4(t)}[\zeta(t-z) - \zeta(t)]dt\\
&\qquad + iK\sqrt{X_0(z)} + C_*\sqrt{X_0(z)}\sqrt{X_4(z)},\\
& z \in \{x \in (-\infty, +\infty),\ \ 0 < y < h\},
\end{aligned}$$

where K is an arbitrary real constant,

$$\begin{aligned}
L_{**} = \{[-\omega, \omega]\backslash L_{00}^1\} \cup \{[-\omega + ih, \omega + ih]\backslash L_{00}^2\},\\
f(ih)\sigma(b)|\sigma(a - ih)| - C_*\sigma^2(ih) = 0
\end{aligned}$$

with the necessary condition (9).

Using the Plemelj formula, we get (10).

The solution of the corresponding homogenous problem is $C^*\sqrt{X_0(z)}$, where C^* is an arbitrary constant. □

Remark 3. Taking into account condition (2), equation (3) can be rewritten as

$$\frac{1}{\pi} \int\limits_{L_{00}^1 \bigcup L_{00}^2} u(t)[\zeta(t-z) - \zeta(t) + 2\zeta(z/2)]dt = f(z), \quad z \in L_{00}^1 \bigcup L_{00}^2.$$

By formula (5) the kernel of the singular integral equation (1) could be introduced in the form

$$\zeta(t-z)-\zeta(t)+2\zeta(z/2)=\frac{\sigma^2(t-z/2)}{\sigma(t-z)\sigma(t)}\,.$$

Remark 4. If we use the Muskhelishvili theory, we will not find an effective solution, but we can make some conclusions about the index of the singular integral equation (1) [1].

We can represent equation (2) as

$$\frac{1}{\pi i}\int\limits_{L_{00}^1\bigcup L_{00}^2}\frac{u(t)}{t-z}dt+\frac{1}{\pi i}\int\limits_{L_{00}^1\bigcup L_{00}^2}u(t)k(t,z)dt=\frac{f(z)+C^0}{i},\quad z\in L_{00}^1\bigcup L_{00}^2,$$

$$k(t,z)=\sum_{\substack{m,n=-\infty\\|m|+|n|\neq 0}}^{+\infty}\left\{\frac{1}{t-z-2n\omega-2mih}-\frac{1}{t-2n\omega-2mih}\right\},$$

$$C^0=\frac{1}{\pi}\int\limits_{L_{00}^1\bigcup L_{00}^2}\frac{u(t)}{t}\,dt.$$

According to the Muskhelishvili theory this equation is equivalent to the Fredholm equation with additional conditions in the parts 2 and 3 of Theorem, which is not given here, and we arrive at the following conclusions:

- the index of the classes 1)–4) is zero.
- the index of the class 5) is -1.
- the index of the class 6) is 1.
- the index of the class 7) is 2.
- the index of the class 8) is -2.

Remark 5. Integral equation (2) is equivalent to the integral equation

$$\frac{1}{\pi}\int\limits_{L}\frac{u(t)}{t-z}\,dt=f(z),\quad z\in L, \tag{21}$$

with the conditions

$$\int\limits_{L_{00}^1\bigcup L_{00}^2}tu(t)dt=0,\qquad \int\limits_{L_{00}^1\bigcup L_{00}^2}u(t)dt=0, \tag{22}$$

Thus we conclude that solutions of different classes of equation (21) are the same as for equation (1), provided that condition (22) is also fulfilled.

Note that the condition $f(0)=0$ is not necessary in this case.

Remark 6. In the author's previous works equation (21) is considered in the case of $b=0$ and solutions for this case are obtained; in some special cases condition (6) is not necessary [7], [8]:

1) For $f(-x)=f(x)$, the integral equation (21) equivalently reduces to the equation

$$\frac{1}{\pi}\int_0^a u(t)[\zeta(t-x)+\zeta(t+x)-2\zeta(t)]dt=f(x), \quad x\in[0,a]. \tag{23}$$

A unique solution of equation (23) of the class $\mathcal{H}(-a,a)$ exists if and only if

$$\int_0^a \frac{f(t)}{\sqrt{X(t)}}\,dt=0,$$

$$\int_0^a \frac{f(t)}{\sqrt{X(t)}}\int_0^a \tau\sqrt{X(\tau)}[\zeta(t-\tau)-\zeta(t+\tau)]d\tau dt=0,$$

and is given by

$$u(x)=-\frac{\sqrt{X(x)}}{\pi}\int_0^a \frac{f(t)}{\sqrt{X(t)}}\,[\zeta(t-x)-\zeta(t+x)]dt, \quad x\in[0,a],$$

where

$$X(x)=\frac{\sigma(x-a)\sigma(x+a)}{\sigma^2(x)}\,.$$

The root $\sqrt{X(z)}$ always indicates the branch which is holomorphic in the z-plane cut along L.

2) For $f(x)=-f(-x)$, equation (21) reduces to the equation

$$\frac{1}{\pi}\int_0^a u(t)[\zeta(t-x)-\zeta(t+x)]dt=f(x), \quad x\in[0,a]. \tag{24}$$

A unique solution of equation (24) of the class $\mathcal{H}(-a,a)$ exists and is given by

$$u(x)=-\frac{\sqrt{X(x)}}{\pi}\int_0^a \frac{f(t)}{\sqrt{X(t)}}\,[\zeta(t-x)+\zeta(t+x)-2\zeta(t)]dt, \quad x\in[0,a].$$

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Author's address:

I. Vekua Institute of Applied Mathematics
Tbilisi State University
2, University St., Tbilisi 380043
Georgia
E-mail: khat@viam.hepi.edu.ge