

FRACTIONAL TYPE OPERATORS IN WEIGHTED GENERALIZED HÖLDER SPACES

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ABSTRACT. Weighted Zygmund type estimates are obtained for the continuity modulus of some convolution type integrals. In the case of fractional integrals this is strengthened to a result on isomorphism between certain weighted generalized Hölder type spaces.

1. INTRODUCTION

A great number of results is known concerning boundedness of convolution type operators in spaces of summable functions, including the weighted case. In the spaces of continuous functions such as $H_0^\omega(\rho)$ the convolution type operators are less investigated. The goal of this paper is to fill a gap to a certain extent in investigations of such a kind.

We consider here the Volterra convolution type operators

$$K\varphi = \int_a^x k(x-t)\varphi(t) dt, \quad a < x < b, \quad (1.1)$$

in the weighted generalized Hölder spaces $H_0^\omega(\rho)$ (see definitions in Sec.2), $-\infty < a < b < \infty$. The kernel $k(x)$ is assumed to be close in a sense to a power function.

The result of the type

$$K : H_0^\omega(\rho) \rightarrow H_0^{\omega_1}(\rho) \quad (1.2)$$

for certain characteristic functions $\omega(h)$ and $\omega_1(h)$ was earlier known in the case of the power kernel $k(x) = x^{\alpha-1}$, $0 < \alpha < 1$, and a power weight function $\rho(t)$. We deal here with arbitrary kernels and weights, i.e. not necessarily power ones.

We introduce a certain class V_λ of kernels and the class w_μ of weight functions for which we manage to give the weighted Zygmund type estimate, that is, to estimate the modulus of continuity $\omega(\rho K_\varphi, h)$ by the modulus of

continuity $\omega(\rho\varphi, h)$. This estimate provides the general result of the type (1.2).

In the case of purely power kernel, i.e. in the case of the fractional integration operator

$$I_{a+}^{\alpha}\varphi = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \varphi(t) dt, \quad 0 < \alpha < 1, \quad (1.3)$$

the result (1.2) is extended to isomorphism:

$$I_{a+}^{\alpha}[H_0^{\omega}(\rho)] = H_0^{\omega\alpha}(\rho) \quad (1.4)$$

with $\omega_{\alpha}(h) = h^{\alpha}\omega(h)$. This is achieved by the preliminary derivation of Zygmund type estimate for fractional differentiation. The latter is treated in a difference form due to A. Marchaud [9] and G.H. Hardy and J.E. Littlewood [2]: (See [17], Sec.13, in this connection.)

$$D_{a+}^{\alpha}f(x) = \frac{f(x)}{\Gamma(1-\alpha)(x-a)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_a^x \frac{f(x)-f(t)}{(x-t)^{1+\alpha}} dt, \quad (1.5)$$

$$0 < \alpha < 1.$$

The paper is organized as follows. In Sec.2 we give necessary preliminaries. Sec.3 contains Zygmund type estimates for the operator (1.2) in the case of kernels in V_{λ} in the non-weighted case first (Theorem 1) and afterwards in the weighted case (Theorem 2). In Theorem 3 we give conditions of Zygmund–Bari–Stechkin type on a characteristic function $\omega(h)$ guaranteeing the result (1.2) for $k(x) \in V_{\lambda}$ and weighted functions in w_{μ} . The characteristic function $\omega_1(h)$ in (1.2) proves to be equal to $hk(h)\omega(h)$. We note corollary of Theorem 3 for $k(x) = x^{\alpha-1}(\ln \frac{\gamma}{x})^{\beta}$, $\gamma > b-a$.

In Sec.4 we establish the weighted Zygmund type estimate for $D_{a+}^{\alpha}f$ with a weight function in w_{μ} (Theorem 4). We prove the assertion $D_{a+}^{\alpha} : H_0^{\omega}(\rho) \rightarrow H_0^{\omega-\alpha}(\rho)$ with $\omega_{-\alpha}(h) = h^{-\alpha}\omega(h)$ under appropriate assumptions on $\omega(h)$ and $\rho(x)$ (Theorem 5).

As a corollary of Theorems 3 and 5 we give conditions for validity of the second index law of E.R. Love within the framework of the spaces $H_0^{\lambda}(x^{\mu})$

Finally, in Sec.5 we prove the isomorphism (1.4) (Theorem 6).

Presented theorems generalize the results of the papers [10]–[12],[18], where the power case for both $k(x) = x^{\alpha-1}$ and $\rho(x) = (x-a)^{\alpha}$ was considered. The presentation of the results of [10] in the non-weighted case can be also found in [18], Sec.13. Note that in [12] the case $\rho(x) = (x-a)^{\mu}(b-x)^{\nu}$ was also considered, not contained in the results of the present paper. The origin of the statement (1.4) is the classical result by G.H. Hardy and J.E. Littlewood [2] for the fractional integration concerning the case

$\omega(h) = h^\lambda, \rho(x) \equiv 1, \alpha + \lambda < 1$. (As for the case $\omega(h) = \prod_{k=1}^n |x - x_k|^{\mu_k}$, see [13] and [17]. Sec.13.)

We also note the papers [4], [5] where Zygmund type estimates are given for the fractional integrodifferentiation in the case of L_p -moduli of continuity.

The question we finally note as open is whether $I_{a+}^\alpha [H_0^\omega(\rho)] = H_0^\omega(\rho)$ in the case of purely imaginary α , under the appropriate assumptions on $\omega(h)$ and $\rho(x)$. We refer to the paper [7] by E.R.Love concerning such fractional integrals (see also [17], Sec.2, n^04).

2. PRELIMINARIES

We follow the papers [14], [15] in the definitions below.

Definition 1. We say that $\psi(x) \in W_\mu = W_\mu([0, l])$ if $\psi(x) \in C([0, l])$, $\psi(0) = 0, \psi(x) > 0$ for $x > 0, \psi(x)$ is almost increasing, while $\psi(x)/x^\mu$ is almost decreasing and there exists a constant $c > 0$ such that

$$\left| \frac{\psi(x) - \psi(y)}{x - y} \right| \leq c \frac{\psi(x^*)}{x^*}, \quad x^* = \max(x, y). \tag{2.1}$$

We remind that a non-negative function $\psi(x), 0 \leq x \leq l, 0 < l \leq \infty$, is called almost increasing (decreasing) if $\psi(x) \leq c\psi(y)$ for all $x \leq y (x \geq y, \text{ resp.})$, this notion being due to S.Bernstein.

Definition 2. We say that $\psi(x) \in W_\mu^*$ if $\psi(x) \in W_\mu$ and $\psi(x)/x^{\mu-\varepsilon}$ is almost increasing for all $\varepsilon > 0$.

We shall also need the following modification of the Definition 2.

Definition 3. We say that a non-negative function $k(x)$ on $[0, l]$ belongs to the class $V_\lambda, \lambda > 0$ if

- i) $k(x) \not\equiv 0, x^\lambda k(x)$ is almost increasing and $x^\lambda k(x)|_{x=0} = 0$;
- ii) there exists $\varepsilon, 0 < \varepsilon < \lambda$, such that $x^{\lambda-\varepsilon} k(x)$ is almost decreasing;
- iii) there exists $c > 0$ such that

$$\left| \frac{k(x) - k(y)}{x - y} \right| \leq c \frac{k(x^*)}{x^*}, \quad x^* = \max(x, y). \tag{2.2}$$

Remark 1. $x^\lambda k(x) \in W_\lambda^* \Rightarrow k(x) \in V_\lambda$ and $k(x) \in V_\lambda \Rightarrow k(x) \in W_\lambda$.

Remark 2. If the almost monotonicity in Definitions 1 and 3 is replaced by the usual monotonicity, then conditions (2.1) and (2.2) are satisfied automatically.

Indeed, let us prove e.g. (2.1), following [14]. If $\varphi(x)/x^\mu$ is decreasing, so that $1 - \varphi(x)/\varphi(y) \leq 1 - x^\mu/y^\mu$ for $y \geq x$, then $\varphi(y) - \varphi(x) \leq \frac{y^\mu - x^\mu}{y^\mu} \varphi(y)$. Since $y^\mu - x^\mu \leq c(y - x)y^{\mu-1}$, we obtain (2.1).

Definition 4 ([1]). A non-negative function $\varphi(t)$ on $[0, l]$ belongs to Zygmund class $Z = Z([0, l])$ if $\int_0^h \frac{\varphi(t)}{t} dt \leq c\varphi(h), 0 \leq h \leq l$.

Definition 5 ([1]). A non-negative function $\varphi(t)$ belongs to Zygmund class $Z_1 = Z_1([0, l])$ if $\int_h^l \frac{\varphi(t)}{t^2} dt \leq c \frac{\varphi(h)}{h}$.

Definition 6. A function $\varphi(x)$ belongs to the generalized Hölder space $H^\omega = H^\omega([a, b])$ if

$$\omega(\varphi, h) \stackrel{\text{def}}{=} \sup_{0 < t < h} \sup_{x, x+t \in [a, b]} |\varphi(x+t) - \varphi(x)| \leq c\omega(h), \quad (2.3)$$

where $\omega(h)$ is a given positive function on $[0, l]$, $\omega(0) = 0$; we set $\|\varphi\|_{H^\omega} = \|\varphi\|_C + \sup_{h>0} [\omega(\varphi, h)/\omega(h)]$.

By H_0^ω we denote the subspace of functions in H^ω which vanish at $x = a$.

The function $\omega(h)$ is called a characteristic of the space H^ω .

Definition 7. By $H_0^\omega(\rho)$ we denote the space of functions $f(x)$ such that $\rho(x)f(x) \in H_0^\omega$, $\|f\|_{H_0^\omega(\rho)} = \|\rho f\|_{H_0^\omega}$, where $\rho(x)$ is a non-negative weight function.

In the sequel we shall use the following inequalities:

1) if $\omega(\varphi, h)$ is the continuity modulus (2.3), then

$$\frac{\omega(\varphi, x)}{x} \leq c \frac{\omega(\varphi, y)}{y}, \quad x \geq y; \quad (2.4)$$

2) if $0 < \alpha < 1$, then

$$\frac{\omega(\varphi, h)}{h^\alpha} \leq c \int_0^h \frac{\omega(\varphi, t)}{t^{1+\alpha}} dt; \quad (2.5)$$

3) if $\psi(x) \in W_\mu$, then

$$\psi(x) \leq c \left(\frac{x}{y}\right)^\mu \psi(y), \quad x \geq y; \quad (2.6)$$

4) if $\psi(x) \in W_\mu$ with $0 < \mu < 1$, then (2.1) holds with x^* replaced both by x and y :

$$\begin{aligned} \left| \frac{\psi(x) - \psi(y)}{x - y} \right| &\leq c \frac{\psi(x)}{x}, \\ \left| \frac{\psi(x) - \psi(y)}{x - y} \right| &\leq c \frac{\psi(y)}{y}; \end{aligned} \quad (2.7)$$

5) if $k(x) \in V_\lambda$, then

$$k(x) \leq c \left(\frac{y}{x}\right)^\lambda k(y), \quad x \leq y, \quad (2.8)$$

and there exists $\varepsilon > 0$ such that

$$k(x) \leq c \left(\frac{y}{x}\right)^{\lambda-\varepsilon} k(y), \quad x \geq y; \quad (2.9)$$

6) if $\lambda \leq 1$, then

$$|x^\lambda - y^\lambda| \leq c(x - y)y^{\lambda-1}, \quad x \geq y > 0, \tag{2.10}$$

and if $\lambda \geq 0$, then

$$|x^\lambda - y^\lambda| \leq c(x - y)x^{\lambda-1}, \quad x \geq y > 0. \tag{2.11}$$

Lemma 1. *Let $k(x) \in V_\lambda$, $\lambda > 0$ and let $\omega(x) \geq 0$ be an almost increasing function. Then $\omega(x)k(x) \leq c \int_x^l \frac{\omega(t)k(t)}{t} dt$ for $0 < x < l/2$.*

Proof. By (2.8) we have

$$\int_x^l \frac{\omega(t)k(t)}{t} dt \geq c\omega(x)x^\lambda k(x) \int_x^l \frac{dt}{t^{1+\lambda}} \geq c\omega(x)x^\lambda k(x) \int_x^{2x} \frac{dt}{t^{1+\lambda}} = c\omega(x)k(x). \quad \blacksquare$$

3. MAPPING PROPERTIES OF CONVOLUTION OPERATORS IN THE SPACE $H_0^\omega(\rho)$

The following theorem provides a Zygmund type estimate for the integral (1.1).

Theorem 1. *Let $k(x) \in V_\lambda$, $0 < \lambda < 1$ and $\varphi(x) \in C([a, b])$, $\varphi(a) = 0$. Then*

$$\omega(K\varphi, h) \leq ch k(h)\omega(\varphi, h) + ch \int_h^{b-a} \frac{k(t)\omega(\varphi, t)}{t} dt. \tag{3.1}$$

Proof. Let $a = 0$ for simplicity. We denote $g(x) = \varphi(x) - \varphi(0)$ and $f(x) = \int_0^x k(x-t)g(t)dt$. For all $x, x+h \in [0, b]$ we have

$$\begin{aligned} f(x+h) - f(x) &= \int_{-h}^x [g(x-t) - g(x)]k(t+h)dt - \\ &- \int_0^x [g(x-t) - g(x)]k(t)dt + g(x) \left[\int_{-h}^x k(t+h)dt - \int_0^x k(t)dt \right]. \end{aligned}$$

So,

$$\begin{aligned} |f(x+h) - f(x)| &\leq \left| \int_{-h}^0 [g(x-t) - g(x)]k(t+h)dt \right| + \\ &+ \left| \int_0^x [g(x-t) - g(x)][k(t) - k(t+h)]dt \right| + \left| g(x) \int_x^{x+h} k(t)dt \right| = A_1 + A_2 + A_3. \end{aligned}$$

Taking (2.8) and increasing of $\omega(\varphi, t)$ into account, we have for A_1 :

$$A_1 \leq \int_0^h \omega(\varphi, t)k(h-t)dt \leq c\omega(\varphi, h)k(h) \int_0^h \left(\frac{h}{h-t}\right)^\lambda dt \leq chk(h)\omega(h). \quad (3.2)$$

For A_2 , applying (2.2) and (2.9), we obtain in the case $h \geq x$:

$$\begin{aligned} A_2 &\leq ch \int_0^x \frac{\omega(\varphi, t)k(t+h)}{t+h} dt \leq chk(h)h^{\lambda-\varepsilon} \int_0^x \frac{\omega(\varphi, t)dt}{(t+h)^{1+\lambda-\varepsilon}} = \\ &= chk(h) \int_0^{x/h} \frac{\omega(\varphi, ht)dt}{(t+1)^{1+\lambda-\varepsilon}} \leq chk(h)\omega(\varphi, h) \int_0^1 \frac{dt}{(t+1)^{1+\lambda-\varepsilon}} \leq \\ &\leq chk(h)\omega(\varphi, h). \end{aligned} \quad (3.3)$$

In the case $h < x$ we write $A_2 \leq \int_0^h + \int_h^x = B_1 + B_2$. For B_1 the estimate (3.3) is valid, while for B_2 we have

$$B_2 \leq ch \int_h^x \frac{\omega(\varphi, t)k(t)}{t} dt. \quad (3.4)$$

As regards A_3 , we have in the case $h \geq x$:

$$\begin{aligned} A_3 &\leq c\omega(\varphi, h)k(x+h)(x+h)^\lambda \int_x^{x+h} \frac{dt}{t^\lambda} \leq \\ &\leq c\omega(\varphi, h)k(h)h^\lambda \int_0^{2h} \frac{dt}{t^\lambda} \leq c\omega(\varphi, h)hk(h). \end{aligned} \quad (3.5)$$

If $h < x$, we use Lemma 1 to obtain

$$A_3 \leq c\omega(\varphi, x)hk(x) \leq ch \int_x^b \frac{\omega(\varphi, t)k(t)dt}{t} \leq ch \int_h^b \frac{\omega(\varphi, t)k(t)}{t} dt. \quad (3.6)$$

Gathering all the estimates for A_i , $i = 1, 2, 3$, we arrive at (3.1). \square

Theorem 2. Let $k(x) \in V_\lambda$, $0 < \lambda < 1$, $\rho(x) = \psi(x-a)$, $\psi(x) \in W_\mu$, $0 < \mu < 2$. Assume that

- i) $\rho(x)\varphi(x) \in C([a, b])$ and $\rho(x)\varphi(x)|_{x=a} = 0$;

ii) $\int_0^{b-a} t^{-\gamma} \omega(\rho\varphi, t) dt < \infty$, $\gamma = \max(1, \mu)$. Then the following Zygmund type estimate holds:

$$\omega(\rho K\varphi, h) \leq ch^\gamma k(h) \int_0^h \frac{\omega(\rho\varphi, t)}{t^\gamma} dt + ch \int_h^{b-a} \frac{\omega(\rho\varphi, t)k(t)}{t} dt, \quad (3.7)$$

if $0 < \mu < 1 + \lambda$ and

$$\omega(\rho K\varphi, h) \leq ch \int_0^h \frac{\omega(\rho\varphi, t)}{t^\mu} dt + h \int_h^{b-a} \frac{\omega(\rho\varphi, t)}{t} dt, \quad (3.8)$$

if $1 + \lambda \leq \mu < 2$.

Proof. Let $\varphi_0(x) = \rho(x)\varphi(x)$ and $a = 0$ for simplicity. We have

$$\begin{aligned} \rho(x)(K\varphi)(x) &= \int_0^x k(x-t)\varphi_0(t)dt + \int_0^x \frac{\psi(x) - \psi(t)}{\psi(t)} k(x-t)\varphi_0(t)dt = \\ &= f_1(x) + f_2(x). \end{aligned}$$

Since $\varphi_0 \in C([0, b])$ and $\varphi_0(0) = 0$, the first term $f_1(t)$ is covered by Theorem 1. To estimate $\omega(f_2, h)$ we represent the difference $f_2(x+h) - f_2(x)$ as

$$\begin{aligned} f_2(x+h) - f_2(x) &= \int_x^{x+h} \frac{\psi(x+h) - \psi(t)}{\psi(t)} \varphi_0(t)k(x+h-t)dt + \\ &+ \int_0^x \frac{\psi(x+h) - \psi(x)}{\psi(t)} \varphi_0(t)k(x+h-t)dt + \\ &+ \int_0^x \frac{\psi(x) - \psi(t)}{\psi(t)} [k(x-t+h) - k(x-t)]\varphi_0(t)dt = I_1 + I_2 + I_3. \end{aligned}$$

Estimate for I_1 . A) Let $0 < \mu < 1$ at first. Taking (2.4), (2.7) and (2.8) into account, we have

$$\begin{aligned} |I_1| &\leq \int_x^{x+h} \frac{(x+h-t)k(x+h-t)\omega(\varphi_0, t)dt}{t} \leq \\ &\leq chk(h) \int_x^{x+h} \frac{\omega(\varphi_0, t-x)dt}{t-x} \leq chk(h) \int_0^h \frac{\omega(\varphi_0, t)dt}{t}. \end{aligned} \quad (3.9)$$

B) If $1 \leq \mu < 2$, then by (2.4),(2.6) and (2.8) we obtain

$$\begin{aligned} |I_1| &\leq c \int_0^h \frac{(h-t)k(h-t)(x+h)^{\mu-1}}{(x+t)^{\mu-1}} \frac{\omega(\varphi_0, t)}{x+t} dt \leq \\ &\leq chk(h) \int_0^h \frac{(x+h)^{\mu-1}}{t(x+h)^{\mu-1}} \omega(\varphi_0, t) dt. \end{aligned} \quad (3.10)$$

In the case $h < x$ we derive from (3.10)

$$|I_1| \leq chk(h) \int_0^h \frac{x^{\mu-1} \omega(\varphi_0, t) dt}{t(x+t)^{\mu-1}} \leq chk(h) \int_0^h \frac{\omega(\varphi_0, t) dt}{t}. \quad (3.11)$$

In the case $h > x$ the inequality (3.10) yields

$$|I_1| \leq chk(h)h^{\mu-1} \int_0^h \frac{\omega(\varphi_0, t) dt}{t(x+t)^{\mu-1}} \leq ch^\mu k(h) \int_0^h \frac{\omega(\varphi_0, t) dt}{t^\mu}. \quad (3.12)$$

So from (3.9), (3.11) and (3.12) there follows the estimate

$$|I_1| \leq ch^\gamma k(h) \int_0^h \frac{\omega(\varphi_0, t) dt}{t^\gamma}, \quad \gamma = \max(1, \mu). \quad (3.13)$$

Estimate for I_2 . A) Let $0 < \mu < 1$. By (2.6) and (2.7) we have

$$|I_2| \leq ch \int_0^x \frac{k(x+h-t)\omega(\varphi_0, t) dt}{t}. \quad (3.14)$$

In the case $h < x$ we represent (3.14) as $|I_2| \leq \int_0^h + \int_h^{(x+h)/2} + \int_{(x+h)/2}^x = I_2' + I_2'' + I_2'''$. It is clear that

$$I_2' \leq chk(h) \int_0^h \frac{\omega(\varphi_0, t) dt}{t}. \quad (3.15)$$

Since $x+h-t \geq t$ in I_2'' , we obtain

$$I_2'' \leq ch \int_h^b \frac{k(t)\omega(\varphi_0, t) dt}{t}. \quad (3.16)$$

Further, $x + h - t \leq t$ in I_2''' , so by (2.4)

$$\begin{aligned} I''' &\leq ch \int_{(x+h-t)/2}^x \frac{k(x+h-t)\omega(\varphi_0, x+h-t)dt}{x+h-t} \leq \\ &\leq ch \int_h^b \frac{\omega(\varphi_0, t)k(t)dt}{t}. \end{aligned} \tag{3.17}$$

If $h \geq x$, then (3.14) immediately yields

$$|I_2| \leq chk(h) \int_0^h \frac{\omega(\varphi_0, t)dt}{t}. \tag{3.18}$$

B) Let now $1 \leq \mu < 2$. Taking (2.1) into account, we have

$$|I_2| \leq ch \int_0^x \frac{(x+h)^{\mu-1}k(x+h-t)\omega(\varphi_0, t)dt}{t^\mu}. \tag{3.19}$$

Hence

$$|I_2| \leq ch^\mu k(h) \int_0^h \frac{\omega(\varphi_0, t)dt}{t^\mu}$$

in the case $h \geq x$. If $h < x$, we represent (3.19) as

$$|I_2| \leq \int_0^h + \int_h^{(x+h)/2} + \int_{(x+h)/2}^x = B_1 + B_2 + B_3.$$

Taking into account that $x+h \leq 2(x+h-t)$ in B_1 , in the case $0 < \mu-1 < \lambda$ we obtain

$$\begin{aligned} |B_1| &\leq ch \int_0^h \frac{(x+h-t)^{\mu-1}k(x+h-t)\omega(\varphi_0, t)dt}{t^\mu} \leq \\ &\leq ch^\mu k(h) \int_0^h \frac{\omega(\varphi_0, t)dt}{t^\mu} \end{aligned} \tag{3.20}$$

by (2.9). If $\mu - 1 > \lambda$, the function $t^{\mu-1}k(t)$ is bounded. So

$$B_1 \leq ch \int_0^h \frac{\omega(\varphi_0, t)dt}{t^\mu}. \tag{3.21}$$

Since $x + h \leq 2(x + h - t)$ again, we have

$$B_2 \leq ch \int_h^{(x+h)/2} \frac{(x+h-t)^{\mu-1} k(x+h-t)}{t^\mu} dt.$$

Here $x + h - t \geq t$, so that

$$B_2 \leq ch \int_h^b \frac{k(t)\omega(\varphi_0, t)dt}{t} \quad (3.22)$$

by (2.9), if $\mu - 1 < \lambda$. If $\mu - 1 \geq \lambda$, by boundedness of $t^{\mu-1}k(t)$ we have

$$B_2 \leq ch \int_h^b \frac{\omega(\varphi_0, t)dt}{t^\mu}. \quad (3.23)$$

To estimate B_3 we notice that $t \geq x + h - t$ in B_3 . So by (2.4) we have

$$B_3 \leq ch \int_{x+h-t/2}^x \frac{k(x+h-t)\omega(\varphi_0, x+h-t)dt}{x+h-t} \leq ch \int_h^b \frac{k(t)\omega(\varphi_0, t)dt}{t}. \quad (3.24)$$

Thus, I_2 admits the estimate

$$|I_2| \leq ch^\gamma k(h) = \int_0^h \frac{\omega(\varphi_0, t)dt}{t^\mu} + ch \int_h^b \frac{k(t)\omega(\varphi_0, t)dt}{t}, \quad \gamma = \max(1, \mu), \quad (3.25)$$

if $\mu < 1 + \lambda$ and

$$|I_2| \leq ch \int_0^h \frac{\omega(\varphi_0, t)dt}{t^\mu} + ch \int_h^b \frac{\omega(\varphi_0, t)dt}{t^\mu} \quad (3.26)$$

if $\mu \geq 1 + \lambda$.

Estimate for I_3 . Let $0 < \mu < 1$. By (2.2) and (2.4) we have

$$|I_3| \leq ch \int_0^x \frac{k(x+h-t)\omega(\varphi_0, t)dt}{t}$$

which coincides with the estimate in (3.14). If $1 \leq \mu < 2$, we derive from (2.1), (2.2) and (2.6):

$$|I_3| \leq ch \int_0^x \frac{x^{\mu-1} k(x+h-t)\omega(\varphi_0, t)dt}{t^\mu} \leq ch \int_0^x \frac{(x+h)^{\mu-1} k(x+h-t)\omega(\varphi_0, t)dt}{t^\mu}.$$

The latter coincides with (3.19). Gathering estimates for I_1, I_2 and I_3 , we obtain (3.7)–(3.8). \square

Theorem 3. *Let $\rho(x) = \psi(x - a), \psi(x) \in W_\mu, 0 < \mu < 2, k(t) \in V_\lambda, 0 < \lambda < 1$. Assume that*

- i) $\mu < \lambda + 1;$
- ii) $t^{-\max(0, \mu-1)}\omega(t) \in Z, tk(t)\omega(t) \in Z_1.$

Then the operator K is bounded from $H_0^\omega(\rho)$ into $H_0^{\omega_k}(\rho)$ with $\omega_k(h) = hk(h)\omega(h)$.

Proof. Let $f = K\varphi$ with $\varphi \in H_0^\omega(\rho)$ and let $a = 0$. To prove that $f \in H_0^{\omega_k}(\rho)$ we remark at first that

$$\int_0^b \frac{\omega(\rho\varphi, t)}{t^\gamma} dt < \infty, \quad \gamma = \max(1, \mu). \tag{3.27}$$

Therefore Zygmund type estimate (3.27) concerning the case $0 < \mu < 1 + \lambda$ holds which gives

$$\frac{\omega(\rho f, h)}{hk(h)\omega(h)} \leq \|\rho\varphi\|_{H_0^\omega} \left\{ \frac{h^{\gamma-1}}{\omega(h)} \int_0^h \frac{\omega(t)}{t^\gamma} dt + \frac{1}{k(h)\omega(h)} \int_h^b \frac{\omega(t)k(t)dt}{t} \right\}. \tag{3.28}$$

Hence by the condition ii) we have

$$\frac{\omega(\rho f, h)}{hk(h)\omega(h)} \leq c\|\varphi\|_{H_0^\omega(\rho)}. \tag{3.29}$$

It remains to prove that $\rho(x)f(x)|_{x=0} = 0$. After the change of variable $t = x - \xi x$ we have

$$|\rho(x)f(x)| \leq x\psi(x) \int_0^1 \frac{|\varphi_0(x - x\xi)|k(x\xi)d\xi}{\psi(x - x\xi)}.$$

Since $\varphi_0(0) = 0$, this yields

$$|\rho(x)f(x)| \leq cx\psi(x) \int_0^1 \frac{\omega(\varphi_0, 1 - \xi)d\xi}{\psi(x - x\xi)}. \tag{3.30}$$

According to (2.6) and (2.8) we see that

$$|\rho(x)f(x)| \leq cxk(x) \int_0^1 \frac{\omega(\varphi_0, 1 - \xi)d\xi}{\xi^\lambda(1 - \xi)^\gamma} = c_1xk(x) \rightarrow 0 \tag{3.31}$$

as $x \rightarrow 0$ in view of (3.27). So $\rho(x)f(x)|_{x=0} = 0$. \square

Corollary 1. *The operator (1.1) with the kernel $k(t) = t^{\alpha-1}(\ln \frac{\gamma}{t})^\beta$, $\gamma > b-a$, $0 < \alpha < 1$, $\beta \geq 0$ is bounded from $H_0^\omega(\rho)$ into $H_0^{\omega^{\alpha,\beta}}(\rho)$, where $\rho(x) = \psi(x-a)$, $\psi(x) \in W_\mu$ and $\omega\alpha, \beta(h) = \omega(h)h^\alpha(\ln \frac{\gamma}{h})^\beta$ under the assumption that $0 < \mu < 2 - \alpha$ and $h^{-\max(0, \mu-1)}\omega(h) \in Z$, $h^\alpha(\ln \frac{\gamma}{h})^\beta\omega(h) \in Z_1$.*

In the case $\psi(x) = x^\mu$ and $\omega(h) = h^\lambda$ the assertion of Corollary 1 was proved in [6] (see [17], Theorem 21.2).

Corollary 2. *The operator of the form $\int_x^b k(t-x)\varphi(t)dt$ is bounded from $H_0^\omega(\rho)$ into $H_0^{\omega^k}(\rho)$ under the assumptions of Theorem 3 if the requirement $\rho f|_{x=a} = 0$ in the definition of the space $H_0^\omega(\rho)$ is replaced by $\rho f|_{x=b} = 0$.*

4. MAPPING PROPERTIES OF FRACTIONAL DIFFERENTIATION IN THE SPACES $H_0^\omega(\rho)$

Now we give Zygmund type estimate for the fractional derivative (1.5).

Theorem 4. *Let $\rho(x) = \psi(x-a)$, $\psi(x) \in W_\mu$, $0 < \mu < 2$, and*

$$\int_0^{b-a} \frac{\omega(\rho f, t)}{t^{\alpha+\mu}} dt < \infty, \quad \gamma = \max(1, \mu).$$

Then

$$\omega(\rho D_{a+}^\alpha f, h) \leq ch^{\gamma-1} \int_0^h \frac{\omega(\rho f, t)}{t^{\alpha+\gamma}} dt. \tag{4.1}$$

Proof. According to (1.5) we have

$$\begin{aligned} \rho(x)(D_{a+}^\alpha f)(x) &= \frac{\rho(x)f(x)}{\Gamma(1-\alpha)(x-a)^\alpha} + \\ &+ \frac{\alpha}{\Gamma(1-\alpha)} \rho(x) \int_0^{x-a} \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt. \end{aligned} \tag{4.2}$$

We set $a = 0$ and denote $\theta(x) = \psi(x) \int_0^x \frac{f(x)-f(x-t)}{t^{1+\alpha}} dt$. To estimate the difference $\theta(x+h) - \theta(x)$ we represent it in the form $\theta(x+h) - \theta(x) = \sum_{k=1}^8 A_k(x)$ (as in [13] in the purely power case), where

$$\begin{aligned} A_1(x) &= \left[1 - \frac{\psi(x)}{\psi(x+h)}\right] \int_0^x \frac{g(x+h) - g(y)}{(x+h-y)^{1+\alpha}} dy, \\ A_2(x) &= [\psi(x+h) - \psi(x)] \int_0^{x+h} \frac{g(y)}{(x+h-y)^{1+\alpha}} \left[\frac{1}{\psi(x+h)} - \frac{1}{\psi(y)}\right] dy, \end{aligned}$$

$$\begin{aligned}
 A_3(x) &= \frac{\psi(x)}{\psi(x+h)} \int_x^{x+h} \frac{g(x+h) - g(y)}{(x+h-y)^{1+\alpha}} dy, \\
 A_4(x) &= \psi(x) \int_x^{x+h} \frac{g(y)}{(x+h-y)^{1+\alpha}} \left[\frac{1}{\psi(x+h)} - \frac{1}{\psi(y)} \right] dy, \\
 A_5(x) &= \int_0^x [g(x) - g(y)] [(x+h-y)^{-1-\alpha} - (x-y)^{-1-\alpha}] dy, \\
 A_6(x) &= \psi(x) \int_0^x g(y) \left[\frac{1}{\psi(x)} - \frac{1}{\psi(y)} \right] [(x+h-y)^{-1-\alpha} - (x-y)^{-1-\alpha}] dy, \\
 A_7(x) &= \frac{1}{\alpha} \frac{\psi(x)}{\psi(x+h)} [g(x+h) - g(x)] [h^{-\alpha} - (x+h)^{-\alpha}], \\
 A_8(x) &= \frac{1}{\alpha} \psi(x) g(x) \left[\frac{1}{\psi(x+h)} - \frac{1}{\psi(x)} \right] [h^{-\alpha} - (x+h)^{-\alpha}].
 \end{aligned}$$

Estimate for A_1 . By (2.1) we have

$$|A_1| \leq c \frac{h}{x+h} \int_0^{x+h} \frac{\omega(g, x+h-y)}{(x+h-y)^{1+\alpha}} dy = c \frac{h}{x+h} \int_0^{x+h} \frac{\omega(g, t)}{t^{1+\alpha}} dt. \tag{4.3}$$

If $h \geq x$, it is obvious that

$$|A_1| \leq c \int_0^h \frac{\omega(g, t)}{t^{1+\alpha}} dt. \tag{4.4}$$

If $h < x$ represent (4.3) as $|A_1| \leq \int_0^h + \int_h^{x+h} = A'_1 + A''_1$. For A'_1 the estimate (4.4) holds. As regards A''_1 , applying (2.4) and (2.5), we have

$$\begin{aligned}
 A''_1 &\leq ch \int_x^{x+h} \frac{\omega(g, t)}{t} \frac{dt}{t^{1+\alpha}} \leq c\omega(g, h) \int_h^\infty \frac{dt}{t^{1+\alpha}} \leq \\
 &\leq c \frac{\omega(g, h)}{h^\alpha} \leq c \int_0^h \frac{\omega(g, t)}{t^{1+\alpha}} dt.
 \end{aligned} \tag{4.5}$$

Estimate for A_2 . A) In the case $0 < \mu < 1$, using (2.7), we obtain

$$|A_2| \leq c \frac{h}{x+h} \int_0^{x+h} \frac{\omega(g, y)}{y(x+h-y)^\alpha} dy = \int_0^{h/2} + \int_{h/2}^{x+h} = A'_2 + A''_2.$$

Obviously, $A'_2 \leq c \int_0^h \frac{\omega(g, t)}{t^{1+\alpha}} dt$. Using (2.4) and (2.5), we derive the following estimate for A''_2 :

$$A''_2 \leq c \frac{\omega(g, h)}{x+h} \int_0^{x+h} \frac{dy}{(x+h-y)^\alpha} = c_1 \frac{\omega(g, h)}{(x+h)^\alpha} \leq c_2 \int_0^h \frac{\omega(g, t)}{t^{1+\alpha}} dt.$$

B) If $1 \leq \mu < 2 - \alpha$, taking (2.1) and (2.6) into account, we obtain for A_2 :

$$|A_2| \leq c \frac{h}{(x+h)^{2-\mu}} \int_0^h \frac{\omega(g, y) dy}{y^\mu (x+h-y)^\alpha} = \int_0^{h/2} + \int_{h/2}^{x+h} = B_1 + B_2.$$

It is obvious that

$$B_1 \leq ch^{\mu-1} \int_0^h \frac{\omega(g, t)}{t^{\mu+\alpha}} dt. \quad (4.6)$$

As regards B_2 , we apply (2.4) and obtain

$$\begin{aligned} B_2 &\leq c \frac{\omega(g, h)}{(x+h)^{2-\mu}} \int_{h/2}^{x+h} \frac{dy}{y^{\mu-1} (x+h-y)^\alpha} \leq \\ &\leq c \frac{\omega(g, h)}{(x+h)^\alpha} \int_0^1 \frac{dt}{t^{\mu-1} (1-t)^\alpha} = c_1 \frac{\omega(g, h)}{(x+h)^\alpha}. \end{aligned}$$

Using then (2.5), we notice that the estimate for B_2 is the same as in (4.6).

Estimate for A_3 . Since $\psi(x)$ is almost increasing, we have

$$|A_3| \leq c \int_x^{x+h} \frac{\omega(g, x+h-y)}{(x+h-y)^{1+\alpha}} dy = c \int_0^h \frac{\omega(g, t)}{t^{1+\alpha}} dt.$$

Estimate for A_4 . Let $0 < \mu < 1$ at first. In view of (2.7) we have

$$|A_4| \leq c \int_0^h \frac{\omega(g, x+h-t)}{t^\alpha (x+h-t)} dt. \quad (4.7)$$

In the case $h < x$ we have $t \leq x+h-t$ in (4.7). So by (2.4) we obtain $|A_4| \leq c \int_0^h \frac{\omega(g,t)}{t^{1+\alpha}} dt$. If $h \geq x$, we represent (4.7) as $|A_4| \leq \int_0^{(x+h)/2} + \int_{(x+h)/2}^h$. Since $t \leq x+h-t$ and $t \geq x+h-t$ in the first and second terms, respectively, by (2.4) we derive that

$$|A_4| \leq c \int_0^{(x+h)/2} \frac{\omega(g,t)}{t^{1+\alpha}} dt + c \int_{(x+h)/2}^h \frac{\omega(g, x+h-t)}{(x+h-t)^{1+\alpha}} dt \leq c \int_0^h \frac{\omega(g,t)}{t^{1+\alpha}} dt.$$

Let now $1 \leq \mu < 2 - \alpha$. Using (2.1) and (2.6), we get

$$|A_4| \leq \frac{c}{(x+h)^{1-\mu}} \int_0^h \frac{\omega(g, x+h-t)}{(x+h-t)^\mu t^\alpha} dt.$$

If $h < x$, by (2.4) we have

$$|A_4| \leq c(x+h)^{\mu-1} \int_0^h \frac{\omega(g,t) dt}{(x+h-t)^{\mu-1} t^{1+\alpha}} \leq c \int_0^h \frac{\omega(g,t)}{t^{1+\alpha}} dt.$$

If $h \geq x$, then $|A_4| \leq \int_0^{(x+h)/2} + \int_{(x+h)/2}^h$. We use (2.4) in the first term and the inequality $t^\alpha \geq (x+h-t)^\alpha$ in the second. This yields

$$\begin{aligned} |A_4| &\leq \int_0^{(x+h)/2} \frac{\omega(g,t)}{t^{1+\alpha}} dt c(x+h)^{\mu-1} \int_{(x+h)/2}^h \frac{\omega(g, x+h-t)}{(x+h-t)^{\alpha+\mu}} dt \leq \\ &\leq c \int_0^h \frac{\omega(g,t)}{t^{1+\alpha}} dt + ch^{\mu-1} \int_x^{(x+h)/2} \frac{\omega(g,t)}{t^{\alpha+\mu}} dt \leq ch^{\mu-1} \int_0^h \frac{\omega(g,t)}{t^{\alpha+\mu}} dt. \end{aligned}$$

Estimate for A_5 . Applying (2.11), we have

$$|A_5| \leq ch \int_0^h \omega(g, x-y) \frac{dy}{(x+h-y)(x-y)^{1+\alpha}} = ch \int_0^x \frac{\omega(g,t) dt}{t^{1+\alpha}(t+h)}.$$

In the case $h \geq x$ it is clear that

$$|A_5| \leq c \int_0^h \frac{\omega(g,t)}{t^{1+\alpha}} dt. \tag{4.8}$$

If $h < x$, then $|A_5| \leq \int_0^h + \int_h^x = A'_5 + A''_5$ with the same estimate as in (4.8) for A'_5 . As regards A''_5 , we have

$$A''_5 \leq c\omega(g, h) \int_h^\infty \frac{dt}{t^{1+\alpha}} \leq c \frac{\omega(g, h)}{h^\alpha} \leq c \int_0^h \frac{\omega(g, t)}{t^{1+\alpha}} dt \quad (4.9)$$

by (2.4) and (2.5).

Estimate for A_6 . A) Let $0 < \mu < 1$ at first. Applying (2.7) and (2.11) we arrive at

$$|A_6| \leq ch \int_0^x \frac{\omega(g, y) dy}{y(x+h-y)(x-y)^\alpha} = \int_0^{x/2} + \int_{x/2}^x = A'_6 + A''_6.$$

For A'_6 we have

$$A'_6 \leq ch \int_0^{x/2} \frac{\omega(g, y) dy}{(y+h)y^{1+\alpha}}. \quad (4.10)$$

If $h < x$, $A'_6 \leq \int_0^{h/2} + \int_{h/2}^{x/2} = K_1 + K_2$. It is evident that K_1 admits the same estimate as in (4.9). For K_2 the application of (2.5) provides the same result: $K_2 \leq c\omega(g, h) \int_{h/2}^\infty \frac{dt}{t^{1+\alpha}} \leq c \int_0^h \frac{\omega(g, t)}{t^{1+\alpha}} dt$. If $h \geq x$, then immediately

$$A'_6 \leq c \int_0^h \frac{\omega(g, t)}{t^{1+\alpha}} dt. \quad (4.11)$$

To estimate A''_6 we remark that $y \geq x - y$ so that

$$A''_6 \leq ch \int_{x/2}^x \frac{\omega(g, x-y) dy}{(x-y)^{1+\alpha}(x+h-y)} = ch \int_0^{x/2} \frac{\omega(g, t) dt}{t^{1+\alpha}(t+h)},$$

which is the same as in (4.10) and so A''_6 admits the same estimates as in (4.5).

B) Let $1 \leq \mu < 2 - \alpha$. Using (2.1), (2.6) and (2.11), we have

$$A_6 \leq chx^{\mu-1} \int_0^x \frac{\omega(g, y) dy}{y^\mu(x+h-y)(x-y)^\alpha} = \int_0^{x/2} + \int_{x/2}^x = U_1 + U_2.$$

If $h < x$, we set $U_1 = \int_0^{h/2} + \int_{h/2}^{x/2} = U'_1 + U''_1$, whence easy calculations yield the inequality

$$U_1 \leq ch \int_0^h \frac{\omega(g, t) dt}{t^{1+\alpha}} + ch^{\mu-1} \int_0^h \frac{\omega(g, t) dt}{t^{\mu+\alpha}} \leq ch^{\mu-1} \int_0^h \frac{\omega(g, t) dt}{t^{\mu+\alpha}}. \tag{4.12}$$

For U_2 by (2.5) we have

$$\begin{aligned} U_2 &\leq ch \int_{x/2}^x \frac{\omega(g, y) dy}{y(x+h-y)(x-y)^\alpha} \leq c\omega(g, h) \int_0^{x/2} \frac{dt}{t^\alpha(t+h)} \leq \\ &\leq c \frac{\omega(g, h)}{h^\alpha} \int_0^\infty \frac{d\xi}{\xi^\alpha(1+\xi)} \leq c \int_0^h \frac{\omega(g, t) dt}{t^{1+\alpha}}. \end{aligned}$$

If $h \geq x$, the estimation of U_1 and U_2 is easy and provides the same as in (4.2). Gathering all the estimates, we obtain $|A_6| \leq ch^{\gamma-1} \int_0^h \frac{\omega(g, t)}{t^{\alpha+\gamma}} dt$, $\gamma = \max(1, \mu)$.

Estimate for A_7 . Applying (2.5) and (2.10) and almost increasing of $\psi(x)$, we easily obtain

$$|A_7| \leq c \int_0^h \frac{\omega(g, t)}{t^{1+\alpha}} dt. \tag{4.13}$$

Estimate for A_8 . Using the inequalities (2.1) and (2.10) for $0 < \mu < 2-\alpha$, we make sure of validity of the estimate (4.13) for A_8 as well.

It remains to consider the first term

$$r(x) = \frac{\psi(x-a)f(x)}{\Gamma(1-\alpha)(x-a)^\alpha} = \frac{g(x)}{\Gamma(1-\alpha)(x-a)^\alpha}$$

in (4.2). Since $g(x) \in H_0^\omega$, we have the estimate

$$|r(x+h) - r(x)| \leq c \int_0^h \frac{\omega(g, t) dt}{t^{1+\alpha}}, \tag{4.14}$$

which is derived by direct calculations under the assumptions of the theorem.

Collecting all the estimates for A_i , $i = 1, \dots, 8$, and (4.14), we obtain the required inequality (4.1). \square

Theorem 5. Let $\rho(x) = \psi(x - a)$, $\psi(x) \in W_\mu$, $0 < \mu < 2 - \alpha$ and let

- 1) $\omega(t) \neq 0$, $t > 0$,
 - 2) $\omega(t)t^{-\alpha}|_{t=0} = 0$,
 - 3) $\omega(t)t^{1-\alpha-\gamma} \in Z$, $\gamma = \max(1, \mu)$.
- (4.15)

Then the operator D_{a+}^α continuously maps $H_0^\omega(\rho)$ into $H_0^{\omega-\alpha}(\rho)$ with $\omega_{-\alpha}(h) = h^{-\alpha}\omega(h)$.

Proof. Let $f(x) \in H_0^\omega(\rho)$ and $\varphi(x) = D_{a+}^\alpha f(x)$. To show that

$$\sup_{0 < h \leq b-a} \frac{h^\alpha \omega(\rho\varphi, h)}{\omega(h)} = c < \infty$$

we observe that the inclusion $\omega(t)t^{1-\alpha-\gamma} \in Z$ implies convergence of the integral $\int_0^{b-a} \omega(t)t^{-\alpha-\gamma} dt$, so Theorem 4 is applicable. Using the estimate (4.1) of Theorem 4, we obtain

$$\frac{h^\alpha \omega(\rho\varphi, h)}{\omega(h)} \leq \frac{h^{\alpha+\gamma-1} \int_0^1 \omega(\rho f, t)t^{-\alpha-\gamma} dt}{\omega(h)} \leq c \|f\|_{H_0^\omega(\rho)}. \quad (4.16)$$

It remains to show that $\rho(x)\varphi(x)|_{x=a} = 0$. By (4.2) we have

$$\begin{aligned} |\rho(x)\varphi(x)| &\leq \frac{\omega(\rho f, x-a)}{\Gamma(1-\alpha)(x-a)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{x-a} \frac{\omega(\rho f, t) dt}{t^{1+\alpha}} + \\ &+ \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{x-a} \frac{|\psi(x-a) - \psi(x-a-t)| \omega(\rho f, x-a-t) dt}{t^{1+\alpha} \psi(x-a-t)} = \\ &= D_1 + D_2 + D_3. \end{aligned} \quad (4.17)$$

Here $D_1 \leq c \|f\|_{H_0^\omega(\rho)} \frac{\omega(x-a)}{(x-a)^\alpha}$, the condition (4.16) implies $\omega(x)x^{-\alpha}|_{x=0} = 0$. So $\lim_{x \rightarrow a} D_1 = 0$. The equality $\lim_{x \rightarrow a} D_2 = 0$ is obvious by the existence of the integral in D_2 .

For the term D_3 in the case $0 < \mu < 1$ we have by (2.7)

$$D_3 \leq c \int_0^{x-a} \frac{\omega(\rho f, x-a-t) dt}{t^\alpha (x-a-t)}.$$

We evaluate this separately for $x-a-t \geq t$ and $x-a-t \leq t$ by means of (2.4):

$$D_3 \leq c \int_0^{(x-a)/2} \frac{\omega(\rho f, t) dt}{t^{1+\alpha}} + c \int_{(x-a)/2}^{x-a} \frac{\omega(\rho f, x-a-t)}{(x-a-t)^{1+\alpha}} dt,$$

whence $\lim_{x \rightarrow a} D_3 = 0$.

If $1 \leq \mu \leq 2 - \alpha$, we use (2.1) and (2.5) and obtain

$$D_3 \leq c \int_0^{x-a} \frac{\omega(\rho f, x-a-t)}{(x-a)^{1-\mu}(x-a-t)^\mu t^\alpha} = \int_0^{(x-a)/2} + \int_{(x-a)/2}^{x-a}$$

and similar to what we did above we have

$$\begin{aligned} D_3 &\leq c \int_0^{(x-a)/2} \frac{(x-a)^{\mu-1} \omega(\rho f, t)}{t^{\alpha+\mu}} dt + c \int_{(x-a)/2}^{x-a} \frac{(x-a)^{\mu-1} \omega(\rho f, x-a-t)}{(x-a-t)^{\alpha+\mu}} dt \leq \\ &\leq (x-a)^{\mu-1} \int_0^{(x-a)/2} \frac{\omega(\rho f, t)}{t^{\alpha+\mu}} dt, \end{aligned}$$

so that $\lim_{x \rightarrow a} D_3 = 0$. Therefore, $\lim_{x \rightarrow a} \rho(x)\varphi(x) = 0$. \square

Corollary 1. *Let $\varphi(x) \in W_\mu$ $0 < \mu < 2 - \alpha$, and let $\omega(t)$ be an almost increasing function on $[0, b - a]$ such that*

- 1) $\omega(0) = 0, \omega(t) \neq 0$ for $t \in (0, b - a]$;
- 2) $\omega(t)t^{-1-\gamma} \in Z, \gamma = \max(1, \mu)$.

Then the operator D_{a+}^α of fractional differentiation continuously maps $H_0^\omega(\rho)$ into H_0^ω with $\rho(x) = \varphi(x - a), \omega_\alpha(h) = h^\alpha \omega(h)$.

Another corollary (of Theorems 5 and 3) will be related to the following Love's index law [8]:

$$I_{0+}^\gamma x^\alpha I_{0+}^\beta x^\gamma I_{0+}^\alpha x^\beta f(x) = f(x), \quad \alpha + \beta + \gamma = 0 \tag{4.18}$$

well known in fractional calculus. This corollary will provide conditions guaranteeing validity of (4.18) for functions $f \in H_0^\omega(\rho)$. For simplicity we restrict ourselves with the cases $\omega(x) = x^\lambda$ and $\rho(x) = x^\mu$. The notation

$$\alpha^* = \begin{cases} \alpha, & \alpha \leq 1 \\ 1, & \alpha \geq 1 \end{cases}$$

is used below.

Corollary 2. *Relation (4.18) is valid for all functions $f(x) \in H_0^\lambda(x^\mu)$ and all $\alpha, \beta, \gamma \in \mathbb{R}^1$ such that $\alpha + \beta + \gamma = 0$, if the number $\lambda \in (0, 1]$ satisfies the conditions $\lambda > -\alpha, (\lambda + \alpha)^* + \beta > 0 [(\lambda + \alpha)^* + \beta]^* + \gamma > 0$ while the weight exponent μ satisfies the conditions*

- 1) $\mu < (\lambda + \alpha)^* + 1$
- 2) $\mu < [(\lambda + \alpha)^* + \beta]^* + 1$
- 3) $\mu < \{[(\lambda + \alpha)^* + \gamma]^* + 1\}$.

5. A THEOREM ON ISOMORPHISM

In Theorem 3 and Corollary 1 of Theorem 5 it was proved that

$$I_{a+}^{\alpha} : H_0^{\omega}(\rho) \longrightarrow H_0^{\omega\alpha}(\rho), \quad (5.1)$$

$$D_{a+}^{\alpha} : H_0^{\omega\alpha}(\rho) \longrightarrow H_0^{\omega}(\rho) \quad (5.2)$$

under the appropriate assumptions on $\omega(h)$ and $\rho(x)$. To derive the assertion $I_{a+}^{\alpha}[H_0^{\omega}(\rho)] = H_0^{\omega\alpha}(\rho)$ it remains to show that any function in $H_0^{\omega\alpha}(\rho)$ is representable by the fractional integral of a function in $H_0^{\omega}(\rho)$. This will be the goal of Theorem 6 below. Preliminarily we state two auxiliary assertions we need.

Lemma 2 ([17], p.185; p.231 in English ed.). *In order a function $f(x)$ to be representable as $f = I_{a+}^{\alpha}\varphi$, $\varphi \in L_p(a, b)$, $-\infty < a < b < \infty$, it is necessary and sufficient that*

- i) $f(x)(x-a)^{-\alpha} \in L_p(a, b)$;
- ii) $\|\psi_{\varepsilon}\|_{L_p} \leq c < \infty$ with c not depending on ε , where

$$\psi_{\varepsilon}(x) = \int_a^{x-\varepsilon} \frac{f(x) - f(t)}{(x-t)^{1+\alpha}} dt,$$

for $a + \varepsilon < x < b$ and $\psi_{\varepsilon}(x) = 0$ for $a < x < a + \varepsilon$.

A close version of Lemma 2 can be found in [16]. (See also [3] for another version under additional assumptions that $f \in L_p$ and $1 < p < 1/2\alpha$).

Lemma 3. *Let $\omega(t)t^{-\delta} \in Z$. There exists $p > 1$ such that $\omega(t)t^{-1-\delta} \in L_p(0, 1)$.*

Proof. It is known that the inclusion $\varphi(t) \in Z$ implies existence of $\varepsilon \in (0, 1)$ such that $t^{-\varepsilon}\varphi(t)$ is almost increasing (see, e.g. [1]). Therefore, there exists $\varepsilon \in (0, 1)$ such that $t^{-\varepsilon-\delta}\omega(t)$ is bounded. So $\omega(t) \leq ct^{\delta+\varepsilon}$ and to have a finite L_p -norm for $\omega(t)/t^{1+\delta}$ we must choose $p < \frac{1}{1-\varepsilon}$. \square

Theorem 6 (On isomorphism). *Let $\psi(x) \in W_{\mu}$, $0 < \mu < 2 - \alpha$ and let $\omega(t)$ be a continuous function such that $\omega(t)t^{1-\gamma} \in Z$, $\omega(t)t^{\alpha} \in Z_1$, $\gamma = \max(1, \mu)$, $0 < \alpha < 1$. Then the fractional integration operator I_{a+}^{α} isomorphically maps the weighted space $H_0^{\omega}(\rho)$ with $\rho(x) = \psi(x-a)$ onto the space $H_0^{\omega\alpha}(\rho)$ with the same weight and the characteristic $\omega_{\alpha}(h) = h^{\alpha}\omega(h)$.*

Proof. In view of (5.1)–(5.2) it is sufficient to prove the representability of a function $f \in H_0^{\omega\alpha}(\rho)$ by a fractional integral. Aiming to apply Lemma 2 we shall prove that there exists $p > 1$ such that conditions i)–ii) of Lemma 2 are satisfied.

The estimate

$$\frac{|f(x)|}{(x-a)^\alpha} \leq c \|f\|_{H_0^{\omega\alpha}} \frac{\omega(x-a)}{(x-a)^\mu} \tag{5.3}$$

is valid for any $f(x) \in H_0^{\omega\alpha}(\rho)$. Really, by (2.6) we have

$$\frac{|f(x)|}{(x-a)^\alpha} \leq c \frac{|\rho(x)f(x)|}{(x-a)^{\mu+\alpha}}, \tag{5.4}$$

which immediately provides (5.3).

Since $\omega(x)/x^{\delta-1} \in Z$, from (5.3) and Lemma 3 we conclude that there exists $p > 1$ such that the condition i) of Lemma 2 is satisfied.

For this p we shall show that a constant $c > 0$ exists such that

$$\|\psi_\varepsilon\|_{L_p} \leq c \leq \infty. \tag{5.5}$$

We set $g(x) = f(x)\psi(x-a)$ and have

$$|\psi_\varepsilon(x)| \leq \frac{1}{\psi(x-a)} \int_a^{x-\varepsilon} \frac{|g(x) - g(t)|}{(x-t)^{1+\alpha}} dt + \int_a^{x-\varepsilon} \frac{|g(t)| \left[\frac{1}{\psi(x-a)} - \frac{1}{\psi(t-a)} \right]}{(x-t)^{1+\alpha}} dt.$$

To estimate F_1 we use (2.6) and obtain

$$F_1 \leq \frac{c}{(x-a)^\mu} \int_a^x \frac{\omega(g, x-t)}{(x-t)^{1+\alpha}} dt \leq \frac{c}{(x-a)^\gamma} \int_a^x \frac{\omega(g, x-t)}{(x-t)^{1+\alpha}} dt. \tag{5.6}$$

Since $g(x) \in H_0^{\omega\alpha}(\rho)$, it is easily proved that (5.6) yields

$$F_1 \leq \frac{c}{(x-a)^\gamma}. \tag{5.7}$$

For F_2 in the case $0 < \mu < 1$ we use (2.6) and (2.7) and obtain

$$F_2 \leq \frac{c}{(x-a)} \int_0^{x-a} \frac{\omega(g, x-t-a)}{t^\alpha(x-t-a)} dt = \int_0^{(x-a)/2} + \int_{(x-a)/2}^{x-a} .$$

Hence after simple calculations

$$\begin{aligned} F_2 &\leq \frac{c}{x-a} \int_0^{(x-a)/2} \frac{\omega(t)dt}{t} + \frac{c}{x-a} \int_{(x-a)/2}^{x-a} \frac{\omega(x-t-a)dt}{x-t-a} \leq \\ &\leq c \frac{\omega(x-a)}{x-a} + \frac{c}{x-a} \int_0^{x-a} \frac{\omega(t)}{t} dt. \end{aligned} \tag{5.8}$$

Since the condition $\omega(t)t^{1-\mu} \in Z$ with $0 < \mu < 1$ implies $\omega(t) \in Z$, we derive from (5.8) the estimate $F_2 \leq c\omega(x-a)/(x-a)$.

Let now $1 \leq \mu < 2 - \alpha$. We use inequalities (2.1) and (2.6) to obtain

$$\begin{aligned} F_3 &\leq \frac{c}{x-a} \int_0^{x-a} \frac{\omega(g, x-t-a)}{t^\alpha(x-t-a)^\mu} dt \leq \frac{c}{x-a} \int_0^{x-a} \frac{\omega(x-t-a)dt}{t^\alpha(x-t-a)^{\mu-\alpha}} = \\ &= \int_0^{(x-a)/2} + \int_{(x-a)/2}^{x-a}. \end{aligned}$$

Calculations and arguments similar to those in the case $0 < \mu < 1$ give the estimate $F_2 \leq c\frac{\omega(x-a)}{(x-a)^\mu}$. Therefore,

$$F_2 \leq c\frac{\omega(x-a)}{(x-a)^\gamma}, \quad \gamma = \max(1, \mu). \quad (5.9)$$

So, from (5.8), (5.9) we obtain $|\psi_\varepsilon(x)| \leq c\frac{\omega(x-a)}{(x-a)^\gamma} \in L_p$. Hence $\|\psi\|_{L_p} \leq c$. \square

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