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ON THE DURRMEYER-TYPE MODIFICATION OF SOME DISCRETE APPROXIMATION OPERATORS

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Abstract. In [10], for continuous functions f from the domain of certain discrete operators L_n the inequalities are proved concerning the modulus of continuity of $L_n f$. Here we present analogues of the results obtained for the Durrmeyer-type modification $\widetilde{L}_n$ of L_n. Moreover, we give the estimates of the rate of convergence of $\widetilde{L}_n f$ in Hölder-type norms

1. Introduction and Notation

Let I be a finite or infinite interval. Consider a sequence $(J_k)_1^\infty$ of some index sets contained in $Z := \{0, \pm1, \pm2, \dots\}$, choose real numbers $\xi_{j,k} \in I$ and fix non-negative functions $p_{j,k}$ continuous on I. Write, formally,

$$L_k f(x) := \sum_{j=J_k} f(\xi_{j,k}) p_{j,k}(x) \quad (x \in I, k \in N := \{1, 2, \dots\}) \tag{1}$$

for univariate (complex-valued) functions f defined on I. If for $f_0(x) \equiv 1$ on I the values $L_k f_0(x)$ ($x \in I$, $k \in N$) are finite, then $L_k f$ are well-defined for every function f bounded on I. Under appropriate additional assumptions, operators (1) are meaningful also for some locally bounded functions f on infinite intervals I. The fundamental approximation properties of operators (1) in the space $C(I)$ of all continuous functions on I can be deduced, for example, via the general Bohman–Korovkin theorems ([5], Sect. 2.2).

Recently, several authors have investigated relations between the smoothness properties of the functions f and $L_k f$ ([1], [10], [15]). For example, taking an arbitrary function $f \in C(I) \cap \mathrm{Dom}(L_n)$, $n \in N$, Kratz and Stadtmüller [10] obtained the following result. Let

$$\sum_{j \in J_k} p_{j,k}(x) \le c_1 \quad \text{for all} \quad x \in I, \quad k \in N, \tag{2}$$

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and let the sum of the above series be independent of x; if, moreover,

$$p'_{j,k} \in C(\mathring{I}), \quad \sum_{j \in J_k} |(\xi_{j.k} - x)p'_{j,k}(x)| \leq c'_1 \quad \text{for all} \quad x \in \mathring{I}, \quad k \in N,$$

where c_1, c'_1 are positive constants and $\mathring{I}$ denotes the interior of I, then the ordinary moduli of continuity of f and $L_n f$ satisfy the inequality

$$\omega(L_n f; \delta) \leq 2(c_1 + c'_1)\omega(f; \delta) \quad (\delta \geq 0).$$

They proved an analogous inequality for the suitable weighted moduli of continuity of f and $L_n f$ when I is an infinite interval and f has the modulus $|f|$ of polynomial growth at infinity. In [12] their result is extended to functions f having $|f|$ of a stronger growth than the polynomial one. [12] also presents some applications of the above-mentioned inequalities in problems of approximation of continuous functions f by $L_n f$ in some Hölder-type norms.

Suppose that for every $j \in J_k$ and every $k \in N$ the integral $\int_I p_{j,k}(t)dt$ coincides with a positive number, say, $1/q_{j,k}$. Denote by $\widetilde{L}_k$ the operators given by

$$\widetilde{L}_k f(x) \equiv \widetilde{L}_k(f)(x) := \sum_{j \in J_k} q_{j,k} p_{j,k}(X) \int_I f(t) p_{j,k}(t) dt \quad (x \in I, \ k \in N) \tag{3}$$

for these measurable (complex-valued) functions f for which the right-hand side of (3) is meaningful. This modification of the classical Bernstein polynomials was first introduced by J.I. Durrmeyer (see [4]). The approximation properties of these polynomials were investigated, for example, in [4], [7], [2]. Some results on the approximation of functions by the Durrmeyer-type modification of the Szász–Mirakyan operators, Baskakov operators or Meyer–König and Zeller operators can be found, for example, in [8], [9], [13], [14], [16].

In this paper we derive Kratz and Stadtmüller type inequalities involving ordinary or weighted moduli of continuity of the functions f and $\widetilde{L}_n f$ on I. Using these inequalities, we obtain estimates of the degree of approximation of f by $\widetilde{L}_n f$ in some Hölder-type norms. Theorems 1–3 show that the smoothness properties of $\widetilde{L}_n f$ are slightly different from those of $L_n f$.

We adopt the following notation. Given any non-negative function w defined on I and any x, $y \in I$, we write $\breve{w}(x, y) := \min\{w(x), w(y)\}$.

For an arbitrary function f defined on I we introduce the quantities

$$\|f\|_w := \sup\{|f(x)|w(x) : \ x \in I\},$$
$$\Omega_w(f; \delta) := \sup\{|f(x) - f(y)||\breve{w}(x, y) : \ x, y \in I, |x - y| \leq \delta\} \ \ (\delta \geq 0).$$

If f is continuous on I and $\|f\|_w < \infty$, we say that $f \in C_w(I)$. The quantity $\Omega_w(f;\delta)$ is called the weighted modulus of continuity of f on I. In case $w(x) = 1$ for all $x \in I$, $\Omega_w(f;\delta)$ becomes $\omega(f;\delta)$ and the symbol $\|f\|$ is used instead of $\|f\|_w$. If the weight w is nondecreasing [nonincreasing] on I, then

$$\Omega_w(f;\delta) := \sup\{|f(x) - f(y)|w(x)\} \quad \big[\Omega_w(f;\delta) := \sup\{|f(x) - f(y)|w(y)\}\big],$$

where the supremum is taken over all $x, y \in I$ such that $0 < y - x \le \delta$.

We denote by W the set of all continuous functions w on I with values not greater than 1, which are positive in the interior of I and satisfy the inequality $\breve{w}(x,y) \le w(t)$ for any three points $x, t, y \in I$ such that $x \le t \le y$ (obviously, this inequality holds if, for example, w is nondecreasing, nonincreasing or concave on I). When I is an infinite interval, we indtroduce, in addition, the set Λ of all positive functions η belonging to W such that $\eta(x) \to 0$ as $|x| \to 0$.

Given two weights $w, \eta \in W$, we define a more general modulus of continuity of f on I by

$$\Omega_{w,\eta}(f;\delta) := \sup\{|f(x) - f(y)|\breve{w}(x,y)\breve{\eta}(x,y) \ : \ x, y \in I, \ |x - y| \le \delta\}.$$

It reduces to $\Omega_w(f;\delta)$ if $\eta \equiv 1$ on I, and to $\Omega_\eta(f;\delta)$ if $w \equiv 1$ on I. Taking into account that the positive function φ is nondecreasing on the interval $(0,1]$ and has values not greater that 1, we put

$$\|f\|_{w,\eta}^{(\varphi)} := \|f\|_{w\eta} + \\ + \sup\Big\{\frac{|f(x) - f(y)|\breve{w}(x,y)\breve{\eta}(x,y)}{\varphi(|x-y|)} \ : \ x, \ y \in I, \ \ < |x - y| \le 1\Big\}.$$

If this quantity is finite, we call it the Hölder-type norm of f on I. Under the assumption $f \in C_\eta(I)$, $\|f\|_{w,\eta}^{(\varphi)} < \infty$ if and only if there exists a positive constant K such that $\Omega_{w,\eta}(f;\delta) \le K\varphi(\delta)$ for every $\delta \in (0,1]$. We write $\|f\|_w^{(\varphi)}$ for $\|f\|_{w,\eta}^{(\varphi)}$ if $\eta \equiv 1$ on I, and $\|f\|_\eta^{(\varphi)}$ if $w \equiv 1$ on I.

Throughout this paper the symbols c_ν ($\nu = 1, 2, \dots$) will mean some positive constants depending only on a given sequence $(L_k)_1^\infty$ and eventually on the considered weights w, η, ρ. The integer part of the real number will be denoted by $[a]$.

2. Smoothness Properties

Let $\widetilde{L}_k$, $k \in N$, be the operators defined by (3) such that $\widetilde{L}_k f_0(x)$ are finite at every $x \in I$. Put

$$r_k(x) := \sum_{j \in J_k} p_{j,k}(x) - 1 \quad (x \in I, \quad k \in N)$$

and make the standing assumption that all functions $p_{j,k}$ ($j \in J_k$, $k \in N$) are absolutely continuous on every compact interval contained in I. Consider measurable functions f locally bounded on I and belonging to $\mathrm{Dom}(\widetilde{L}_n)$ for some $n \in N$. Write, as in Section 1, $\overset{\circ}{I} = \mathrm{Int}\, I$.

Theorem 1. *Suppose that condition* (2) *is satisfied and*

$$\sum_{j\in J_k} q_{j,k}|p'_{j,k}(x)| \int_I |t-x|p_{j,k}(t)dt \le \frac{c_2}{w(x)} \tag{4}$$

for $x \in \overset{\circ}{I}$ *and all* $k \in N$, w *being a function of the class* W. *Then*

$$\Omega_w(\widetilde{L}_n f;\delta) \le c_3\omega(f;\delta) + \|f\|_w \omega(r_n;\delta) \quad (\delta \ge 0), \tag{5}$$

where $c_3 = 2(c_1\|w\| + c_2)$.

Proof. Let x, $y \in I$ $0 < y - x \le \delta$ and let $x_0 := (x+y)/2$. Clearly,

$$\widetilde{L}_n f(x) - \widetilde{L}_n f(y) = \sum_{j\in J_n} q_{j,n}(p_{j,n}(x) - p_{j,n}(y)) \int_I (f(t) - f(x_0))p_{j,n}(t)dt + \\ + f(x_0)(r_n(x) - r_n(y)). \tag{6}$$

Taking into account (2) and the well-known inequality $|f(t) - f(x_0)| \le (1 + [|t-x_0|\delta^{-1}])\omega(f;\delta)$, we obtain $|\widetilde{L}_n f(x) - \widetilde{L}_n f(y)| \le (2c_1 + A_n(x,y)) \times \omega(f;\delta) + |f(x_0)|\omega(r_n;\delta)$, where

$$A_n(x,y) := \sum_{j\in J_n} q_{j,n}|p_{j,n}(x) - p_{j,n}(y)|\delta^{-1} \int_{I\setminus I_\delta} |t-x_0|p_{j,n}(t)dt \le \\ \le \delta^{-1} \int_x^y \Big(\sum_{j\in J_n} q_{j,n}|p'_{j,n}(s)| \int_{I\setminus I_\delta} |t-x_0|p_{j,n}(t)dt \Big) ds$$

and $I_\delta := I \cap (x_0 - \delta, x_0 + \delta)$. If $x < s < y$ and $|t - x_0| \ge y - x$, then $|t-x_0| \le 2|t-s|$. Hence, applying (4), we get

$$A_n(x,y) := 2\delta^{-1} \int_x^y \Big(\sum_{j\in J_n} q_{j,n}|p'_{j,n}(s)| \int_I |t-s|p_{j,n}(t)dt \Big) ds \le 2c_2\delta^{-1} \int_x^y \frac{1}{w(s)} ds,$$

and inequality (5) follows.

The result of Theorem 1 is interesting if $\omega(f;\delta) < \infty$. This holds, for example, for functions $f \in C(I)$ on the compact interval I. If I is an infinite interval, the assumption $\omega(f;\delta) < \infty$ implies the restriction $f(x) = O(|x|)$ as $|x| \to \infty$. So, in this case, it is convenient to use the weighted modulus of continuity $\Omega_\eta(f;\delta)$ with some $\eta \in \Lambda$. If $f \in C_\eta(I)$, then this modulus is a nondecreasing function of δ on the interval $[0,\infty)$. It is easy to verify that, for every $\delta>0$ and for all $x,y\in I$ there holds the inequality

$$|f(x)-f(y)|\check{\eta}(X,y)\le(1+[\delta^{-1}|x-y|])\Omega_\eta(f;\delta). \tag{7}$$

Moreover, in case $\rho\in\Lambda$ and $\rho(x)/\eta(x)\to 0$ as $|x|\to\infty$ we have $\Omega_\rho(f;\delta)\to 0$ as $\delta\to 0+$, whenever $f\in C_\eta(I)$ is uniformly continuous on each finite interval contained in I.

Note that under the assumptions $\eta\in\Lambda$, $f\in C_\eta(I)$ and $\widetilde{L}_k(1/\eta)(x)<\infty$ we have $|L_kf(x)|<\infty$. If, moreover, $\rho\in\Lambda$ and

$$\widetilde{L}_k\Big(\frac{1}{\eta}\Big)(x)\le\frac{c_4}{\rho(x)}\quad\text{for all}\quad x\in I\ \text{ and }\ k\in N \tag{8}$$

then $\|\widetilde{L}_kf\|_\rho<\infty$.

In the next two theorems it is assumed that I is an infinite interval. ■

Theorem 2. *Let condition* (2) *be satisfied. Suppose, moreover, that there exist functions* $w\in W$, ρ, $\eta\in\Lambda$, $\rho\le\eta$ *such that* (4), (8) *and*

$$\sum_{j\in J_k}q_{j,k}|p'_{j,k}(x)|\int_I\frac{|t-x|}{\eta(t)}p_{j,k}(t)dt\le$$
$$\le\frac{c_5}{w(x)\rho(x)}\quad\textit{for a.e.}\ \ x\in\overset{\circ}{I}\quad\textit{and}\ \ k\in N \tag{9}$$

hold. Then

$$\Omega_{w,\rho}(\widetilde{L}_nf;\delta)\le c_6\Omega_\eta(f;\delta)+\|f\|_{w\rho}\omega(r_n;\delta)\quad(\delta\ge 0), \tag{10}$$

where $c_6=2((c_1+c_4)\|w\|+c_2+c_5)$.

Proof. Consider $x,y\in I$ such that $0<y-x\le\delta$. Retain the symbol x_0 used in the proof of Theorem 1 and start with identity (6). In view of (7), $|\widetilde{L}_nf(x)-\widetilde{L}_nf(y)|\le B_n(x,y)\Omega_\eta(f;\delta)+|f(x_0)||r_n(x)-r_n(y)|$, where

$$B_n(x,y):=\sum_{j\in J_n}q_{j,n}|p_{j,n}(x)-p_{j,n}(y)|\int_I(1+[\delta^{-1}|t-x_0|])\frac{1}{\check{\eta}(t,x_0)}p_{j,n}(t)dt.$$

Observing that for every $t\in I$

$$\frac{\check{\rho}(x,y)}{\check{\eta}(t,x_0)}\le 1+\frac{\check{\rho}(x,y)}{\eta(t)} \tag{11}$$

and applying (2), we obtain

$$B_n(x,y)\check{\rho}(x,y) \le 2c_1 + \sum_{j\in J_n} q_{j,n}|p_{j,n}(x)-p_{j,n}(y)| \int\limits_I \frac{\check{\rho}(x,y)}{\eta(t)} p_{j,n}(t)dt +$$

$$+\delta^{-1}\sum_{j\in J_n} q_{j,n}\int\limits_x^y |p'_{j,n}(s)|ds \int\limits_{I-I_\delta}\left(1+\frac{\check{\rho}(x,y)}{\eta(t)}\right)|t-x_0|p_{j,n}(t)dt.$$

Further, the inequality $|t-x_0| \le 2|t-s|$ $(t \in I\backslash I_\delta,\ x<s<y)$ and assumptions (4), (8), (9) lead to

$$B_n(x,y)\check{\rho}(x,y) \le 2(c_1+c_4) + 2\delta^{-1}\int\limits_x^y \frac{c_2+c_5}{w(s)}ds.$$

The desired estimate is now evident.

For functions f for which $|f|$ is of the polynomial growth at infinity our result can be stated as follows. ■

Theorem 3. *Let conditions* (2), (4) *be satisfied and let* $\eta(x) = (1+|x|)^{-\sigma}$ $x \in I$ $\sigma > 0$. *Suppose that inequality* (9) *in which* $\rho = \eta$ *holds. Then*

$$\Omega_{w,\eta}(\widetilde{L}_n f;\delta) \le c_7\Omega_\eta(f;\delta) + \|f\|_{w\eta}\omega(r_n;\delta) \quad (\delta \ge 0),$$

where $c_7 = 2(c_1 + 2\cdot 3^\sigma c_1 + c_2 + 2c_5)$.

Proof. To see this it is enough to make a slight modification in the evaluation of the term $B_n(x,y)$ occurring in the proof of Theorem 2. Namely, let us divide the interval I into two sets I_n and $I\backslash I_h$, where $I_h := I\cap(x_0-h, x_0+h)$, $h = y-x$. If $t \in I_h$, then $[\delta^{-1}|t-x_0|] = 0$ and

$$\frac{\check{\eta}(x,y)}{\eta(t)} \le 3^\sigma\check{\eta}(x,y)\left(\frac{1}{\eta(x)}+\frac{1}{\eta(y)}\right) \le 2\cdot 3^\sigma.$$

This inequality, (11) and (2) imply

$$B_n(x,y)\check{\eta}(x,y) \le 2(1+2\cdot 3^\sigma)c_1 +$$

$$+\sum_{j\in J_n} q_{j,n}|p'_{j,n}(s)ds| \int\limits_{I\backslash I_h}\left(\frac{|t-x_0|}{\delta}+\frac{\check{\eta}(x,y)}{\eta(t)}\Big(1+\frac{|t-x_0|}{y-x}\Big)\right)p_{j,n}(t)dt.$$

Observing that $|t-x_0| \le 2|t-s|$, $|t-x_0| \le y-x$ whenever $t \in I\backslash I_h$, $x<s<y$, we obtain, on account of (4) and (9) (with $\rho=\eta$),

$$B_n(x,y)\check{\eta}(x,y) \le 2(1+2\cdot 3^\sigma)c_1 +$$
$$+\frac{2}{\delta}\int_x^y \frac{c_2}{w(s)}ds + 4\frac{\check{\eta}(x,y)}{y-x}\int_x^y \Big(\sum_{j\in J_n} q_{j,n}|p'_{j,n}(s)| \int_I \frac{|t-s|}{\eta(t)}p_{j,n}(t)dt\Big)ds \le$$
$$\le 2(1+2\cdot 3^\sigma)c_1 + \frac{2}{y-x}\int_x^y \frac{c_2+c_5}{w(s)}ds\,.$$

Thus

$$B_n(x,y)\check{w}(x,y)\check{\eta}(x,y) \le 2(1+2\cdot 3^\sigma)c_1\|w\| + 2c_2 + 4c_5. \quad \blacksquare$$

Remark 1. For many known operators the functions $r_k(x)\equiv 0$ on I, the quantities $\mu_{2,k}(x) := \sum_{j\in J_k}(\xi_{j,k}-x)^2 p_{j,k}(x)$ are finite at every $x\in I$ and positive in $\overset{\circ}{I}$; moreover,

$$p'_{j,k}(x)\mu_{2,k}(x) = p_{j,k}(x)(\xi_{j,k}-x) \tag{12}$$

for every $x \in\overset{\circ}{I}$ and every $k\in N$. In view of identity (12) and the Cauchy–Schwartz inequality the left-hand side of (4) can be estimated from above by $(\widetilde{\mu}_{2,k}(x)/\mu_{2,k}(x))^{1/2}$, where $\widetilde{\mu}_{2,k}(x) := \sum_{j\in J_k} q_{j,k}|p_{j,k}(x)|\int_I (t-x)^2 p_{j,k}(t)dt$. Therefore, in this case, assumption (4) can be replaced by

$$\frac{\widetilde{\mu}_{2,k}(x)}{\mu_{2,k}(x)} \le \frac{c_2^2}{w^2(x)} \quad \text{for all} \quad x\in\overset{\circ}{I}, \quad k\in N. \tag{13}$$

Analogously, the left-hand side of (9) can be estimated by

$$\frac{1}{\mu_{2,k}(x)}\Big(\widetilde{\mu}_{2,k}(x)\sum_{j\in J_k} q_{j,k}(\xi_{j,k}-x)^2 p_{j,k}(x)\int_I \frac{p_{j,k}(t)}{\eta^2(t)}dt\Big)^{1/2}.$$

Hence, if

$$\frac{1}{\mu_{2,k}(x)}\sum_{j\in J_k} q_{j,k}p_{j,k}(x)(\xi_{j,k}-x)^2 \int_I \frac{p_{j,k}(t)}{\eta^2(t)}dt \le \frac{c_8^2}{\rho^2(x)} \tag{14}$$

for all $x\in\overset{\circ}{I}$, $k\in N$, then (9) holds with $c_5 = c_2\cdot c_8$.

Remark 2. Let $w \in W$, $\eta \in \Lambda$. Define the weighted modulus $\Phi_w(f;\delta)$ and $\Phi_{w,\eta}(f;\delta)$ as in Section 1, replacing $\check{w}(x,y)$ by

$$\overline{w}(x,y) := \begin{cases} 0 & \text{if} \quad w(x)=0 \quad \text{or} \quad w(y)=0, \\ \Big(\dfrac{1}{w(x)}+\dfrac{1}{w(y)}\Big)^{-1} & \text{otherwise,} \end{cases}$$

and $\check{\eta}(x,y)$ by $\overline{\eta}(x,y)$, respectively. Since $\overline{w}(x,y) \le \check{w}(x,y)$ for every pair of points $x,y \in I$, Theorem 1 remains valid for $\Phi_w(\widetilde{L}_n f;\delta)$. Further, in this case, inequality (7) becomes $|f(x)-f(y)|\overline{\eta}(x,y) \le 2(1+[\delta^{-1}|x-y|])\Phi_\eta(f;\delta)$. Consequently, under the assumptions of Theorem 2, the modulus $\Phi_{w,\rho}(\widetilde{L}_n f;\delta)$ and $\Phi_\eta(f;\delta)$ satisfy inequality (10) with the constant $2c_6$ instead of c_6.

Note that, for the weight $\eta(x)=(1+|x|)^{-\sigma}$ with the parameter $\sigma>0$, the modulus $\Phi_\eta(f;\delta)$ is equivalent to the one introduced in [10], p. 331 (see also [12]).

3. Approximation Properties

Considering still the functions f as in Section 2 we first estimate the ordinary weighted norm of the difference $\widetilde{L}_n f - f$.

Theorem 4. *Let condition* (2) *be satisfied and let*

$$\rho(x)\widetilde{L}_k\Big(\frac{1}{\eta^2}\Big)(x) \le \frac{c_9}{\eta(x)} \quad \textit{for all} \quad x \in I, \;\; k \in N, \tag{15}$$

$$\rho(x)\widetilde{\mu}_{2,k}(x) \le c_{10}\eta(x)\delta_k^2 \quad \textit{for all} \quad x \in I, \;\; k \in N, \tag{16}$$

where $(\delta_k)_1^\infty$ *is a sequence of positive numbers,* η *is a positive function on* I *and* ρ *is a non-negative one such that* $\rho \le \eta$. *Then*

$$\|\widetilde{L}_n f - f\|_\rho \le c_{11}\Omega_\eta(f;\delta_n) + \|f\|_\rho \|r_n\|, \tag{17}$$

where $c_{11} = c_1 + (c_1c_9)^{1/2} + (c_9c_{10})^{1/2} + c_{10}$.

Proof. Start with the obvious identity

$$\widetilde{L}_n f(x) - f(x) = \sum_{j\in J_n} q_{j,n}p_{j,n}(x)\int_I (f(t)-f(x))p_{j,n}(t)dt + f(x)r_n(x)$$

and take a positive number δ. In view of (7) and the inequality $(\check{\eta}(x,t))^{-1} \le (\eta(x))^{-1}+(\eta(t))^{-1}$ we have $|\widetilde{L}_n f(x)-f(x)| \le \gamma_n(x)\Omega_\eta(f;\delta)+|f(x)|\cdot\|r_n\|$, where

$$\gamma_n(x) := \sum_{j\in J_n} q_{j,n}p_{j,n}(x)\int_I (1+[\delta^{-1}|t-x|])\Big(\frac{1}{\eta(x)}+\frac{1}{\eta(t)}\Big)p_{j,n}(t)dt.$$

Further, by (2), (15) and (16) and the Cauchy–Schwartz inequality we obtain

$$\gamma_n(x)\rho(x) \le c_1 + \widetilde{L}_n\Big(\frac{1}{\eta}\Big)(x)\rho(x) + \delta^{-2}\frac{\rho(x)}{\eta(x)}\widetilde{\mu}_{2,n}(x) +$$
$$+\rho(x)\delta^{-1}\sum_{j\in J_n} q_{j,n}p_{j,n}(x)\int\limits_I \frac{|t-x|}{\eta(t)}p_{j,n}(t)dt \le$$
$$\le c_1 + \Big(c_1\widetilde{L}_n\Big(\frac{1}{\eta^2}\Big)(x)\Big)^{1/2}\rho(x) + c_{10}\delta^{-2}\delta_n^2 +$$
$$+\rho(x)\delta^{-1}(\widetilde{\mu}_{2,n}(x))^{1/2}\Big(\widetilde{L}_n\Big(\frac{1}{\eta^2}\Big)(x)\Big)^{1/2} \le$$
$$\le c_1 + (c_1c_9)^{1/2} + c_{10}\delta^{-2}\delta_n^2 + (c_9c_{10})^{1/2}\delta^{-1}\delta_n.$$

Choosing $\delta = \delta_n$, we get (17) at once. ∎

Remark 3. In the case when $\eta(x) = 1$ for all $x \in I$, the constant c_{11} in (17) is equal to $c_1 + c_{10}$. If we use the modulus $\Phi_\eta(f;\delta)$ (defined in Remark 2) instead of $\Omega_\eta(f;\delta)$, the constant c_{11} should be multiplied by 2.

Passing to approximation in the Hölder-type norm we note that, for an arbitrary $\nu_n \in (0,1]$,

$$\|\widetilde{L}_n f - f\|_{w,\eta}^{(\varphi)} \le \Big(1 + \frac{2}{\varphi(\nu_n)}\Big)\|\widetilde{L}_n f - f\|_{w\eta} +$$
$$+\sup\Big\{\frac{1}{\varphi(\delta)}\big(\Omega_{w,\eta}(\widetilde{L}_n f;\delta) + \Omega_{w,\eta}(f;\delta)\big) \;:\; 0<\delta\le\nu_n\Big\} \tag{18}$$

(see, for example, [11], [12]). This inequality, Theorem 4 and the estimates obtained in Section 2 allow us to state a few standard results. We will formulate only one of them. Namely, combining inequality (18) with Theorems 1 and 2 gives

Theorem 5. *Let conditions* (2), (4) *be satisfied and let* $(\delta_k)_1^\infty$ *be a sequence of numbers from* $(0,1]$ *for which* (16) *holds with* $\rho = w$ *and* $\eta \equiv 1$ *on* I. *Then*

$$\|\widetilde{L}_n f - f\|_w^{(\varphi)} \le c_{12}\sup\Big\{\frac{\omega(f;\delta)}{\varphi(\delta)} \;:\; 0<\delta\le\delta_n\Big\} + \|f\|_w\Delta_n^{(\varphi)}\ ,$$

where $c_{12} = 3c_1 + 2c_2 + 3c_{10} + (1+2c_1)\|w\|$ *and*

$$\Delta_n^{(\varphi)} = 3\|r_n\|/\varphi(\delta_n) + \sup\{\omega(r_n;\delta)/\varphi(\delta) \;:\; 0<\delta\le\delta_n\}.$$

Remark 4. Clearly, if the assumptions of Theorems $1--5$ hold for positive integers k belonging to a certain subset N_1 of N, then the corresponding assertions remain valid only if $n \in N_1$.

4. Examples

1) The Bernstein polynomials $B_k f \equiv L_k f$ are defined by (1) with $\xi_{j,k} = j/k$, $p_{j,k} = \binom{k}{j} x^j (1-x)^{k-j}$, $I = [0,1]$, $J_k = \{0,1,2,\dots,k\}$. The corresponding Bernstein–Durrmeyer polynomials $\widetilde{L}_k f \equiv \widetilde{L}_k f$ are of the form (3) in which $q_{j,k} = k+1$ for all $j \in J_k$, $k \in N$. In this case $r_k(x) = 0$ for all $x \in I$, the constant c_1 in (2) equals 1, $\mu_{2,k}(x) = x(1-x)/k$ and equality (12) is true. Since $\widetilde{\mu}_{2,k}(x) = \frac{2x(1-x)(k-3)+2}{(k+2)(k+3)}$ ($x \in I$, $k \in N$) (see [4]), we easily state that condition (13) is satisfied with $c_2 = 1$, $w(x) = (x(1-x))^{1/2}$. Hence, in view of Theorem 1 (and Remark 1), for every $f \in C(I)$ and every $n \in N$, $\Omega_w(\widetilde{B}_n f; \delta) \le 3\omega(f;\delta)$ $(\delta \ge 0)$ Further, $\widetilde{\mu}_{2,k}(x) \le \frac{1}{2k}$ for all $x \in I$, $k \in N$ (see [4], p. 327). Therefore (16) holds with $\rho(x) = \eta(x) = 1$ for all $x \in I$, $\delta_k = k^{-1/2}$ and $c_{10} = 1/2$. Thus Theorem 4 gives $\|\widetilde{B}_n f - f\| \le \frac{3}{2}\omega(f; n^{-1/2})$ for all $n \in N$ (cf. [4], Theorem II.2). Also, Theorem 5 applies with $w(x) = (x(1-x))^{1/2}$, $\delta_n = n^{-1/2}$, $c_{12} = 8$ and $\Delta_n^{(\varphi)} = 0$.

2) The Meier–König and Zeller operators $M_k \equiv L_k$ are defined by $\xi_{j,k} = j/(j+k)$, $p_{j,k}(x) = \binom{k+j-1}{j} x^j (1-x)^k$, $x \in I = [0,1)$, $j \in J_n = N_0$, $N_0 := \{0,1,\dots\}$. Their Durrmeyer modification $\widetilde{M} \equiv \widetilde{L}_k$ are of the form (3) in which $q_{j,k} = (k+j)(k+j+1)/k$. Condition (2) holds with $c_1 = 1$. Since

$$p'_{j,k}(x)\frac{x(1-x)^2}{k} = p_{j,k+1}(x)\Big(\frac{j}{k+j} - x\Big)^2 \quad (0 < x < 1),$$

the left-hand side of (4) can be estimated from above by

$$\frac{k}{x(1-x)^2}\bigg(\Big\{\sum_{j=0}^{\infty}\Big(\frac{j}{k+j} - x\Big)^2 p_{j,k+1}(x)\Big\} \times$$
$$\times \Big\{\sum_{j=0}^{\infty} q_{j,k} p_{j,k+1}(x)\int_0^1 (t-x)^2 p_{j,k}(t)\,dt\Big\}\bigg)^{1/2}$$

for all $x \in (0,1)$, $k \in N$. If $k \ge 3$, the expression in the first curly brackets is not greater than $2x(1-x)^2/k$ (see [3]); straightforward calculation shows that the expression in the second ones does not exceed $7(1-x)^2/k$. Thus, for the functions $f \in C(I) \cap \mathrm{Dom}(\widetilde{M}_n)$ and $\widetilde{M}_n f$ $(n \ge 3)$, inequality (5) applies with $c_3 = 10$, $w(x) = x^{1/2}$ and $r_n(x) = 0$ for all $x \in I$.

3) The Baskakov–Durrmeyer operators $\widetilde{U}_{k,c} \equiv \widetilde{L}_k$ (with a parameter $c \in N_0$) are defined by (3) in which $I = [0,\infty)$, $J_k = N_0$, $p_{j,k}(x) = (-1)^j x^j \psi_{k,c}^{(j)}(x)/j!$, $\psi_{k,c}(x) = e^{-kx}$ if $c = 0$, and $\psi_{k,c}(x) = (1+cx)^{-k/c}$ if $c \ge 1$, $q_{j,k} = k - c$ for $k > c$ (see [9]). Now $r_k(x) = 0$ for all $x \in I$, $k \in N$,

$c_1 = 1$, $\mu_{2,k}(x) = x(1+cx)/k$ for all $x \in I$, $k > c$ and condition (12) holds with $\xi_{j,k} = j/k$. Further,

$$\widetilde{\mu}_{2,k} = \frac{2x(1+cx)(k+3c)+2}{(k-2c)(k-3c)} \quad \text{for} \quad x \in I, \quad k > 3c.$$

Hence Theorem 1 (via Remarks 1, 4) applies for $n > 3c$, with $w(x) = (x/(1+x))^{1/2}$, $c_3 = 2(1+c_2)$, $c_2 = (2(1+3c)(1+6c)/(1+c))^{1/2}$.

4) The Szász–Mirakyan–Durrmeyer operators $\widetilde{S}_k$ are the special case of operators $\widetilde{U}_{k,c}$ defined in 3), with $c = 0$. From 3) we know that, for these operators, conditions (2) and (13) hold with $c_1 = 1$, $c_2 = 2^{1/2}$ and $w(x) = (x/(1+x))^{1/2}$. Consider $f \in C_\eta(I)$ with the weight $\eta(x) = (1+x)^{-\sigma}$ where $\sigma \in N$. It is easy to see that, for $k \geq 2\sigma$,

$$\int_0^\infty \frac{1}{\eta^2(t)} p_{j,k}(t)dt = \frac{k^j}{j!}\int_0^\infty (1+t)^{2\sigma} t^j e^{-kt} dt \leq 2^{2^{2\sigma-1}}\Big(\frac{1}{k} + \frac{k^j}{j!}\int_0^\infty t^{2\sigma+j} e^{-kt} dt\Big) =$$
$$= 2^{2^{2\sigma-1}} \frac{1}{k}\Big(1 + \frac{(2\sigma+j)!}{j!} k^{-2\sigma}\Big) \leq 2^{2^{2\sigma-1}} \frac{1}{k}\Big(1 + \Big(\frac{j}{k}+1\Big)^{2\sigma}\Big).$$

Consequently, the left-hand side of (14) is not greater than

$$\frac{2^{2^{2\sigma-1}}}{\mu_{2,k}(x)} \sum_{j=0}^\infty \Big(\frac{j}{k} - x\Big)^2 p_{j,k}(x) \Big(1 + 2^{2^{2\sigma-1}}\Big((1+x)^{2\sigma} + \Big(\frac{j}{k} - x\Big)^{2\sigma}\Big)\Big) =$$
$$= 2^{2^{2\sigma-1}}\Big(1 + 2^{2^{2\sigma-1}}(1+x)^{2\sigma}\Big) + \frac{4^{2\sigma-1}}{\mu_{2,k}(x)} \sum_{j=0}^\infty \Big(\frac{j}{k} - x\Big)^{2\sigma+2} p_{j,k}(x) \leq$$
$$\leq c_{13}(1+x)^{2\sigma}$$

(see [10], p. 334). Applying Theorem 3 (together with Remarks 1, 4), we get the estimate

$$\Omega_{w,\eta}(\widetilde{S}_n f; \delta) \leq c_{14}\Omega_\eta(f;\delta) \quad (\delta \geq 0, \ \ n \geq 2\sigma). \tag{19}$$

Since $\widetilde{\mu}_{2,k}(x) \leq 2(1+x)/k$, conditions (15) and (16) are satisfied with $\rho(x) = (1+x)^{-\sigma-1}$ and $\delta_k = k^{-1/2}$. Consequently, Theorem 4 gives

$$\|\widetilde{S}_n f - f\|_\rho \leq c_{15}\Omega_\eta(f; n^{-1/2}) \quad \text{for all} \ \ n \in N.$$

Combining this result and (19) with the general inequality (18), we easily verify that, for $n \geq 2\sigma$,

$$\|\widetilde{S}_n f - f\|_{w,\rho}^{(\varphi)} \leq c_{16} \sup\Big\{\frac{1}{\varphi(\delta)}\Omega_\eta(f;\delta) \ : \ 0 < \delta \leq n^{-1/2}\Big\}.$$

5) The generalized Favard operators $F_k \equiv L_k$ are deefined by (1) with $\xi_{j,k} = j/k$, $J_k = Z$, $I = (-\infty, \infty)$ and

$$p_{j,k}(x) \equiv p_{j,k}(\gamma; x) = (\sqrt{2\pi} k\gamma_k)^{-1} \exp\Big(-\frac{1}{2}\gamma_k^{-2}\Big(\frac{j}{k} - x\Big)^2\Big),$$

$\gamma = (\gamma_k)_1^\infty$ being a positive null sequence satisfying

$$k^2\gamma_k^2 \geq \frac{1}{2}\pi^{-2}\log k \quad \text{for} \;\; k \geq 2, \quad \gamma_1^2 \geq \frac{1}{2}\pi^{-2}\log 2$$

(see [6]). Denote by $\widetilde{F}_k$ their Durrmeyer modification of form (3) in which $q_{j,k} = k$ for all $j \in Z$ and $k \in N$. As is known ([6], [12]), for all $x \in I$ and $k \in N$,

$$|r_k(x)| \equiv |r_k(\gamma; x)| = \Big| \sum_{j=-\infty}^{\infty} p_{j,k}(\gamma; x) - 1\Big| \leq 2 \;\text{ or }\; |r_k(\gamma; x)| \leq 7\pi\gamma_k.$$

$\mu_{2,k}(x) \equiv \mu_{2,k}(\gamma; x) \leq 51\gamma_k^2$; moreover, $\omega(r_k(\gamma; x)| \leq 16\pi\delta$ for every $\delta \geq 0$ (see [10], p. 336). It is easy to see that

$$\widetilde{\mu}_{2,k}(x) \equiv \widetilde{\mu}_{2,k}(\gamma; x) = \mu_{2,k}(\gamma; x) + \gamma_k^2(1 + r_k(\gamma; x)) \leq 54\gamma_k^2.$$

Observing that

$$p'_{j,k}(\gamma; x) = \gamma_k^{-2}\Big(\frac{j}{k} - x\Big)p_{j,k}(\gamma; x)$$

and applying the Cauchy–Schwartz inequality, we estimate the left-hand side of (4) by

$$k\gamma_k^{-2} \sum_{j=-\infty}^{\infty} \Big|\frac{j}{k} - x\Big| p_{j,k}(\gamma; x) \int_{-\infty}^{\infty} |t - x| p_{j,k}(\gamma; t) dt \leq$$
$$\leq \gamma_k^{-2}(\mu_{2,k}(\gamma; x))^{1/2}(\widetilde{\mu}_{2,k}(\gamma; x))^{1/2},$$

i.e., $w(x) = 1$ for all real x and $c_2 = 52,5$. Thus Theorem 1 yields the estimate

$$\omega(\widetilde{F}_n f; \delta) \leq 111\omega(f; \delta) + 16\pi\delta\|f\| \quad (\delta \geq 0)$$

for every $n \in N$ and every $f \in C(I)$. Clearly, this inequality is interesting if $f \in C(I)$ is bounded on I.

Consider now $f \in C_\eta(I)$ where $\eta(x) = \exp(-\sigma x^2)$ $\sigma > 0$. If $\sigma\gamma_k^2 \geq 3/32$, then

$$\exp(\sigma x^2) \exp\Big(-\frac{1}{2}\gamma_k^{-2}\Big(\frac{j}{k} - x\Big)^2\Big) \exp\Big(-\frac{1}{2}\gamma_k^{-2}\Big(\frac{j}{k} - t\Big)^2\Big) \leq$$
$$\leq \exp(4\sigma x^2) \exp\Big(-\frac{1}{8}\gamma_k^{-2}\Big(\frac{j}{k} - x\Big)^2\Big) \exp\Big(-\frac{1}{8}\gamma_k^{-2}\Big(\frac{j}{k} - t\Big)^2\Big);$$

whence

$$\widetilde{F}_k(1/\eta)(x) \le 2(1 + r_k(2\gamma; x)) \exp(4\sigma x^2).$$

Analogously, one can show that the left-hand side of (9) is not greater than

$$2\gamma_k^{-2}\mu_{2,k}(\gamma; x))^{1/2}(\widetilde{\mu}_{2,k}(2\gamma; x))^{1/2} \exp(4\sigma x^2)$$

provided that $\sigma\gamma_k^2 \le 3/64$. Further (see [12]),

$$r_k(2\gamma; x) \le 2/15, \quad \mu_{2,k}(2\gamma; x) \le 23\gamma_k^2$$

and

$$\widetilde{\mu}_{2,k}(2\gamma; x) = \mu_{2,k}(2\gamma; x) + (2\gamma_k)^2(1 + r_k(2\gamma; x)) \le \frac{413}{15}\gamma_k^2.$$

Thus Theorem 2 applies with $w(x) \equiv 1$, $\rho(x) = \exp(-4\sigma x^2)$, $c_4 = 68/15$, $c_5 = 75$ (i.e. $c_6 = 271$) and n such that $\sigma\gamma_n^2 \le 3/64$. In the same way one can show that Theorem 4 is true with $\rho(x) = \rho_1(x) := \exp(-7\sigma x^2)$, $\delta_n = \gamma_n$, $\sigma\gamma_n^2 \le 3/64$ and a positive absolute constant c_{11}. From these results the estimate of $\|\widetilde{F}_n f - f\|_{\rho_1}^{(\varphi)}$ follows at once via inequality (18).

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