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TWO-WEIGHTED ESTIMATES FOR SOME INTEGRAL TRANSFORMS IN THE LEBESGUE SPACES WITH MIXED NORM AND IMBEDDING THEOREMS

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Abstract. Two-weighted inequalities are proved for anisotropic potentials. These estimates are used to obtain the refinements of the well-known imbedding theorems in the scale of weighted Lebesgue spaces.

Two-weighted inequalities are obtained for anisotropic potentials in Lebesgue spaces with mixed norm. These estimates are used to prove imbedding theorems for different metrics and different dimensions for weighted spaces of anisotropic Bessel potentials.

Nonweighted cases were previously treated in [1–3]. One-weighted estimates for isotropic Bessel potentials can be found in [4].

1. A measurable almost everywhere positive function $\varrho : \mathbb{R}^1 \to \mathbb{R}^1$ will be called a weight function. Let $w = (w_1, w_2, \dots, w_n)$ be a vector-function where w_i $(i = 1, 2, \dots, n)$ is a weight function. By definition a measurable function $f(x) = f(x_1, x_2, \dots, x_n)$ given on the n-dimensional space $\mathbb{R}^n$ belongs to L^p_w, $p = (p_1, p_2, \dots, p_n)$, $1 < p_i < \infty$ $(1 = 1, 2, \dots, n)$, if the norm

$$\|f\|_{L^p_w(\mathbb{R}^n)} =$$
$$=\bigg(\int\limits_{-\infty}^{\infty} w_1^{p_1}(x_1)dx_1\bigg(\int\limits_{-\infty}^{\infty} w_2^{p_2}(x_2)dx_2\dots\bigg(\int\limits_{-\infty}^{\infty} f(x)w_n^{p_n}(x_n)dx_n\bigg)^{\frac{p_{n-1}}{p_n}}\bigg)\dots\bigg)^{\frac{p_1}{p_2}}\bigg)^{\frac{1}{p_1}}$$

is finite.

We shall introduce a class of pairs of weight functions.

For a given number r, $1 < r < \infty$, we write $r' = \frac{r}{r-1}$.

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Definition 1. A pair of weight functions (ϱ,σ) given on $\mathbb{R}^1$ will be said to belong to the class $G^{s,r}_\beta$, $0<\beta<1$, $1<r<s<\infty$, if the conditions

$$\sup\Big(\int\limits_I \varrho^s(t)dt\Big)^{\frac{1}{s}}\Big(\int\limits_{\mathbb{R}^1}\frac{\sigma^{-r'}(t)}{(|I|+|t_I-t|)^{\beta r'}}dt\Big)^{\frac{1}{r'}}<\infty, \tag{1.1}$$

$$\sup\Big(\int\limits_I \sigma^{-r'}(t)dt\Big)^{\frac{1}{r'}}\Big(\int\limits_{\mathbb{R}^1}\frac{\varrho^{s}(t)}{(|I|+|t_I-t|)^{s\beta}}dt\Big)^{\frac{1}{s}}<\infty, \tag{1.2}$$

are fulfilled, where the supremum is taken over all bounded one-dimensional intervals I, with centre and length, t_I and $|I|$ respectively.

In the sequel we shall proceed from

Theorem A [5–7]. *The fractional integral*

$$I_\gamma(f)(x)=\int\limits_{-\infty}^{\infty}\frac{f(\tau)}{|t-\tau|^{1-\gamma}}d\tau,\quad 0<\gamma<1,$$

generates a continuous operator from $L^r_\sigma(\mathbb{R}^1)$ into $L^s_\varrho(\mathbb{R}^1)$ if and only if $(\varrho,\sigma)\in G^{s,r}_{1-\gamma}$.

Let numbers $a_j>0$ $(j=1,2,\dots,n)$ be given. For $x=(x_1,x_2,\dots,x_n)$ we set

$$|x|_a=\Big(\sum_{i=1}^n |x|^{\frac{2}{a_j}}\Big)^{\frac{1}{2}}.$$

It is obvious that for $a_j=1$ $(j=1,2,\dots,n)$ we obtain an usual Euclidian distance.

Theorem 1. *Let $w=(w_1,w_2,\dots,w_n)$, $v=(v_1,v_2,\dots,v_n)$, where w_i and v_i $(i=1,2,\dots,n)$ are weight functions given on $\mathbb{R}^n$. We set*

$$\mathcal{K}f(x)=\int\limits_{\mathbb{R}^n}\frac{f(y)}{|x-y|_a^{\mu}}dy,$$

where

$$\mu=\sum_{j=1}^n a_j(1-\gamma_j),\quad 0<\gamma_j<1\ \ (j=1,2,\dots,n).$$

If $1<p_i<q_i<\infty$, $(v_i,w_i)\in G^{q_i,p_i}_{1-\gamma_i}$ $(i=1,2,\dots,n)$, then there exists a positive number c such that the inequality $\|\mathcal{K}f\|_{L^q_v}\le c\|f\|_{L^p_w}$ holds for any $f\in L^p_w$.

The proof of Theorem 1 will be based on several lemmas. The first lemma is a weighted analogue of the well–known Hardy–Littlewood inequality (see [8], Theorem 382).

Lemma 1. *Let* $(\varrho,\sigma) \in G^{s,r}_{1-\gamma}$, $1 < r < s < \infty$, $0 < \gamma < 1$. *Then there exists a constant* $c > 0$, *such that the inequality*

$$\Big|\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{\varphi(x)\psi(y)}{|x-y|^{1-\gamma}} dx\,dy\Big| \le c\|\varphi\|_{L^r_\sigma}\|\psi\|_{L^{s'}_{1/\varrho}} \tag{1.3}$$

holds for any arbitrary $\varphi \in L^r_\sigma(\mathbb{R}^1)$ *and* $\psi \in L^{s'}_{1/\varrho}(\mathbb{R}^1)$.

Proof. By virtue of the Hölder inequality we have

$$\Big|\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{\varphi(x)\psi(y)}{|x-y|^{1-\gamma|}} dx\,dy\Big| \le$$
$$\le \|\varphi\sigma\|_{L^r}\Big(\int\limits_{\mathbb{R}^1}\Big(\int\limits_{\mathbb{R}^1} \frac{|\psi(y)|dy}{|x-y|^{1-\gamma}}\Big)^{r'} \frac{1}{\sigma^{r'}(x)}dx\Big)^{\frac{1}{r'}}.$$

From the condition $(\varrho,\sigma) \in G^{s,r}_{1-\gamma}$ readily follows that $(\frac{1}{\sigma},\frac{1}{\varrho}) \in G^{r',s'}_{1-\gamma}$. Using Theorem A we obtain the estimate

$$\Big|\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{\varphi(x)\psi(y)}{|x-y|^{1-\gamma}} dx\,dy\Big| \le c\|\varphi\|_{L^r_\sigma}\cdot\|\psi\|_{L^{s'}_{1/\varrho}}. \ \blacksquare$$

Bellow we shall set $\varrho = (\frac{1}{v_1},\frac{1}{v_2},\dots,\frac{1}{v_n})$ for $v = (v_1, v_2, \dots, v_n)$.

Lemma 2. *Let* $1 < p_i < q_i < \infty$, $(v_i, w_i) \in G^{q_i,p_i}_{1-\gamma_i}$, $0 < \gamma_i < 1$. *Then there exists a positive constant* c *such that*

$$\Big|\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{\varphi(x)\psi(y)}{|x-y|^{\mu}_a} dx\,dy\Big| \le c\|\varphi\|_{L^p_w}\cdot\|\psi\|_{L^{q'}_\varrho}$$

for arbitrary $\varphi \in L^p_w(\mathbb{R}^n)$ *and* $\psi \in L^{q'}_\varrho(\mathbb{R}^n)$.

Proof. We shall apply the reduction technique to the one-dimensional case.

Let $x' = (x_1, x_2, \dots, x_{n-1})$, $y' = (y_1, y_2, \dots, y_{n-1})$ and

$$\mu' = \sum_{j=1}^{n-1} a_j(1-\gamma_j), \quad a' = (a_1, a_2, \dots, a_{n-1}).$$

It readily follows that

$$|x-y|^{\mu}_a = |x-y|^{\mu'}_a |x-y|^{a_n(1-\gamma_n)}_a \ge |x'-y'|^{\mu'}_{a'} |x_n-y_n|^{1-\gamma_n}.$$

Therefore by Lemma 1 we obtain

$$\Big|\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\frac{\varphi(x)\psi(y)}{|x-y|_a^{\mu}}dx\,dy\Big|\leq\int\limits_{\mathbb{R}^{n-1}}\int\limits_{\mathbb{R}^{n-1}}\frac{dx'dy'}{|x'-y'|_a^{\mu'}}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{|\varphi(x)|\,|\psi(y)|}{|x_n-y_n|^{1-\gamma_n}}dx_ndy_n\leq$$
$$\leq c\int\limits_{\mathbb{R}^{n-1}}\int\limits_{\mathbb{R}^{n-1}}\frac{F(x')H(y')}{|x'-y'|_{a'}^{\mu'}}dx'dy',$$

where

$$F(x')=\Big(\int\limits_{\mathbb{R}^1}|\varphi(x)|^{p_n}w_n^{p_n}(x_n)dx_n\Big)^{\frac{1}{p_n}},$$
$$H(x')=\Big(\int\limits_{\mathbb{R}^1}|\psi(y)|^{q'_n}v_n^{-q'_n}(y_n)dy_n\Big)^{\frac{1}{q'_n}}.$$

Further reduction leads us to the proof of Lemma 2. $\square$

Proof of Theorem 1. By the property of the norm and also by Lemma 2 we have

$$\|\mathcal{K}f\|_{L_v^q}=\sup\Big|\int\limits_{\mathbb{R}^n}\mathcal{K}f(x)g(x)dx\Big|,$$

where the least upper bound is taken over all functions g for which

$$\|g\|_{L_\varrho^{q'}}\leq 1,\quad \varrho=(\frac{1}{v_1},\frac{1}{v_2},\dots,\frac{1}{v_n}).$$

Next, by Lemma 2 we have

$$\Big|\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\frac{f(y)g(x)}{|x-y|_a^{\mu}}dx\,dy\Big|\leq c\|f\|_{L_w^p}\cdot\|g\|_{L_\varrho^{q'}}\leq c\|f\|_{L_w^p}.\ \blacksquare$$

Theorem 2. *Let* $1\leq m\leq n$, $v_+=(v_1,v_2,\dots,v_m)$, $w_+=(w_1,w_2,\dots,w_m)$, $w=(w_1,w_2,\dots,w_m,1,\dots,1)$, $1<p_i<q_i<\infty$ $(i=1,2,\dots,m)$, $q_+=(q_1,q_2,\dots,q_m)$, $p_+=(p_1,p_2,\dots,p_m)$, $1<p_i<\infty$ $(i=m+1,\dots,n)$.

Next we set

$$\mu=\sum_{i=1}^{m}a_j(1-\gamma_j)+\sum_{j=m+1}^{n}\frac{a_j}{p'_j},\tag{1.4}$$

where $a_j>0$, $0<\gamma_j<1$ $(j=1,2,\dots,m)$

If $(v_i,w_i)\in G_{1-\gamma_i}^{q_i,p_i}$ $(i=1,2,\dots,m)$, *then there exists a constant* $c>0$ *such that for any* $f\in L_w^p(\mathbb{R}^n)$ *and arbitrary* $(x_{m+1}^0,\dots,x_n^0)$ *the function*

$$h(x)=\int\limits_{\mathbb{R}^n}\frac{f(y)}{|x-y|_a^{\mu}}dy$$

belongs to the space $L^{q^+}_{v_+}(\mathbb{R}^m)$ *and the inequality holds* $\|h\|_{L^{q^+}_{v_+}} \le c\|f\|_{L^p_w}$, *where the constant* c *is independent of* f.

The proof of Theorem 2 is based on the following

Lemma 3. *Let the conditions of Theorem 2 be fulfilled. If* $\varrho=(\frac{1}{v_1},\frac{1}{v_2},\dots,\frac{1}{v_n})$, $q'_+ = (q'_1,q'_2,\dots,q'_m)$, *then there exists a constant* $c>0$ *such that for all* $f:\mathbb{R}^n\to\mathbb{R}^1$ *and* $g:\mathbb{R}^m\to\mathbb{R}^1$ *we have*

$$\Big|\int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^n}\frac{g(x)f(y)}{|x-y|_a^{\mu}}\Big| \le c\|f\|_{L^p_w(\mathbb{R}^n)}\cdot\|g\|_{L^{q'_+}_{\varrho_+}(\mathbb{R}^m)}. \tag{1.5}$$

Proof. Let $y'=(y_1,y_2,\dots,y_m)$, $y''=(y_{m+1},\dots,y_n)$, $p_+=(p_1,p_2,\dots,p_m)$, $p_-=(p_{m+1},\dots,p_n)$, $p'_-=(p'_{m+1},\dots,p'_n)$.

Obviously

$$\Big|\int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^n}\frac{g(x)f(y)}{|x-y|_a^{\mu}}dx\,dy\Big| \le \int\limits_{\mathbb{R}^m}|g(x)|\Big(\int\limits_{\mathbb{R}^m}\Big(\int\limits_{\mathbb{R}^{n-m}}\frac{|f(y)|dy''}{|x-y|_a^{\mu}}\Big)dy'\Big)dx. \tag{1.6}$$

By virtue of the Hölder inequality

$$\int\limits_{\mathbb{R}^{n-m}}\frac{|f(y)|dy''}{|x-y|_a^{\mu}} \le \|f\|_{L^{p_-}(\mathbb{R}^{n-m})}\cdot\|\,|x-y|_a^{-\mu}\|_{L^{p'_-}(\mathbb{R}^{n-m})}. \tag{1.7}$$

We introduce some notation:

$$\varphi(y')=\|f(y',\cdot)\|_{L^{p_-}(\mathbb{R}^{n-m})},\ \varphi_1(x,y')=\|\,|x-y|_a^{-\mu}\|_{L^{p'_-}(\mathbb{R}^{n-m})},$$
$$T=\Big(\sum_{j=1}^{m}|x_j-y_j|^{\frac{2}{a_j}}\Big)^{\frac12}.$$

Let us prove that there exists a positive number c_1 such that

$$\varphi_1(x,y')\le c_1 T^{\left(\sum_{j=m+1}^{n}\frac{a_j}{p'_j}-\mu\right)}. \tag{1.8}$$

We have

$$\varphi_1(x,y')=\Big\|\Big(T^2+\sum_{j=m+1}^{n}|x_j-y_j|^{\frac{2}{a_j}}\Big)^{-\frac{\mu}{2}}\Big\|_{L^{p'_-}(\mathbb{R}^{n-m})}=$$
$$=\Big\|\Big(T^2+\sum_{j=m+1}^{n}|y_j|^{\frac{2}{a_j}}\Big)^{-\frac{\mu}{2}}\Big\|_{L^{p'_-}(\mathbb{R}^{n-m})}.$$

The change of the variable $y_j = T^{a_j} t_j$ in the latter expression leads to the equality

$$\varphi(x,y') = T^{\sum_{j=m+1}^{n} \frac{a_j}{p'_j} - \mu} \Big\| \Big(1 + \sum_{j=m+1}^{n} |t_j|^{\frac{2}{a_j}}\Big)^{-\frac{\mu}{2}} \Big\|_{L^{p'_-}(\mathbb{R}^{n-m})}.$$

Now it is obvious that

$$\Big(1 + \sum_{j=m+1}^{n} |t_j|^{\frac{2}{a_j}}\Big)^{-\frac{\mu}{2}} = \Big(1 + \sum_{j=m+1}^{n} |t_j|^{\frac{2}{a_j}}\Big)^{-\frac{1}{2}\sum_{k=m+1}^{n} \frac{a_k}{p'_k} + \sum_{i=1}^{m} a_i(1-\gamma_i)} =$$
$$= \prod_{j=m+1}^{n} \Big(1 + \sum_{j=m+1}^{n} |t_j|^{\frac{2}{a_j}}\Big)^{-\frac{1}{2}(\frac{a_j}{p'_j}+\varepsilon)} \leq \prod_{j=m+1}^{n} \big(1 + |t_j|^{\frac{2}{a_j}}\big)^{-\frac{1}{2}(\frac{a_j}{p'_j}+\varepsilon)},$$

where $\varepsilon > 0$.

Therefore

$$\Big\| \Big(1 + \sum_{j=m+1}^{n} |t_j|^{\frac{2}{a_j}}\Big)^{-\frac{\mu}{2}} \Big\|_{L^{p'_-}(\mathbb{R}^{n-m})} \leq$$
$$\leq \Big\| \prod_{j=m+1}^{n} \big(1 + |t_j|^{\frac{2}{a_j}}\big)^{-\frac{1}{2}(\frac{a_j}{p'_j}+\varepsilon)} \Big\|_{L^{p'_-}(\mathbb{R}^{n-m})}.$$

On the other hand,

$$\int_{\mathbb{R}^1} \frac{dt_j}{\big(1 + |t_j|^{\frac{2}{a_j}}\big)^{\frac{1}{2}(\frac{a_j}{p'_j}+\varepsilon)p'_j}} \leq \int_{\mathbb{R}^1} \frac{dt_j}{(1+|t_j|)^{1+\frac{\varepsilon p'_j}{a_j}}} < \infty.$$

Thus we have proved the estimate (1.8). It implies

$$\varphi(x,y') \leq \Big(\sum_{j=1}^{m} |x_j - y_j|^{\frac{2}{a_j}}\Big)^{-\frac{1}{2}\sum_{j=1}^{m} a_j(1-\gamma_j)} \leq c_1 \prod_{j=1}^{m} |x_j - y_j|^{(\gamma_j - 1)}. \quad (1.9)$$

Using the generalized Hölder inequality and Lemma 1 with (1.6), (1.7), (1.8) and (1.9) we obtain:

$$\Big| \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^n} \frac{g(x)f(y)}{|x-y|_a^{\mu}} dy \Big| \leq c_2 \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{|g(x)|\varphi(y')}{\prod_{j=1}^{m} |x_j - y_j|^{1-\gamma_j}} dx\, dy \leq$$
$$\leq c_3 \|\varphi(y')\|_{L^{p_+}_{w_+}(\mathbb{R}^m)} \|g\|_{L^{q'}_{\varrho}(\mathbb{R}^m)}.$$

This implies that

$$\Big| \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^n} \frac{g(x)f(y)}{|x-y|_a^{\mu}} dy \Big| \leq c_3 \|f\|_{L^p_w(\mathbb{R}^n)} \|g\|_{L^{q_+}_{\varrho}(\mathbb{R}^m)}. \quad \blacksquare$$

Proof of Theorem 2. Using the standard arguments, the validity of Theorem 2 readily follows from Lemma 3. □

2. In this paragraph we shall prove the imbedding theorems for different metrics and different dimensions for weighted spaces of anisotropic Bessel potentials.

Definition 2 (see[2]). Let $r=(r_1,r_2,\dots,r_n)$, $p=(p_1,p_2,\dots,p_n)$, $r_j>0$ $(j=1,2,\dots,n)$. It will be said that $f\in L^{p,r}_w(\mathbb{R}^n)$ if

$$f(x)=\int\limits_{\mathbb{R}^n} G_r(x-y)g(y)dy,$$

where G_r is the anisotropic Bessel–Macdonald kernel and $g\in L^p_w(\mathbb{R}^n)$. By the definition, $\|f\|_{L^{p,r}_w}=\|g\|_{L^p_w}$.

The kernel G_r is characterized by its Fourier transform as follows (see[2]) $(2\pi)^{\frac{n}{2}}\widehat{G}_2(\lambda)=[1+\sigma^2(\lambda)]^{-\frac{r^*}{2}}$ where the function $\sigma(\lambda)$ is determined by the equation

$$\sum_{j=1}^n \frac{\lambda_j^2}{\sigma^{2a_j}}=1,\quad a_j=\frac{r^*}{r_j},\quad \frac{1}{r^*}=\frac{1}{n}\sum_{j=1}^n\frac{1}{r_j}.$$

The kernel G_r obeys, along each j – the coordinate direction, the estimates

$$|G_r(x)|\le c|x_j|^{-r_j\left(\sum_{i=1}^n \frac{1}{r_i}-1\right)}. \tag{2.1}$$

Now we shall prove the imbedding theorem of different metrics.

Theorem 3. *Let* $1<p_j<q_j<\infty$, $(v_i,w_i)\in G^{q_i,p_i}_{1-\gamma_i}$ $(i=1,2,\dots,n)$. *Put*

$$\varkappa=1-\sum_{j=1}^n\frac{\gamma_j}{r_j}$$

and $\varrho=\varkappa r$, $r=(r_1,r_2,\dots,r_n)$.

Then each function $f\in L^{p,r}_w(\mathbb{R}^n)$ *belongs to the space* $L^{q,\varrho}_v(\mathbb{R}^n)$ *and the inequality* $\|f\|_{L^{q,\varrho}_v}\le c\|f\|_{L^{p,r}_w}$ *holds, where the constant* c *is independant of* f.

Proof. We have

$$f(x)=\int\limits_{\mathbb{R}^n} G_\varrho(x-y)h(y)dy,$$

where

$$h(x)=\int\limits_{\mathbb{R}^n} G_{r(1-\varkappa)}(x-y)g(y)dy$$

and $g\in L^p_w(\mathbb{R}^n)$.

Now it will be shown that $h \in L^q_v(\mathbb{R}^n)$. Due to (2.1) we have

$$|G_{r(1-\varkappa)}(x)| \le c|x_i|^{-r_j(1-\varkappa)\left[\sum_{i=1}^n \frac{1}{r_i(1-\varkappa)}-1\right]},$$

or

$$|G_{r(1-\varkappa)}(x)| \le c|x_i|^{-r_j \sum_{i=1}^n \frac{1-\gamma_i}{r_i}}.$$

Let $a_j = \frac{1}{r_j}$ and

$$\max_{1\le j\le n} |x_j|^{\frac{1}{a_j}\sum_{i=1}^n a_i(1-\gamma_i)} = |x_{j_0}|^{\frac{1}{a_{j_0}}\sum_{i=1}^n a_i(1-\gamma_i)}.$$

As can be easily verified,

$$\Big(\sum_{i=1}^n |x_j|^{\frac{2}{a_j}}\Big)^{\frac{1}{2}\sum_{i=1}^n a_i(1-\gamma_i)} \le c_1|x_{j_0}|^{\frac{1}{a_{j_0}}\sum_{i=1}^n a_i(1-\gamma_i)}.$$

Therefore

$$|G_{r(1-\varkappa)}(x)| \le c_2|x|_a^{-\mu}, \tag{2.2}$$

where

$$\mu = \sum_{j=1}^n a_j(1-\gamma_j).$$

Hence

$$|h(x)| \le \int\limits_{\mathbb{R}^n} |g(y)|\,|x-y|_a^{-\mu}dy.$$

Applying Theorem 1, we obtain $\|h\|_{L^q_v} \le c_3\|f\|_{L^p_w}$, which implies $\|f\|_{L^{q,\varrho}_v} \le \|f\|_{L^{p,r}_w}$. ■

Using Theorem 2 one may prove an imbedding theorem of different dimensions in a similar manner.

Theorem 4. *Let $1 < p_i < \infty$ $(i = 1,2,\dots,n)$, $1 < p_i < q_i < \infty$ $(i = 1,2,\dots,m)$, $1 \le m \le n$. It is also assumed that $(v_i, w_i) \in G^{q_i,p_i}_{1-\gamma_i}$, $0 < \gamma_i < 1$ $(i = 1,2,\dots)$.*

If

$$\varkappa = 1 - \sum_{j=1}^m \frac{\gamma_j}{r_j} - \sum_{j=m+1}^n \frac{1}{r_j p_j}, \tag{2.3}$$

then for an arbitrary function f from the space $L^{p,r}_w(\mathbb{R}^n)$ the function $F(x_1,x_2,\dots,x_m) = f(x_1,x_2,\dots,x_m,x^0_{m+1},\dots,x^0_n)$ belongs to the space $L^{q,\varrho}_v(\mathbb{R}^m)$ and the inequality $\|F\|_{L^{q,\varrho}_v(\mathbb{R}^m)} \le c\|f\|_{L^{p,r}_w(\mathbb{R}^n)}$ holds where the constant c is independent of f.

Proof. In the case under consideration the kernel $G_{(1-\varkappa)r}$ admits the estimate $|G_{(1-\varkappa)r}(x)| \leq c|x|_a^{-\mu}$, where $a=(a_1,a_2,\dots,a_n)$, $a_i=\frac{1}{r_j}$ and

$$\mu=\sum_{i=1}^{m}\frac{1-\gamma_i}{r_i}+\sum_{j=m+1}^{n}\frac{1}{r_j p_j'}. \tag{2.4}$$

Hence we can apply Theorem 2. The rest of the proof is as for the preceding theorem. □

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