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ON PERFECT MAPPINGS FROM $\mathbb{R}$ TO $\mathbb{R}$

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Abstract. Perfect mappings from $\mathbb{R}$ to $\mathbb{R}$ and also mappings close to perfect ones are considered. Some of their properties are given.

This paper deals with some properties of perfect mappings from the space of real numbers with natural topology $\mathbb{R}$ to $\mathbb{R}$. For example, it is shown that no perfect mapping from $\mathbb{R}$ to $\mathbb{R}$ has finite partial limits at $+\infty$ and $-\infty$, that no bounded mapping from $\mathbb{R}$ to $\mathbb{R}$ is perfect, and so on. We also consider mappings from $\mathbb{R}$ to $\mathbb{R}$ (not necessarily continuous) with closed, compact and bounded fibers and mapppings under which the image of any closed subset of $\mathbb{R}$ is closed in $\mathbb{R}$.

Recall that a mapping $f : X \to Y$ of topological spaces is called a closed mapping [1] if f is continuous and the following condition is satisfied:

$(CL\downarrow)$ for every closed subset F of the space X the image $f(F)$ is closed in Y.

Recall also that a mapping $f : X \to Y$ of topological spaces is said to be perfect [1] if X is a Hausdorff space, f is a closed mapping and f satisfies the following condition:

(CM_p^{-1}) for any point $y \in Y$ the fiber $f^{-1}(y)$ is compact.

Note that if $f : X \to Y$ is a mapping from the Hausdorff space X to the space Y (not necessarily continuous), satisfying the conditions $(CL\downarrow)$ and (CM_p^{-1}) (see Example 1 below), then for every compact subspace Z of the space Y the inverse image $f^{-1}(Z)$ is compact. (The proof of this fact repeats the proof of Theorem 3.7.2 [11].) This implies that every function $f : X \to \mathbb{R}$ from the Hausdorff space X to $\mathbb{R}$, satisfying the conditions $(CL\downarrow)$ and (CM_p^{-1}), is a Borel function.

In the sequel $\mathcal{K}$ will denote the class of mappings (not necessarily continuous) from $\mathbb{R}$ to $\mathbb{R}$.

We shall define the following classes of mappings:

$\mathcal{K}(C) \equiv \{f \in \mathcal{K} | f \text{ is continuous }\}$;

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$\mathcal{K}(CL\downarrow) \equiv \{f \in \mathcal{K}|$ for any closed subset F of $\mathcal{K}$ the image $f(F)$ is closed in $\mathbb{R}\}$;

$\mathcal{K}(CM_p^{-1}) \equiv \{f \in \mathcal{K}|$ for any point $y \in \mathbb{R}$ the fiber $f^{-1}(y)$ is compact $\}$;

$\mathcal{K}(CL_p^{-1}) \equiv \{f \in \mathcal{K}|$ for any point $y \in \mathbb{R}$ the fiber $f^{-1}(y)$ is closed in $\mathbb{R}\}$;

$\mathcal{K}(B_p^{-1}) \equiv \{f \in \mathcal{K}|$ for any point $y \in \mathbb{R}$ the fiber $f^{-1}(y)$ is a bounded subset of $\mathbb{R}\}$.

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a closed mapping and let*

$$\lim_{x\to+\infty} f(x) = a \in \mathbb{R} \quad (\lim_{x\to-\infty} f(x) = a \in \mathbb{R}).$$

Then there exists a real number M such that for any $x > M$ (for any $x < M$) we have $f(x) = a$.

Proof. Let $\lim_{x\to+\infty} f(x) = a \in \mathbb{R}$ (the other case is analogous). Assume that our assertion is not true. Then for any real number M there must exist a real number $x_M > M$ such that $f(x_M) \neq a$. In particular, $\exists x_1 : x_1 > 1 : f(x_1) \neq a; \exists x_2 > x_1 + 1 : f(x_2) \neq a$ and so on $\exists x_k : x_{k-1} + 1 : f(x_k) \neq a; \exists x_2 > x_1 + 1 : f(x_2) \neq a$ and so on. Thus we have a sequence $(x_k)_{k\geq 1}$ of real numbers such that $x_k > k$ and $f(x_k) \neq a$. for any natural number k.

It is obvious that $\lim_{k\to\infty} x_k = +\infty$, the set $\{x_k\}_{k=1}^{\infty}$ is closed in $\mathbb{R}$ and $a \in \{f(x_k)\}_{k=1}^{\infty}$. But, by assumption, $\lim_{x\to+\infty} f(x) = a$. Hence, by virtue of the continuity of f, $\lim_{k\to\infty} f(x_k) = a$. Therefore a belongs to the closure of the set $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}$. Since $a \in \{f(x_k)\}_{k=1}^{\infty}$, the set $\{f(x_k)\}_{k=1}^{\infty}$ is not closed in $\mathbb{R}$.

On the other hand, since the set $\{x_k\}_{k=1}^{\infty}$ is closed in $\mathbb{R}$, the set $f(\{x_k\})_{k=1}^{\infty}$ must be closed too by the condition. This is the contradiction. ■

Corollary 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a closed mapping and let*

$$\lim_{x\to+\infty} f(x) = a \in \mathbb{R} \quad (\lim_{x\to-\infty} f(x) = a \in \mathbb{R}).$$

Then the set $f^{-1}(a)$ is not bounded.

Proof. Let $\lim_{x\to+\infty} f(x) = a \in \mathbb{R}$ ($\lim_{x\to-\infty} f(x) = a \in \mathbb{R}$). Then, by the above theorem, there exists a real number M such that $f^{-1}(a) \supset (M; +\infty)(f^{-1}(a) \supset (-\infty; M))$. Therefore $f^{-1}(a)$ is not bounded. ■

Corollary 2. *If $f : \mathbb{R} \to \mathbb{R}$ is a perfect mapping, then there are no limits of the function f at $+\infty$ and $-\infty$. In particular, no perfect mapping $f : \mathbb{R} \to \mathbb{R}$ has asymptotes.*

Proof. Let $\lim_{x\to+\infty} f(x) = a \in \mathbb{R}$ ($\lim_{x\to-\infty} f(x) = a \in \mathbb{R}$). Then, by the previous corollary, $f^{-1}(a)$ is not bounded. But since f is a perfect mapping, the set $f^{-1}(a)$ must be compact and hence bounded. This is the contradiction. ■

Theorem 2. *Let $f \in \mathcal{K}(CL\downarrow) \cap \mathcal{K}(CL_p^{-1})$ (see Example 2 below). Assume that $a \in \mathbb{R}$. If there exists a left-hand limit $\lim_{x\to a-} f(x)$ (respectively, if there exists a right-hand limit $\lim_{x\to a+} f(x)$), then $\lim_{x\to a-} f(x) = f(a)$ (respectively, $\lim_{x\to a+} f(x) = f(a)$).*

Proof. Let $\lim_{x\to a-} f(x) = b \in \mathbb{R}$. Assume that $b \neq f(a)$. (The other case is analogous.)

We have two possible cases:

1) $\exists\delta > 0 \quad (\delta \in \mathbb{R}) : \forall x \in (a-\delta; a) : f(x) = b$;

2) $\forall\delta > 0 \quad (\delta \in \mathbb{R}) \ \ \exists x_\delta \in (a-\delta; a) : f(x_\delta) \neq b$.

Let us consider each of them separately.

Case 1. We have $f^{-1}(b) \supset (a-\delta; a)$. By assumption, $f(a) \neq b$. Therefore $a \notin f^{-1}(b)$. On the other hand, since $f^{-1}(b) \supset (a-\delta; a)$, the point a belongs to the closure of the set $f^{-1}(b)$. Hence the set $f^{-1}(b)$ is not closed. This is the contradiction. Therefore case 1 is impossible.

Case 2. Denote $\frac{|b-f(a)|}{2} \equiv \varepsilon$. Clearly, $\varepsilon > 0$. Since $\lim_{x\to a-} f(x) = b$, there exists a positive real number δ such that for $f(x) \in (f(a)-\varepsilon; f(a)+\varepsilon)$ for any $x \in (a-\delta; a)$. Therefore $b \notin (f(a)-\varepsilon; f(a)+\varepsilon)$.

By the condition $\exists x_1 \in (a-\delta; a) : f(x_1) \neq b$.

Assume that $z_1 \equiv a-\delta$ and $z_2 \equiv \max\{a-\frac{\delta}{2}; x_1\}$. Then there exists $x_2 \in (z_2; a)$ such that $f(x_2) \neq b$.

Let the points $x_1, x_2, \dots, x_{k-1}$ be already constructed. Denote $z_k \equiv \max\{a-\frac{\delta}{k}; x_{k-1}\}$. According to the condition there exists a point x_k such that $x_k \in (z_k; a)$ and $f(x_k) \neq b$.

Therefore for any natural k we have a point $x_k \in \mathbb{R}$ such that $a-\frac{\delta}{k} < x_k < a$ and $f(x_k) \neq b$.

Since $\lim_{k\to\infty}(a-\frac{\delta}{k}) = a$, we have $\lim_{k\to\infty} x_k = a$.

Since $\lim_{x\to a-} f(x) = b$, we have $\lim_{k\to\infty} f(x_k) = b$.

The set $\{a\} \cup \{x_k\}_{k=1}^{\infty}$ is obviously closed in $\mathbb{R}$.

Now let us consider the set $f(\{a\} \cup \{x_k\}_{k=1}^{\infty}) = \{f(a)\} \cup \{f(x_k)\}_{k=1}^{\infty}$. Since for any $k \geq 1 \quad f(x_k) \neq b$ and, by assumption, $f(a) \neq b$, we obtain $b \notin f(\{a\} \cup \{x_k\}_{k=1}^{\infty})$. On the other hand, since $\lim_{k\to\infty} f(x_k) = b$, b belongs to the closure of the set $f(\{a\} \cup \{x_k\}_{k=1}^{\infty}) = \{f(a)\} \cup \{f(x_k)\}_{k=1}^{\infty}$. Hence the set $f(\{a\} \cup \{x_k\}_{k=1}^{\infty})$ is not closed in $\mathbb{R}$.

But the set $\{a\} \cup \{x_k\}_{k=1}^{\infty}$ is closed in $\mathbb{R}$ and, by the condition, the set $f(\{a\} \cup \{x_k\}_{k=1}^{\infty})$ must be closed in $\mathbb{R}$. This is the contradiction. ■

Corollary 3. *Let $f \in \mathcal{K}(CL\downarrow) \cap \mathcal{K}(CL_p^{-1})$. Then any point of discontinuity of the mapping f must be only of the second kind.*

Theorem 3. *Let $f \in \mathcal{K}(CL\downarrow) \cap \mathcal{K}(B_p^{-1})$ (see Example 3 below). Then the function f does not have finite partial limits at $+\infty$ and $-\infty$.*

Proof. Assume, conversely, that there is a sequence of real numbers $x_1, x_2, \dots, x_n, \dots$ such that $\lim_{n\to\infty} x_n = +\infty$ and $\lim_{n\to\infty} f(x_n)$ exists. Let $\lim_{n\to\infty} f(x_n) = a \in \mathbb{R}$. (The case concerning $-\infty$ is analogous.) Here we have two possible cases:

(a) the set $\{ n \mid f(x_n) = a \}$ is empty or finite;

(b) the set $\{ n \mid f(x_n) = a \}$ is infinite.

Consider each of them separately.

Case (a). There exists a natural number n_0 such that $f(x_n) \neq a$ for any $n \geq n_0$. Then $a \neq f(\{x_n\}_{n=n_0}^{\infty}) = \{f(x_n)\}_{n=n_0}^{\infty}$. But $\lim_{n\to\infty} f(x_n) = a$. Therefore the set $f(\{x_n\}_{n=n_0}^{\infty})$ is not closed in $\mathbb{R}$. On the other hand, since $\lim_{n\to\infty} x_n = +\infty$, the set $\{x_n\}_{n=n_0}^{\infty}$ is closed in $\mathbb{R}$ and, hence, by the condition, the set $f(\{x_n\}_{n=n_0}^{\infty})$ must be closed in $\mathbb{R}$. This is the contradiction.

Case (b). There exists a subsequence $(x_{n_i})_i \geq 1$ of the sequence $(x_n)_{n\geq 1}$ such that $f(x_{n_i}) = a$ for any $i \geq 1$. Since $\lim_{n\to\infty} x_n = +\infty$, we have $\lim_{i\to\infty} x_{n_i} = +\infty$. Hence the set $\{x_{n_i}\}_{i=1}^{\infty}$ is not bounded. On the other hand, $f^{-1}(a) \supset \{x_{n_i}\}_{i=1}^{\infty}$ and therefore the set $f^{-1}(a)$ will not be bounded. This is the contradiction. ■

Remark 1. Corollary 2 is a consequence of Theorem 3.

Corollary 4. *No perfect mapping from $\mathbb{R}$ to $\mathbb{R}$ has finite partial limits at $+\infty$ and $-\infty$.*

Corollary 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that at least one of the following two conditions is satisfied:*

1) $\exists M,\ L \in \mathbb{R} : \forall x > M : |f(x)| < L$;

2) $\exists M,\ L \in \mathbb{R} : \forall x < M : |f(x)| < L$.

Then the function f is not perfect. in particular, no bounded function from $\mathbb{R}$ to $\mathbb{R}$ is perfet.

Proof. If $f : \mathbb{R} \to \mathbb{R}$, satisfies for example, the first condition, then there exist at least two (possibly equal to each other) partial limits of f at $+\infty$: $\overline{\lim}_{x\to+\infty} f(x)$ and $\underline{\lim}_{x\to+\infty} f(x)$. But this contradicts Corollary 4. ■

Corollary 6. *If $f : \mathbb{R} \to \mathbb{R}$ is a perfect mapping, then for any positive real numbers M and L there exist points $x''_M < -M$ and $x'_M > M$ such that $|f(x''_M)| > L$ and $|f(x'_M)| > L$.*

Theorem 4. *Let $f \in \mathcal{K}(CL \downarrow) \cap \mathcal{K}(CL_p^{-1})$ (see Example 2 below). Assume that $a \in \mathbb{R}$ and let $(x_n)_{n\geq 1}$ be a sequence of real numbers such that $\lim_{n\to\infty} x_n = a$. Assume also that the limit $\lim_{n\to\infty} f(x_n)$ exists. Then $\lim_{n\to\infty} f(x_n) = f(a)$.*

Proof. Assume that $\lim_{n\to\infty} f(x_n) = b \in \mathbb{R}$ and $b \neq f(a)$. The following two cases are possible:

1) $\exists n_0 : \forall n \geq n_0 \quad f(x_n) = b$;

2) $\forall n\ \exists m(n) > n : f(x_{m(n)}) \neq b$.

Case 1. Clearly, $f^{-1}(b) \supset \{x_n\}_{n=n_0}^{\infty}$ and since, by assumption, $f(a) \neq b$, we have $a \notin f^{-1}(b)$. From $f^{-1}(b) \supset \{x_n\}_{n=n_0}^{\infty}$ and $\lim_{n\to\infty} x_n = a$ it follows that the point a belong to the closure of the set $f^{-1}(b)$. hence the set $f^{-1}(b)$ is not closed in $\mathbb{R}$. This is the contradiction.

Case 2. There exists a subsequence $(x_{n_i})_{i\geq 1}$ of the sequence $(x_n)_{n\geq 1}$ such that for any natural $i \geq 1$ we have $f(x_{n_i}) \neq b$. Clearly, $\lim_{i\to\infty} f(x_{n_i}) = b$. Since $\lim_{n\to\infty} x_n = a$, we have $\lim_{i\to\infty} x_{n_i} = a$. Hence the set $\{a\} \cup \{x_{n_i}\}_{i=1}^{\infty}$ is closed in $\mathbb{R}$. Consider the image $f(\{a\} \cup \{x_{n_i}\}_{i=1}^{\infty}) = \{f(a)\} \cup \{f(x_{n_i})\}_{i=1}^{\infty}$. Since $f(x_{n_i}) \neq b$ and $f(a) \neq b$ for any $i \geq 1$, we obtain $b \notin \{f(a)\} \cup \{f(x_{n_i})\}_{i=1}^{\infty}$. On the other hand, since $\lim_{i\to\infty} f(x_{n_i}) = b$, b belongs to the closure of the set $\{f(a)\} \cup \{f(x_{n_i})\}_{i=1}^{\infty}$. Hence this set is not closed. This is the contradiction. ■

We shall conclude the paper with several examples:

Example 1. Let $f_1 : \mathbb{R} \to \mathbb{R}$ be a function defined by the formula

$$f_1(x) = \begin{cases} \log_{\frac{1}{2}} |x|, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

We shall show that $f_1 \in \mathcal{K}(CL\downarrow) \cap \mathcal{K}(CM_p^{-1})$ but $f_1 \notin \mathcal{K}(C)$. Indeed, it is obvious that $f_1 \in \mathcal{K}(CM_p^{-1})$ and $f_1 \notin \mathcal{K}(C)$. Therefore it remains to show that $f_1 \in \mathcal{K}(CL\downarrow)$.

Let F be a closed subset of $\mathbb{R}$ and a be an accumulation point of the set $f(F)$. Then there exists a sequence of real numbers $(y_k)_{k\geq 1}$ such that for every $k \geq 1$ we have $y_k \in f(F)$, $y_k \neq a$, $y_k \neq 0$, and $\lim_{k\to\infty} y_k = a$. Clearly, for every $k \geq 1$ there exists a point $x_k \in F$ such that $f(x_k) = y_k$. We have $a = \lim_{k\to\infty} y_k = \lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} \log_{\frac{1}{2}} |x_k| \Rightarrow \lim_{k\to\infty} |x_k| = (\frac{1}{2})^a$. Then either there exists a subsequence $(x_{n_i})_{i\geq 1}$ of the sequence $(x_n)_{n\geq 1}$ such that $\lim_{i\to\infty} x_{n_i} = (\frac{1}{2})^a$ or there exists a subsequence $(x_{n_j})_{j\geq 1}$ of the sequence $(x_n)_{n\geq 1}$ such that $\lim_{j\to\infty} x_{n_j} = -(\frac{1}{2})^a$.

Let $(x_{n_j})_{j\geq 1}$ be a subsequence of the sequence $(x_n)_{n\geq 1}$ such that

$$\lim_{j\to\infty} x_{n_j} = -\left(\frac{1}{2}\right)^a.$$

(If such a subsequence does not exist, one may consider a subsequence $(x_{n_i})_{i\geq 1}$ of the sequence $(x_n)_{n\geq 1}$ with $\lim_{i\to\infty} x_{n_i} = (\frac{1}{2})^a$).

Since F is closed and for $x_{n_j} \in F$, $\ -(\frac{1}{2})^2 \in F$ for each $j \geq 1$. But $f(-(\frac{1}{2})^a) = a$. Therefore $a \in f(F)$. Hence the set $f(F)$ is closed in $\mathbb{R}$.

Example 2. Let the mapping $f_2 : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f_2(x) = \begin{cases} 1, & \text{for } \ x \le 0, \\ \log_{\frac{1}{2}} x, & \text{for } x > 0. \end{cases}$$

We shall show that $f_2 \in \mathcal{K}(CL\downarrow) \cap \mathcal{K}(CL_p^{-1})$ but $f_2 \notin \mathcal{K}(C)$ and $f_2 \notin \mathcal{K}(B_p^{-1})$.

It is obvious that $f_2 \notin \mathcal{K}(C)$ and $f_2 \in \mathcal{K}(CL_p^{-1})$.

Since $f_2^{-1}(1) = (-\infty; 0] \cup \{\frac{1}{2}\}$, $\quad f_2 \notin \mathcal{K}(B_p^{-1})$.

Now let us prove that $f_2 \in \mathcal{K}(CL\downarrow)$. Assume that $F \subseteq \mathbb{R}$ is a closed subset of $\mathbb{R}$ and $a \in \mathbb{R}$ is an accumulation point of the set $f(F)$. Then there exists a sequence of real numbers $(y_k)_{k\ge1}$ such that for any $k \ge 1$ we have $y_k \in f(F)$, $y_k \ne a$, $y_k \ne 1$, and $\lim_{k\to\infty} y_k = a$.

Since for any $k \ge 1$ we have $y_k \ne 1$ and $y_k \in f(F)$, there exists, for each $k \ge 1$, a positive real number x_k such that $x_k \in F$ and $f(x_k) = \log_{\frac{1}{2}} x_k = y_k$. Thus

$$\lim_{k\to\infty} y_k = a \Rightarrow \lim_{k\to\infty} \Big(\frac{1}{2}\Big)^{y_k} = \Big(\frac{1}{2}\Big)^{a} \Rightarrow \lim_{k\to\infty} x_k = \Big(\frac{1}{2}\Big)^{a}.$$

Since F is closed and for each $k \ge 1$, we have $x_k \in F$, $(\frac{1}{2})^a \in F$. But $f((\frac{1}{2})^a) = a$. Hence $a \in f(F)$. Therefore the set $f(F)$ is closed in $\mathbb{R}$.

Example 3. Let the mapping $f_3 : \mathbb{R} \to \mathbb{R}$ be determined by the formula

$$f_3(x) = \begin{cases} x, & \text{for } \ x \in (-\infty; 0] \cup [1; +\infty), \\ 0, & \text{for } \ x \in (0; 1). \end{cases}$$

We shall show that $f_3 \in \mathcal{K}(CL\downarrow) \cap \mathcal{K}(B_p^{-1})$ but $f_3 \notin \mathcal{K}(C)$ and $f_3 \notin \mathcal{K}(CL_p^{-1})$. That $f_3 \notin \mathcal{K}(C)$ and $f_3 \in \mathcal{K}(B_p^{-1})$ is obvious.

Since $f_3^{-1}(0) = [0; 1)$, we have $f_3 \notin \mathcal{K}(CL_p^{-1})$. Let us show that $f_3 \in \mathcal{K}(CL\downarrow)$. For this take any closed subset F of $\mathbb{R}$ and consider an accumulation point a of the set $f(F)$. Since $\overline{f(F)} \subset \overline{f(\mathbb{R})} = (-\infty; 0] \cup [1; +\infty)$, we have $a \in (-\infty; 0] \cup [1; +\infty)$.

If $a \in (-\infty; 0]$ (if $a \in [1; +\infty)$), then there exists a sequence of real numbers $(y_k)_{k\ge1}$ such that for every $k \ge 1$ we have $y_k \in f(F)$, $y_k \ne a$, $y_k < 0$ (respectively, $y_k > 1$) and $\lim_{k\to\infty} y_k = a$.

Since each $y_k \in f(F)$, for every $k \ge 1$ there exists $x_k \in F$ such that $y_k = f(x_k) = x_k$. Therefore $a = \lim_{k\to\infty} y_k = \lim_{k\to\infty} x_k$. Since F is closed, $a \in F$. But, clearly, $f(a) = a$.

Hence $a \in f(F)$. Therefore $f(F)$ is a closed subset of $\mathbb{R}$.

Reference

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