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BOUNDARY VALUE PROBLEMS OF ELECTROELASTICITY WITH CONCENTRATED SINGULARITIES

T. BUCHUKURI AND D. YANAKIDI

Abstract. We investigate the solutions of boundary value problems of linear electroelasticity, having growth as a power function in the neighbourhood of infinity or in the neighbourhood of an isolated singular point. The number of linearly independent solutions of this type is established for homogeneous boundary value problems.

The basic equations of the static state of an electroelastic medium are written in terms of displacement and electric potential components as follows [1, 2]:

$$c_{ijkl}\frac{\partial^2 u_k}{\partial x_j\partial x_l}+e_{kij}\frac{\partial^2\varphi}{\partial x_j\partial x_k}+F_i=0, \tag{1}$$

$$-e_{ikl}\frac{\partial^2 u_k}{\partial x_i\partial x_l}+\mathcal{E}_{ik}\frac{\partial^2\varphi}{\partial x_i\partial x_k}=0,\quad i=1,2,3, \tag{2}$$

where $u=(u_1,u_2,u_3)$ is a displacement vector, φ is an electric field potential, c_{ijkl}, e_{kij}, $\mathcal{E}_{ik}$, are constants, $F=(F_1,F_2,F_3)$ is mass force. It is assumed that the constants c_{ijkl}, e_{kij} , $\mathcal{E}_{ik}$ satisfy the conditions

$$c_{ijkl}=c_{jikl}=c_{klij},\quad e_{kij}=e_{kji},\quad \mathcal{E}_{ij}=\mathcal{E}_{ji},\quad i,j,k,l=1,2,3. \tag{3}$$

System (1), (2) can be written in the matrix form. We introduce the operator

$$A(\partial x)=\|A_{ij}(\partial x)\|_{4\times4},\quad A_{ik}(\partial x)=c_{ijkl}\frac{\partial^2}{\partial x_j\partial x_l},\quad i,k=1,2,3,$$

$$A_{i4}(\partial x)=e_{kij}\frac{\partial^2}{\partial x_k\partial x_j},\quad i=1,2,3, \tag{4}$$

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$$A_{4k}(\partial x) = -e_{ikl}\frac{\partial^2}{\partial x_i \partial x_l}, \quad k = 1,2,3, \quad A_{44}(\partial x) = \mathcal{E}_{ik}\frac{\partial^2}{\partial x_i \partial x_k}.$$

Introducing the four-component vectors $U = (U_1, U_2, U_3, U_4) = (u_1, u_2, u_3, \varphi)$ and $\chi = (F_1, F_2, F_3, 0)$, system (1), (2) is rewritten as

$$A(\partial x)U + \chi = 0. \tag{5}$$

It is easy to show that the operator $A(\partial x)$ is a second order homogeneous operator of the elliptic type.

Assume that an electroelastic medium occupies a bounded domain Ω^+ of the three-dimensional space $\mathbb{R}^3$. Let $\Omega^- = \mathbb{R}^3 \backslash \bar{\Omega}^+$, $S = \partial\Omega^+ = \partial\Omega^-$.

Assume that the surface S is partitioned into four parts: S_{11}, S_{12}, S_{13}, S_{14}, where $S_{1i} \cap S_{1j} = \varnothing$, $i \neq j$ and $\cup_{i=1}^4 S_{1i} = S$. Also assume that we have another partitioning of S into two parts: S_{21} and S_{22}; $S_{21} \cap S_{22} = \varnothing$, $S_{21} \cup S_{22} = S$.

We shall consider a boundary value problem for system (5) when the following conditions are given: displacements on the part S_{11} of the boundary S, boundary mechanical stresses on the part S_{12}, normal components of the displacement vector and tangential components of the mechanical stress vector on the part S_{13}, and normal components of the boundary stress vector and tangential components of the displacement vector on the part S_{14}. These conditions can be written in the form

$$u_i\big|_{S_{11}}(y) = f_i(y), \quad i = 1,2,3; \tag{6}$$

$$\tau_{ji}n_j\big|_{S_{12}}(y) = g_i(y), \quad i = 1,2,3; \tag{7}$$

$$u_i n_i\big|_{S_{13}}(y) = f^{(n)}(y), \quad (\tau_{ji}n_j - n_k\tau_{jk}n_j)\big|_{S_{13}}(y) = g_i^{(\tau)}(y), \quad i = 1,2,3, \tag{8}$$

$$n_i\tau_{ji}n_j\big|_{S_{14}}(y) = g^{(n)}(y), \quad (u_i - n_k u_k n_i)\big|_{S_{14}}(y) = f_i^{(\tau)}(y), \quad i = 1,2,3. \tag{9}$$

Here f_i, g_i, $f^{(n)}$, $g^{(n)}$, $f_i^{(\tau)}$, $g_i^{(\tau)}$ are the known functions.

In addititiion to the above "mechanical" boundary conditions we should also be given "electric" conditions

$$\varphi\big|_{S_{21}}(y) = \psi(y), \tag{10}$$

$$\mathcal{D}_i n_i\big|_{S_{22}}(y) = h(y). \tag{11}$$

In the above formulas τ_{ji} denotes the mechanical stress tensor, $\mathcal{D}_i$ the electric induction vector. These values are related with the unknown displacement vector U and the electric potential φ by the relations

$$\tau_{ij} = \frac{1}{2}c_{ijkl}\Big(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\Big) + e_{kij}\frac{\partial \varphi}{\partial x_k}, \tag{12}$$

$$\mathcal{D}_i = \frac{1}{2} e_{ikl} \Big(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \Big) - \mathcal{E}_{ik} \frac{\partial \varphi}{\partial x_k}. \tag{13}$$

We shall apply the term "the basic internal regular boundary value problem of electrostatics" to the following problem:

Find in the domain Ω^+ the four-component vector $U = (u, \varphi)$ of the class $C^2(\Omega^+) \cap C^1(\bar{\Omega}^+)$ which is a solution of system (5) and satisfies conditions (6)-(11). Denote this problem by $(\mathcal{E})^+$.

The external boundary value problem $(\mathcal{E})^-$ is formulated absolutely in the same manner. In that case the vector U is sought for in the domain Ω^-.

From the basic problem $(\mathcal{E})^\pm$ one can obtain, as the particular case, various problems. Denote by $(p, q)^\pm$ the problem $(\mathcal{E})^\pm$ when $S_{1p} = S$ ($S_{1i} = \varnothing$, if $i \neq p$) and $S_{2q} = S$ ($S_{2i} = \varnothing$, if $i \neq q$).

Denote by $(\mathcal{E})_0^\pm$ the problem $(\mathcal{E})^\pm$ with the homogeneous boundary conditions $f_i = 0$, $g_i = 0$, $f_i^{(\tau)} = 0$, $g_i^{(\tau)} = 0$, $f^{(n)} = 0$, $g^{(n)} = 0$, $\psi = 0$, $h = 0$, when $\chi = 0$. The notation $(p, q)_0^\pm$ has the same meaning as above.

The following uniqueness theorem is valid:

Theorem 1. *If $U = (u, \varphi)$ is a solution of the problem $(\mathcal{E})_0^\pm$, then it has the form*

$$U_i(x) = \varepsilon_{ijk} a_i x_k + b_i, \quad \varphi = \varphi_0, \quad i = 1, 2, 3,$$

where φ_0, a_i, b_i are arbitrary constants and ε_{ijk} is the Levy-Civita symbol. In that case if $S_{11} \neq \varnothing$ and is not a subset of some plane, then $a_i = 0$, $b_i = 0$, $(i = 1, 2, 3)$; if $S_{21} \neq \varnothing$, then $\varphi_0 = 0$.

The proof is based on the Green formula

$$\sum_{i,k=1}^{4} \int_{\Omega^+} U_i(x) A_{ik}(\partial x) V_k(x)\, dx = \sum_{i,k=1}^{4} \int_{S} U_i(y) T_{ik}(\partial y, n) V_k(y)\, d_y S - \\ - \int_{\Omega^+} E(U, V)(x)\, dx, \tag{14}$$

where $T_{ik}(\partial y, n)$ are the components of the boundary stress operator

$$T_{ik}(\partial y, n) = c_{ijkl} n_j(y) \frac{\partial}{\partial y_l}, \quad i, k = 1, 2, 3,$$
$$T_{i4}(\partial y, n) = e_{kij} n_j(y) \frac{\partial}{\partial y_k}, \quad i = 1, 2, 3, \tag{15}$$
$$T_{4k}(\partial y, n) = -e_{ikl} n_i(y) \frac{\partial}{\partial y_l}, \quad k = 1, 2, 3, \quad T_{44}(\partial y, n) = \mathcal{E}_{ik} n_i(y) \frac{\partial}{\partial y_k};$$

$$E(U, V) = c_{ijkl} \frac{\partial u_i}{\partial x_j} \frac{\partial v_k}{\partial x_k} + e_{kij} \frac{\partial u_i}{\partial x_j} \frac{\partial \psi}{\partial x_k} - e_{ikl} \frac{\partial \varphi}{\partial x_i} \frac{\partial v_k}{\partial x_l} + \mathcal{E}_{ik} \frac{\partial \varphi}{\partial x_i} \frac{\partial \psi}{\partial x_k},$$
$$U = (u, \varphi), \quad V = (v, \varphi). \tag{16}$$

If $U=(u,\varphi)$ is a solution of the problem $(\mathcal{E})_0^+$, then

$$\forall y\in S\ :\ \sum_{i,k=1}^{4} U_i(y)T_{ik}(\partial y,n)U_k(y)=0. \tag{17}$$

Applying (17), from the formula (14) we conclude that

$$\forall x\in\Omega^+\ :\ E(U,U)=0, \tag{18}$$

where U is a solution of the problem $(\mathcal{E})_0^+$. From (18) we obtain

$$\forall x\in\Omega^+\ :\ \frac{\partial u_i(x)}{\partial x_j}+\frac{\partial u_j(x)}{\partial x_i}=0,\quad \frac{\partial\varphi(x)}{\partial x_i}=0,\quad i,j=1,2,3. \tag{19}$$

This formula immediately implies that all the statements of Theorem 1 are valid.

Theorem 2. *If U is a solution of the problem $(\mathcal{E})_0^-$ and satisfies the conditions*

$$U_i(x)=O(|x|^{-1}),\ \ i=1,2,3,4,\quad \frac{\partial U_i(x)}{\partial x_j}=o(|x|^{-1}),\ \ j=1,2,3, \tag{20}$$

in the neighbourhood of a point at infinity, then $\forall x\in\Omega^-\ :\ U(x)=0$.

The theorem is proved by the reasoning used in proving Theorem 1 for the domain $\Omega_r\equiv\Omega^-\backslash\overline{B(0,r)}$ where $B(0,r)$ is the ball with centre at the point 0 and radius r and with passage to the limit as $r\to\infty$.

We can formulate Theorem 2 more precisely, since it turns out that conditions (20) can be considerably weakened.

The results of [3] imply, as the particular case,

Lemma 1. *Let U be a solution of the system*

$$A(\partial x)U=0 \tag{5_0}$$

of the class $C^2(\Omega^-)$ in the domain Ω^- and one of the conditions below

$$\lim_{r\to\infty} r^{-(p+4)}\int\limits_{B(0,r)\backslash B(0,r/2)}|U(y)|\,dy=0, \tag{21}$$

$$U(y)=o(|y|^{p+1}),\quad |y|\to\infty, \tag{22}$$

$$\int\limits_{\Omega^-}\frac{|U(y)|}{1+|y|^{p+4}}\,dy<+\infty \tag{23}$$

be fulfilled for some nonnegative integer number p. Then for any nonnegative integer q we have the representation

$$\forall x \in \Omega^- \ : \ U_j(x) = \sum_{|\alpha|\le p} c_j^{(\alpha)} x^\alpha + \sum_{|\beta|\le q} d_k^{(\beta)} \partial^\beta \Phi_{jk}(x) + \psi_j(x), \qquad j = 1,2,3,4, \tag{24}$$

$c_j^{(\alpha)} = const$, $d_k^{(\beta)} = const$, $\psi_j \in C^2(\Omega^-)$, and in the neighbourhood of infinity

$$\partial^\gamma \psi_j(x) = O(|x|^{-2-|\gamma|-q}). \tag{25}$$

Here $\Phi = \|\Phi_{jk}\|_{4\times4}$ is the matrix of fundamental solutions of equations (5_0).

Theorems 2 and 3 imply directly the following uniqueness theorem:

Theorem 3. *Let U be a solution of the problem $(\mathcal{E})_0^-$ and satisfy at infinity the condition*

$$U(x) = o(1). \tag{26}$$

Then $\forall x \in \Omega^- \ : \ U(x) = 0$.

Consider now the boundary value problem: Find in the domain Ω^- the solution U of the Problem $(\mathcal{E})^-$, satisfying at infinity the condition

$$U(x) = o(|x|^{p+1}). \tag{27}$$

This problem will be denoted by $(\mathcal{E}_p)^-$ and the corresponding homogeneous problem by $(\mathcal{E}_p)_0^-$.

Let $K(p)$ be the number of linearly independent polynomial solutions of system (5_0) with a degree not higher that p. Repeating the reasoning given in [4] for an equation of classical elasticity we can readily prove that

$$K(p) = 4\left[\binom{p+2}{2} + \binom{p+1}{2}\right] = 4(p+1)^2. \tag{28}$$

Now it is easy to prove the following

Lemma 2. *Let the homogeneous problem $(\mathcal{E})_0^-$ has a solution satisfying condition* (26) *(it will be trivial by virtue of Theorem* 3*). Then the homogeneous problem $(\mathcal{E}_p)_0^-$ has at most $K(p) = 4(p+1)^2$ linearly independent solutions.*

Proof. Let $U^{(1)},\dots,U^{(r)}$ $(r>4(p+1)^2)$ be solutions of the homogeneous problem $(\mathcal{E}_p)_0^-$. By virtue of Lemma 1 $U^{(i)}=P^{(i)}+V^{(i)}$ where $P^{(i)}$ is a polynomial solution of system (5_0) of a degree not higher than p and $V^{(i)}$ is a solution of system (5_0) satisfying condition (26). Then by the condition of the lemma there exist numbers c_i not all equal to zero such that $\sum_{i=1}^{r}c_iP^{(i)}=0$. Consider the vector $W\equiv\sum c_iU^{(i)}=\sum c_iV^{(i)}$. W is a linear combination of $U^{(i)}$ and hence will be a solution of the homogeneous problem $(\mathcal{E}_p)_0^-$, but at the same time W is a linear combination of solutions $V^{(i)}$, therefore satisfying condition (26), and hence, on account of Theorem 3, $W=0$. Thus solutions $U^{(i)}$ are linearly dependent. ■

Lemma 2 immediately yields

Corollary 1. *If the nonhomogeneous problem $(\mathcal{E})^-$ has the unique solution for any f_i, g_i, $f^{(n)}$, $g^{(n)}$, $f_i^{(\tau)}$, $g_i^{(\tau)}$, ψ, and h belonging to the class C^∞, then the homogeneous problem $(\mathcal{E}_p)_0^-$ has exactly $K(p)=4(p+1)^2$ linearly independent solutions, while the nonhomogeneous problem $(\mathcal{E}_p)^-$ has the solution U for arbitrary boundary data and this solution is represented as $U=U_0+U^{(p)}$, where U_0 is the solution of the problem $(\mathcal{E})^-$ satisfying condition* (26) *and $U^{(p)}$ is an arbitrary solution of the problem $(\mathcal{E}_p)_0^-$.*

The problem $(\mathcal{E})^+$ is treated with sufficient completeness in [5]. This paper also contains the proof of the existence of a generalized solution in Sobolev spaces. Using the well-known regularization theorems [6], from these results we easily obtain the existence of classical solutions for sufficiently smooth S and boundary data. In particular, we have

Lemma 3. *Let the boundary S of the domain Ω^+ and the boundary data belong to the class $C^\infty(\bar\Omega^+)$. Then:*

problems $(1.1)^+$ *and* $(4.1)^+$ *have the unique solution of the class $C^\infty(\bar\Omega^+)$;*

the problem $(3.1)^+$ *has the unique solution of the class $C^\infty(\bar\Omega^+)$ if S is not the rotation surface;*

the problem $(2.1)^+$ *has a solution of the class $C^\infty(\bar\Omega^+)$ if and only if the conditions*

$$\int\limits_S g_i(y)\,d_yS=0,\quad i=1,2,3, \tag{29}$$

$$\int\limits_S \varepsilon_{ijk}y_jg_k(y)\,d_yS=0,\quad i=1,2,3, \tag{30}$$

are fulfilled to within a term of the form $U=(u_1^{(0)},u_2^{(0)},u_3^{(0)},0)$ where

$$U_i^{(0)}(x)=\varepsilon_{ijk}x_ja_k+b_i, \tag{31}$$

a_i and b_i are arbitrary constants $(i=1,2,3)$;

problems $(1.2)^+$ *and* $(4.2)^+$*, and also problem* $(3.2)^+$ *if* S *is not the rotation surface, have a solution if and only if*

$$\int\limits_S h(y)\,d_yS = 0; \tag{32}$$

the solution is defined to within a term of the form $U = (0, 0, 0, \varphi_0)$ *where* φ_0 *is an arbitrary number;*

problem $(2.2)^+$ *has a solution if and only if conditions* (29), (30), (32) *are fulfilled; the solution is defined to within a term of the form* $U = (u, \varphi_0)$ *where* u *is written as* (31) *and* φ_0 *is an arbitrary number.*

We are interested in investigating not smooth solutions of the problems $(\mathcal{E})^+$, but such solutions that at some given points have singularities not higher than given power orders.

Let $x^{(1)}, \dots, x^{(r)}$ be points lying in the domain Ω^+, $M_r \equiv \{x^{(1)}, \dots, x^{(r)}\}$.

Consider the problem with concentrated singularities: Find the solution U of equation (5) which belongs to the class $C^2(\Omega^+ \backslash M_r) \cap C^1(\bar{\Omega}^+ \backslash M_r)$, satisfies the boundary conditions (6)-(11) and, in the neighbourhood of the point $x^{(i)}$, the condition $|U(x)| \leq \frac{c}{|x - x^{(i)}|^{p_i}}$, $i = 1, \dots, r$, where p_i are given nonnegative numbers. Denote this problem by $(\mathcal{E})^p_{cs}$.

The investigation of this problem is largely based on one proposition following from the theorem proved in a more general situation in [3].

Lemma 4. *Let* $\Omega \subset \mathbb{R}^3$, $y \in \Omega$, U *be a solution of* (5_0) *of the class* $C^2(\Omega \backslash \{y\})$ *in the domain* $\Omega \backslash \{y\}$ *and, for some* $c > 0$ *and* $p \geq 0$, $|U(x)| \leq \frac{c}{|x-y|^p}$. *Then*

$$U_j(x) = U_j^{(0)}(x) + \sum_{k=1}^{4} \sum_{|\alpha| \leq [p]-1} a_k^{(\alpha)} \partial^\alpha \Phi_{jk}(x - y), \quad j = 1, \dots, 4,$$

where $U^{(0)}$ *is the solution of system* (5_0) *of the class* $C^2(\Omega)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ *is the multiindex,* $[p]$ *is the integer part of the number* p, $a_k^{(\alpha)} = const$, $\Phi = \|\Phi_{jk}\|_{4\times4}$ *is the matrix of fundamental solutions of equation* (5).

Using this lemma, by a reasoning analogous to that from [4], we prove

Lemma 5. *Let* $\mathcal{F}_p$ *be a finite-dimensional space stretched onto the system of vectors* $\{\partial^\alpha \Phi^{(k)}(\cdot - x^{(i)});\ k = 1, 2, 3, 4;\ |\alpha| \leq [p_i] - 1\}$, *where* $\Phi^{(k)} = (\Phi_{1k}, \Phi_{2k}, \Phi_{3k}, \Phi_{4k})$. *Then* $\dim \mathcal{F}_p = 4 \sum_{i=1}^{r} [p_i]^2$.

Proof. We assume $V = (v, \varphi)$ and introduce the notation

$$\epsilon_{ij}^{(V)} = \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big), \quad E^{(V)} = -\frac{\partial \varphi}{\partial x_k},$$

$$\tau_{ij}^{(V)} = c_{ijkl}\epsilon_{kl}^{(V)} - e_{kij}E_k^{(V)}, \quad \mathcal{D}_i^{(V)} = e_{ikl}\epsilon_{kl}^{(V)} + \mathcal{E}_{ik}E_k^{(V)}.$$

Let $\{\omega^{(k)};\ k = 1, \dots, \dim \mathcal{F}_p\}$ be base spaces $\mathcal{F}$. Denote by $U^{(k)}$ a solution of system (5_0) satisfying the boundary conditions (6)-(11) for

$$f_i = \omega_i^{(k)}, \quad g_i = \tau_{ji}^{(\omega^{(k)})}n_j, \quad f^{(n)} = \omega_i^{(k)}n_i,$$
$$g_i^{(\tau)} = \tau_{ji}^{(\omega^{(k)})}n_j - n_l\tau_{jl}^{(\omega^{(k)})}n_jn_i, \quad g^{(n)} = n_i\tau_{ji}^{(\omega^{(k)})}n_j,$$
$$f_i^{(\tau)} = \tau_{ji}^{(\omega^{(k)})}n_j - n_l\tau_{jl}^{(\omega^{(k)})}n_jn_i, \quad \psi = \omega_4^{(k)}, \quad h = \mathcal{D}_i^{(\omega^{(k)})}n_i.$$

Consider the vectors $V^{(k)} = U^{(k)} - \omega^{(k)}$. Obviously, the vector $V^{(k)}$ is a solution of the homogeneous problem $(\mathcal{E}_p)_0^+$. We shall prove that the system of vectors $\{V^{(k)}, \psi^{(i)}; k = 1, \dots, \dim \mathcal{F}_p; i = 1, \dots, q\}$, where $\{\psi^{(i)}\}$ a linearly independent system of solutions of the homogeneous problem $(\mathcal{E})_0^+$, is linearly independent. Indeed, if

$$\sum_{k=1}^{\dim \mathcal{F}_p} c_kV^{(k)} + \sum_{i=1}^{q} d_i\psi^{(i)} = 0,$$

then

$$\sum_{k=1}^{\dim \mathcal{F}_p} c_k\omega^{(k)} = \sum_{k=1}^{\dim \mathcal{F}_p} c_kU^{(k)} + \sum_{i=1}^{q} d_i\psi^{(i)}.$$

By virtue of (28) $\sum c_k\omega^{(k)} = 0$ and therefore $c_k = 0$ and $ud_i = 0$. Now from Lemma 2 we obtain the proof of Theorem 4. ■

Theorem 4. *If the problem $(\mathcal{E})^+$ has a solution for any boundary data of class C^∞, then the homogeneous problem $(\mathcal{E}_0)_{cs}^p$ has exactly $4\sum_{i=1}^r [p_i]^2 + q$ linearly independent solutions, where q is the number of linearly independent solutions of the problem $(\mathcal{E}_0)^+$ and $[p_i]$ is the integer part of the number p_i.*

This theorem readily implies

Corollary 2. *The homogeneous problems $(1.1)_{cs}^p$ and $(4.1)_{cs}^p$ have exactly $4\sum_{i=1}^r [p_i]^2$ linearly independent solutions and if the boundary S is not the surface of rotation, then Problem $(3.1)_{cs}^p$, too, has the same number of linearly independent solutions.*

Similar theorems hold for the other problems as well.

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Authors' address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, Z. Rukhadze St., Tbilisi 380093
Republic of Georgia