

**TWO-WEIGHTED  $L_p$ -INEQUALITIES FOR SINGULAR  
INTEGRAL OPERATORS ON HEISENBERG GROUPS**

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ABSTRACT. Some sufficient conditions are found for a pair of weight functions, providing the validity of two-weighted inequalities for singular integrals defined on Heisenberg groups.

Estimates for singular integrals of the Calderon–Zygmund type in various spaces (including weighted spaces and the anisotropic case) have attracted a great deal of attention on the part of researchers. In this paper we will deal with singular integral operators  $T$  on the Heisenberg group  $H^n$  which have an essentially different character as compared with operators of the Calderon–Zygmund type. We have obtained the two-weighted  $L_p$ -inequality with monotone weights for singular integral operators  $T$  on  $H^n$ . Applications are given.

Let  $H^n$  be the Heisenberg group (see [1], [2]) realized as a set of points  $x = (x_0, x_1, \dots, x_{2n}) = (x_0, x') \in \mathbb{R}^{2n+1}$  with the multiplication

$$xy = \left(x_0 + y_0 + \frac{1}{2} \sum_{i=1}^n (x_i y_{n+i} - x_{n+i} y_i), \quad x' + y'\right).$$

The corresponding Lie algebra is generated by the left-invariant vector fields

$$\begin{aligned} X_0 &= \frac{\partial}{\partial x_0}, & X_i &= \frac{\partial}{\partial x_i} + \frac{1}{2} x_{n+i} \frac{\partial}{\partial x_0}, \\ X_{n+i} &= \frac{\partial}{\partial x_{n+i}} - \frac{1}{2} x_i \frac{\partial}{\partial x_0}, & i &= 1, \dots, n, \end{aligned}$$

which satisfy the commutation relation

$$\begin{aligned} [X_i, X_{n+i}] &= \frac{1}{4} X_0, \\ [X_0, X_i] &= [X_0, X_{n+i}] = [X_i, X_j] = [X_{n+i}, X_{n+j}] = [X_i, X_{n+j}] = 0, \end{aligned}$$

$$i, j = 1, \dots, n \quad i \neq j.$$

The dilation  $\delta_t : \delta_t x = (t^2 x_0, tx')$ ,  $t > 0$ , is defined on  $H^n$ . The Haar measure on this group coincides with the Lebesgue measure  $dx = dx_0 dx_1 \cdots dx_{2n}$ . The identity element in  $H^n$  is  $e = 0 \in \mathbb{R}^{2n+1}$ , while the element  $x^{-1}$  inverse to  $x$  is  $(-x)$ .

The function  $f$  defined in  $H^n$  is said to be  $H$ -homogeneous of degree  $m$ , on  $H^n$ , if  $f(\delta_t x) = t^m f(x)$ ,  $t > 0$ . We also define the norm on  $H^n$

$$|x|_H = [x_0^2 + (\sum_{i=1}^{2n} x_i^2)^2]^{1/4}$$

which is  $H$ -homogeneous of degree one. This also yields the distance function, namely, the distance

$$\begin{aligned} d(x, y) &= d(y^{-1}x, e) = |y^{-1}x|_H, \\ |y^{-1}x|_H &= [(x_0 - y_0 - \frac{1}{2} \sum_{i=1}^n (x_i y_{n+i} - x_{n+i} y_i))^2 + \\ &\quad + (\sum_{i=1}^{2n} (x_i - y_i)^2)^2]^{1/4}. \end{aligned}$$

$d$  is left-invariant in the sense that  $d(x, y)$  remains unchanged when  $x$  and  $y$  are both left-translated by some fixed vector in  $H^n$ . Furthermore,  $d$  satisfies the triangle inequality  $d(x, z) \leq d(x, y) + d(y, z)$ ,  $x, y, z \in H^n$ . For  $r > 0$  and  $x \in H^n$  let

$$B(x, r) = \{y \in H^n; |y^{-1}x|_H < r\} \quad (S(x, r) = \{y \in H^n; |y^{-1}x|_H = r\})$$

be the  $H$ -ball ( $H$ -sphere) with center  $x$  and radius  $r$ .

The number  $Q = 2n + 2$  is called the homogeneous dimension of  $H^n$ . Clearly,  $d(\delta_t x) = t^Q dx$ .

Given functions  $f(x)$  and  $g(x)$  defined in  $H^n$ , the Heisenberg convolution ( $H$ -convolution) is obtained by

$$(f * g)(x) = \int_{H^n} f(y)g(y^{-1}x)dy = \int_{H^n} f(xy^{-1})g(y)dy,$$

where  $dy$  is the Haar measure on  $H^n$ .

The kernel  $K(x)$  admitting the estimate  $|K(x)| \leq C|x|_H^{\alpha-Q}$  is summable in the neighborhood of  $e$  for  $\alpha > 0$  and in that case  $K * g$  is defined for the function  $g$  with bounded support. If however the kernel  $K(x)$  has a singularity of order  $Q$  at zero, i.e.,  $|K(x)| \sim |x|_H^{-Q}$  near  $e$ , then there arises a singular integral on  $H^n$ .

Let  $\omega(x)$  be a positive measurable function on  $H^n$ . Denote by  $L_p(H^n, \omega)$  a set of measurable functions  $f(x)$ ,  $x \in H^n$ , with the finite norm

$$\|f\|_{L_p(H^n, \omega)} = \left( \int_{H^n} |f(x)|^p \omega(x) dx \right)^{1/p}, \quad 1 \leq p < \infty.$$

We say that a locally integrable function  $\omega : H^n \rightarrow (0, \infty)$  satisfies Muckenhoupt's condition  $A_p = A_p(H^n)$  (briefly,  $\omega \in A_p$ ),  $1 < p < \infty$ , if there is a constant  $C = C(\omega, p)$  such that for any  $H$ -ball  $B \subset H^n$

$$\left( |B|^{-1} \int_B \omega(x) dx \right) \left( |B|^{-1} \int_B \omega^{1-p'}(x) dx \right) \leq C, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

where the second factor on the left is replaced by  $\text{ess sup}\{\omega^{-1}(x) : x \in B\}$  if  $p = 1$ .

Let  $K(x)$  be a singular kernel defined on  $H^n \setminus \{e\}$  and satisfying the conditions:  $K(x)$  is an  $H$ -homogeneous function of degree  $-Q$ , i.e.,  $K(\delta_t x) = t^{-Q} K(x)$  for any  $t > 0$  and  $\int_{S_H} K(x) d\sigma(x) = 0$ , where  $d\sigma(x)$  is a measure element on  $S_H = S(e, 1)$ .

Denote by  $\omega_K(\delta)$  the modulus of continuity of the kernel on  $S_H$ :

$$\omega_K(\delta) = \sup\{|K(x) - K(y)| : x, y \in S_H, |y^{-1}x|_H \leq \delta\}.$$

It is assumed that

$$\int_0^1 \omega_K(t) \frac{dt}{t} < \infty.$$

We consider the singular integral operator  $T$ :

$$Tf(x) = \int_{H^n} K(xy^{-1})f(y)dy =: \lim_{\varepsilon \rightarrow 0^+} \int_{|xy^{-1}|_H > \varepsilon} K(xy^{-1})f(y)dy.$$

As is known,  $T$  acts boundedly in  $L_p(H^n)$ ,  $1 < p < \infty$  (see [3], [4]). For singular integrals with Cauchy–Szegő kernels the weighted estimates were established in the norms of  $L_p(H^n, \omega)$  with weights  $\omega$  satisfying the condition  $A_p$  [5]. These results extend to the more general kernels considered above [4].

**Theorem 1** [4]. *Let  $1 < p < \infty$  and  $\omega \in A_p$ ; then  $T$  is bounded in  $L_p(H^n, \omega)$ .*

In the sequel we will use

**Theorem 2.** *Let  $1 \leq p \leq q < \infty$  and  $U(t), V(t)$  be positive functions on  $(0, \infty)$ .*

1) *The inequality*

$$\left( \int_0^\infty U(t) \left| \int_0^t \varphi(\tau) d\tau \right|^q dt \right)^{1/q} \leq K_1 \left( \int_0^\infty |\varphi(t)|^p v(t) dt \right)^{1/p}$$

with the constant  $K_1$  not depending on  $\varphi$  holds iff the condition

$$\sup_{t>0} \left( \int_t^\infty U(\tau) d\tau \right)^{p/q} \left( \int_0^t V(\tau)^{1-p'} d\tau \right)^{p-1} < \infty$$

is fulfilled;

2) *The inequality*

$$\left( \int_0^\infty U(t) \left| \int_t^\infty \varphi(\tau) d\tau \right|^q dt \right)^{1/q} \leq K_2 \left( \int_0^\infty |\varphi(t)|^p V(t) dt \right)^{1/p}$$

with the constant  $K_2$  not depending on  $\varphi$  holds iff the condition

$$\sup_{t>0} \left( \int_0^t U(\tau) d\tau \right)^{p/q} \left( \int_t^\infty V(\tau)^{1-p'} d\tau \right)^{p-1} < \infty$$

is fulfilled.

Note that Theorem 2 was proved by G.Talenti, G.Tomaselli, B.Muckenhoupt [7] for  $1 \leq p = q < \infty$ , and by J.S.Bradley [8], V.M.Kokilashvili [9], V.G.Maz'ya [10] for  $p < q$ .

We say that the weight pair  $(\omega, \omega_1)$  belongs to the class  $\tilde{A}_{pq}(\gamma)$ ,  $\gamma > 0$ , if either of the following conditions is fulfilled: a)  $\omega(t)$  and  $\omega_1(t)$  are increasing functions on  $(0, \infty)$  and

$$\sup_{t>0} \left( \int_t^\infty \omega(\tau) \tau^{-1-\gamma q/p'} d\tau \right)^{p/q} \left( \int_0^{t/2} \omega(\tau)^{1-p'} \tau^{\gamma-1} d\tau \right)^{p-1} < \infty;$$

b)  $\omega(t)$  and  $\omega_1(t)$  are decreasing functions on  $(0, \infty)$  and

$$\sup_{t>0} \left( \int_0^{t/2} \omega_1(\tau) \tau^{\gamma-1} d\tau \right)^{p/q} \left( \int_t^\infty \omega(\tau)^{1-p'} \tau^{-1-\gamma p'/q} d\tau \right)^{p-1} < \infty.$$

**Theorem 3.** *Let  $1 < p < \infty$  and the weight pair  $(\omega, \omega_1) \in \tilde{A}_p(Q) \equiv \tilde{A}_{pp}(Q)$ . Then for  $f \in L_p(H^n, \omega(|x|_H))$  there exists  $Tf(x)$  for almost all  $x \in H^n$  and*

$$\int_{H^n} |Tf(x)|^p \omega_1(|x|_H) dx \leq C \int_{H^n} |f(x)|^p \omega(|x|_H) dx, \quad (1)$$

where the constant  $C$  does not depend on  $f$ .

**Corollary.** *If  $\omega(t)$ ,  $t > 0$  is increasing (decreasing) and the function  $\omega(t)t^{-\beta}$  is decreasing (increasing) for some  $\beta \in (0, Q(p-1))$  ( $\beta \in (-Q, 0)$ ), then  $T$  is bounded on  $L_p(H^n, \omega(|x|_H))$ .*

*Proof of Theorem 3.* Let  $f \in L_p(H^n, \omega(|x|_H))$  and  $\omega, \omega_1$  be positive increasing functions on  $(0, \infty)$ . We will prove that  $Tf(x)$  exists for almost all  $x \in H^n$ . We take any fixed  $\tau > 0$  and represent the function  $f$  in the norm of the sum  $f_1 + f_2$ , where

$$f_1(x) = \begin{cases} f(x), & \text{if } |x|_H > \tau/2 \\ 0, & \text{if } |x|_H \leq \tau/2 \end{cases}, \quad f_2(x) = f(x) - f_1(x).$$

Let  $\omega(t)$  be a positive increasing function on  $(0, \infty)$  and  $f \in L_p(H^n, \omega(|x|_H))$ . Then  $f_1 \in L_p(H^n)$  and therefore  $Tf_1(x)$  exists for almost all  $x \in H^n$ . Now we will show that  $Tf_2$  converges absolutely for all  $x : |x|_H \geq \tau$ . Note that  $C(K) = \sup_{x \in S_H} |K(x)| < \infty$ . Hence

$$\begin{aligned} |Tf_2(x)| &\leq C(K) \int_{|y|_H \leq \tau/2} \frac{|f(y)|}{|xy^{-1}|_H^Q} dy \leq \\ &\leq \left(\frac{2}{\tau}\right)^{\frac{Q}{p}} \int_{|y|_H \leq \tau/2} \frac{|f(y)|\omega(|y|_H)^{\frac{1}{p}}}{\omega(|y|_H)^{\frac{1}{p}}} dy, \end{aligned} \quad (2)$$

since  $|xy^{-1}|_H \geq |x|_H - |y|_H \geq \tau/2$ . Thus, by the Hölder inequality we can estimate (2) as

$$|Tf_2(x)| \leq C\tau^{-Q/p} \|f\|_{L_p(H^n, \omega(|x|_H))} \left( \int_0^{\tau/2} \omega(t)^{1-p'} t^{Q-1} dt \right)^{1/p'}.$$

Therefore  $Tf_2(x)$  converges absolutely for all  $x : |x|_H \geq \tau$  and thus  $Tf(x)$  exists for almost all  $x \in H^n$ . Assume  $\bar{\omega}_1(t)$  to be an arbitrary continuous increasing function on  $(0, \infty)$  such that  $\bar{\omega}_1(t) \leq \omega_1(t)$ ,  $\bar{\omega}_1(0) = \omega_1(0+)$  and  $\bar{\omega}_1(t) = \int_0^t \varphi(\tau) d\tau + \bar{\omega}_1(0)$ ,  $t \in (0, \infty)$  (it is obvious that such  $\bar{\omega}_1(t)$  exists; for example,  $\bar{\omega}_1(t) = \int_0^t \omega_1'(\tau) d\tau + \omega_1(t)$ ).

We observe that the condition a) implies

$$\exists C_1 > 0, \quad \forall t > 0, \quad \omega_1(t) \leq C_1 \omega(t/2). \quad (3)$$

Indeed, from

$$\begin{aligned} &\exists C_2 > 0, \quad \forall t > 0, \\ &\left( \int_t^\infty \varphi(\tau) \tau^{-Q(p-1)} d\tau \right) \left( \int_0^{t/2} \omega(\tau)^{1-p'} \tau^{Q-1} d\tau \right)^{p-1} \leq C_2 \end{aligned} \quad (4)$$

we obtain (3), since

$$\int_t^\infty \omega_1(\tau)\tau^{-1-Q(p-1)}d\tau \geq C\omega_1(t)t^{-Q(p-1)},$$

$$\left(\int_0^{t/2} \omega(\tau)^{1-p'}\tau^{Q-1}d\tau\right)^{p-1} \leq C\omega(t/2)^{-1}t^{Q(p-1)}$$

and, besides,

$$\frac{1}{Q(p-1)}\int_t^\infty \varphi(\tau)\tau^{-Q(p-1)}d\tau = \int_t^\infty \varphi(\tau)d\tau \int_\tau^\infty \lambda^{-1-Q(p-1)}d\lambda =$$

$$= \int_t^\infty \lambda^{-1-Q(p-1)}d\lambda \int_t^\lambda \varphi(\tau)d\tau \leq \int_t^\infty \omega_1(\tau)\tau^{-1-Q(p-1)}d\tau.$$

We have

$$\|Tf\|_{L_p, \bar{\omega}_1(H^n)} \leq \left(\int_{H^n} |Tf(x)|^p dx \int_0^{|x|_H} \varphi(t)dt\right)^{1/p} +$$

$$+ \left(\bar{\omega}_1(0) \int_{H^n} |Tf(x)|^p dx\right)^{1/p} = A_1 + A_2.$$

If  $\omega(0+) > 0$ , then  $L_p(H^n, \omega(|x|_H)) \subset L_p(H^n)$ , and if  $\omega(0+) = 0$ , then  $\bar{\omega}(t) \leq \omega_1(t) \leq C\omega(t/2)$  implies  $\bar{\omega}_1(0) = 0$ . Therefore in the case  $\omega(0+) = 0$  we have  $A_2 = 0$ .

If  $\omega(0) > 0$ , then  $f \in L_p(\mathbb{R}^n)$  and we have

$$A_2 \leq C\left(\bar{\omega}_1(0) \int_{H^n} |f(x)|^p dx\right)^{1/p} \leq C\left(\int_{H^n} |f(x)|^p \omega_1(|x|_H) dx\right)^{1/p} \leq$$

$$\leq C\|f\|_{L_p(H^n, \omega(|x|_H))}.$$

Now we can write

$$A_1 \leq \left(\int_0^\infty \varphi(t)dt \int_{|x|_H > t} |Tf(x)|^p dx\right)^{1/p} \leq A_{11} + A_{12}.$$

where

$$A_{11}^p = \int_0^\infty \varphi(t)dt \int_{|x|_H > t} \left| \int_{|y|_H > t/2} K(x, y^{-1})f(y)dy \right|^p dx,$$

$$A_{12}^p = \int_0^\infty \varphi(t)dt \int_{|x|_H > t} \left| \int_{|y|_H < t/2} K(x, y^{-1})f(y)dy \right|^p dx.$$

The relation

$$\int_{|y|_H > t/2} |f(y)|^p dy \leq \frac{1}{\omega(t/2)} \int_{|y|_H > t/2} |f(y)|^p \omega(|y|_H) dy$$

implies  $f \in L_p(\{y \in H^n : |y|_H > t\})$  for any  $t > 0$ .

Hence, on account of (3), we have

$$\begin{aligned} A_{11} &\leq C \left( \int_0^\infty \varphi(t) dt \int_{|x|_H > t/2} |f(x)|^p dx \right)^{1/p} = \\ &= C \left( \int_{H^n} |f(x)|^p dx \int_0^{2|x|_H} \varphi(t) dt \right)^{1/p} \leq \\ &\leq C \left( \int_{H^n} |f(x)|^p \omega_1(2|x|_H) dx \right)^{1/p} \leq C \|f\|_{L_{p,\omega}(|x|_H)}(H^n). \end{aligned}$$

Obviously, if  $|x|_H > t$ ,  $|y|_H < t/2$ , then  $\frac{1}{2}|x|_H \leq |y^{-1}x|_H \leq \frac{3}{2}|x|_H$ . Therefore

$$\begin{aligned} &\int_{|x|_H > t} \left| \int_{|y|_H < t/2} K(xy^{-1})f(y)dy \right|^p dx \leq \\ &\leq C(K) \int_{|x|_H > t} \left( \int_{|y|_H < t/2} |xy^{-1}|_H^{-Q} |f(y)| dy \right)^p dx \leq \\ &\leq 2^{Qp} C(K) \int_{|x|_H > t} |x|_H^{-Qp} dx \left( \int_{|y|_H < t/2} |f(y)| dy \right)^p. \end{aligned}$$

Taking the  $H$ -polar coordinates  $x = \delta_\varrho \bar{x}$ ,  $\varrho = |x|_H$ ,  $\bar{x} \in S_H$  we can write

$$\int_{|x|_H > t} |x|_H^{-Qp} dx = \int_{S_H} d\sigma(\bar{x}) \int_0^\infty \varrho^{Q-1-Qp} d\varrho = Ct^{Q-Qp}.$$

For  $\alpha > Q(1 + \frac{1}{p'})$ , by virtue of the Hölder inequality, we have

$$\begin{aligned} \int_{|y|_H < t/2} |f(y)| dy &= \alpha \int_{S_H} d\sigma(\bar{y}) \int_0^{t/2} \varrho^{Q-\alpha-1} |f(\delta_\varrho \bar{y})| d\varrho \int_0^\varrho s^{\alpha-1} ds = \\ &= \alpha \int_0^{t/2} s^{\alpha-1} ds \int_{s < |y|_H < t/2} |f(y)| |y|_H^{-\alpha} dy \leq \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^{t/2} s^{\alpha-1} ds \left( \int_{s < |y|_H < t/2} |f(y)|^p |y|_H^{-Qp} dy \right)^{1/p} \times \\
&\quad \times \left( \int_{s < |y|_H < t/2} |y|_H^{(Q-\alpha)p'} dy \right)^{1/p'} \leq \\
&\leq C \int_0^{t/2} s^{Q+\frac{Q}{p'}} \left( \int_{s < |y|_H < t/2} |f(y)|^p |y|_H^{-Qp} dy \right)^{1/p} ds.
\end{aligned}$$

Consequently

$$\begin{aligned}
A_{12} &\leq C \left\{ \int_0^\infty \varphi(2t) t^{-Q(p-1)} \times \right. \\
&\quad \left. \times \left[ \int_0^t s^{Q(1+\frac{1}{p'})} \left( \int_{|y|_H \geq s} |f(y)|^p |y|_H^{-Qp} dy \right)^{1/p} ds \right]^p dt \right\}^{1/p}.
\end{aligned}$$

By (4) and Theorem 2

$$\begin{aligned}
A_{12} &\leq C \left[ \int_0^\infty s^{Qp(1+\frac{1}{p'})} \left( \int_{|y|_H > s} |f(y)|^p \times \right. \right. \\
&\quad \left. \left. \times |y|_H^{-Qp} dy \right) \omega(s) s^{-(Q-1)(p-1)} ds \right]^{1/p} = \\
&= C \left( \int_0^\infty s^{-1+Qp} \omega(s) ds \int_{|y|_H > s} |f(y)|^p |y|_H^{-Qp} dy \right)^{1/p} = \\
&= C \left( \int_{H^n} |f(y)|^p |y|_H^{-Qp} \int_0^{|y|_H} \omega(s) s^{-1+Qp} ds \right)^{1/p} \leq \\
&\leq C \left( \int_{H^n} |f(y)|^p \omega(|y|_H) dy \right)^{1/p}.
\end{aligned}$$

Hence we obtain (1) for  $\omega_1(t) = \bar{\omega}_1(t)$ . Now, by the Fatou theorem, the inequality (1) is fulfilled.  $\square$

Theorem 3 was earlier announced in [11].

A similar reasoning can be used to prove the analogue of Theorem 3 for the operator  $T_\alpha : f \rightarrow T_\alpha f$  where

$$T_\alpha f(x) = \int_{H^n} |xy^{-1}|_H^{\alpha-Q} f(y) dy, \quad 0 < \alpha < Q.$$

Namely, we have

**Theorem 4.** Let  $0 < \alpha < Q$ ,  $1 < p < \frac{Q}{\alpha}$ ,  $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$  and the weights  $(\omega, \omega_1)$  be monotone positive functions on  $(0, \infty)$ . Then the inequality

$$\left( \int_{H^n} |T_\alpha f(x)|^q \omega_1(|x|_H) dx \right)^{1/q} \leq C \left( \int_{H^n} |f(x)|^p \omega(|x|_H) dx \right)^{1/p}$$

holds if and only if  $(\omega, \omega_1) \in \tilde{A}_{p,q}(Q)$ .

*Remark.* In the case of a homogeneous group the analogue of Theorem 4 is also valid (see [12]).

For monotone weights one can find the weighted  $L_p$ -estimates for a Calderon-Zygmund operator in [13] and [14], and for the anisotropic case in [15].

As known [16], if  $f \in C_0^\infty(H^n)$ , then the function

$$g(x) = C_n \int_{H^n} |xy^1|_H^{-2n} f(y) dy$$

is a solution of the equation  $L_0 g = f$ , where  $L_0 = -\sum_{i=1}^{2n} X_j^2$ . In particular, our results lead to

**Theorem 5.** Let  $1 < p < \infty$ ,  $(\omega, \omega_1) \in \tilde{A}(Q)$ ,  $f \in L_p(H^n, \omega(|x|_H))$ , and  $L_0(g) = f$ . Then

$$\begin{aligned} \|X_0 g\|_{L_p(H^n, \omega_1(|x|_H))} &\leq c \|f\|_{L_p(H^n, \omega(|x|_H))}, \\ \|X_i X_j g\|_{L_p(H^n, \omega_1(|x|_H))} &\leq C \|f\|_{L_p(H^n, \omega(|x|_H))}, \\ &i, j = 1, 2, \dots, 2n. \end{aligned}$$

**Theorem 6.** Let  $1 < p < q < \infty$ ,  $\frac{1}{p} - \frac{1}{q} = \frac{1}{Q}$ ,  $(\omega, \omega_1) \in \tilde{A}_{pq}(Q)$ ,  $f \in L_p(H^n, \omega(|x|_H))$ , and  $L_0 g = f$ . Then

$$\|X_i g\|_{L_q(H^n, \omega_1(|x|_H))} \leq C \|f\|_{L_p(H^n, \omega(|x|_H))}, \quad i = 1, 2, \dots, 2n.$$

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