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LIMIT BEHAVIOR OF SOLUTIONS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. A classification of classes of equivalent linear differential equations with respect to ω-limit sets of their canonical representatives is introduced. Some consequences of this classification to the oscillatory behavior of solution spaces are presented.

1. INTRODUCTION

Many authors dealt with the behavior of solutions of differential equations to the (mostly right) end of the interval of definition – the limit behavior (often considered for the independent variable tending to ∞). Asymptotic, oscillatory and other qualitative properties of solutions of linear differential equations were intensively studied e.g. by N.V.Azbelev and Z.B.Caljuk [1], J.H.Barrett [2], G.D.Birkhoff [3], O.Borůvka [4], W.A.Coppel [5], M.Greguš [6], G.B.Gustafson [7], M.Hanan [8], I.T.Kiguradze and T.A.Chanturia [9], G.Sansone [14], C.A.Swanson [15], and many others.

The aim of this paper is to introduce a certain classification of the limit behavior of solutions of linear differential equations, a classification which is invariant with respect to the most general pointwise transformations of these equations. This classification has natural consequences to the oscillatory and asymptotic behavior of solutions. The main tool is based on the geometric approach introduced in [11] which enables us to convert some "non-compact" problems into "compact" ones. This method was applied for solving some open problems [12], and it has recently been explained systematically in detail together with other methods and results concerning linear differential equations in the monograph [13].

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2. Background and preliminary results

Let $C^n(I)$ denote the set of all functions defined on an open interval $I \subseteq \mathbb{R}$ with continuous derivatives up to and including the order n. For $n \geq 2$, let $\mathcal{L}_n$ stand for all ordinary linear differential equations of the form

$$P_n \equiv y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0 \quad \text{on} \quad I,$$

I being an open interval of the reals, p_i are real continuous functions defined on I for $i = 0, 1, \ldots, n-1$, i.e. $p_i \in C^0(I)$, $p_i : I \to \mathbb{R}$.

Consider $Q_n \in \mathcal{L}_n$,

$$Q_n \equiv z^{(n)} + q_{n-1}(t)z^{(n-1)} + \cdots + q_0(t)z = 0 \quad \text{on} \quad J.$$

We say that the equation P_n is globally equivalent to the equation Q_n if there exist two functions,

$$f \in C^n(J), \quad f(t) \neq 0 \quad \text{for each} \quad t \in J, \quad \text{and}$$
$$h \in C^n(J), \quad h'(t) \neq 0 \quad \text{for each} \quad t \in J, \quad \text{and} \quad h(J) = I,$$

such that whenever $y : I \to \mathbb{R}$ is a solution of P_n then

$$z : J \to \mathbb{R}, \quad z(t) := f(t) \cdot y(h(t)), \quad t \in J, \tag{1}$$

is a solution of Q_n.

Let $\mathbf{y}(x) = (y_1(x), \ldots, y_n(x))^T$ denote an n-tuple of linearly independent solutions of the equation P_n considered as a column vector function or as a curve in n-dimensional euclidean space $\mathbb{E}_n$ with the independent variable x as the parameter and $y_1(x), \ldots, y_n(x)$ as its coordinate functions; M^T denotes the transpose of the matrix M.

If $\mathbf{z}(t) = (z_1(t), \ldots, z_n(t))^T$ denotes an n-tuple of linearly independent solutions of the equation Q_n, then the global transformation (1) can be equivalently written as

$$\mathbf{z}(t) = f(t) \cdot \mathbf{y}(h(x)) \tag{1'}$$

or, for an arbitrary regular constant $n \times n$ matrix A,

$$\mathbf{z}(t) = Af(t) \cdot \mathbf{y}(h(x)) \tag{1''}$$

expressing only that another n-tuple of linearly independent solutions of the same equation Q_n is taken.

Denote the n-tuple $\mathbf{v} = (v_1, \ldots, v_n)^T$,

$$\mathbf{v}(x) := \mathbf{y}(x)/\|\mathbf{y}(x)\|,$$

where $\|\mathbf{y}(x)\| := (y_1^2(x) + \cdots + y_n^2(x))^{1/2}$ is the euclidean norm of $\mathbf{y}$ in $\mathbb{E}_n$. It was shown (see [11] or [13]) that $\mathbf{v} \in C^n(I)$, $\mathbf{v} : I \to \mathbb{E}^n$, and the Wronski determinant of $\mathbf{v}$ is different from zero on I. Of course, $\|\mathbf{v}(x)\| = 1$, i.e.

$\mathbf{v} \in \mathbb{S}_{n-1}$, where $\mathbb{S}_{n-1}$ is the unit sphere in $\mathbb{E}_n$. Denote by T_n the differential equation from $\mathcal{L}_n$ which has this $\mathbf{v}$ as its n-tuple of linearly independent solutions. Evidently T_n is globally equivalent to P_n. Moreover (see again [11] or [13]), if

$$\mathbf{u}(s) := \mathbf{v}(g(s)),$$

where the function g satisfies

$$g(s) : J \to I \subseteq \mathbb{R}, \quad g(J) = I, \quad |(g^{-1}(x))'| = \|\mathbf{v}'(x)\|$$

for the inverse g^{-1} to g, and hence $g \in C^n(J)$, $g'(s) \neq 0$ on J, we have $\|\mathbf{u}'(s)\| = 1$, i.e. this $\mathbf{u}$ is the length reparametrization of the curve $\mathbf{v}$. Of course, $\|\mathbf{u}(s)\| = \|\mathbf{v}(g(s))\| = 1$. If R_n denotes the differential equation admitting $\mathbf{u}$ as its n-tuple of linearly independent solutions on $J \subseteq \mathbb{R}$, then the above considered equation P_n is globally equivalent both to equation T_n and to R_n; equation R_n is also called the canonical equation of the whole class of equations from $\mathcal{L}_n$ globally equivalent to P_n. Canonical equations are characterized by admitting n-tuples of linearly independent solutions $\mathbf{u}$ satisfying

$$\|\mathbf{u}(s)\| = 1, \qquad \|\mathbf{u}'(s)\| = 1;$$

for more details see [13].

The following result describes the connection between the behavior of curves $\mathbf{y}$, $\mathbf{v}$ and $\mathbf{u}$ and the zeros of solutions of the corresponding equations P_n, T_n and R_n, see [11] or [13].

Proposition 1. *Let P_n, T_n and R_n be equations from $\mathcal{L}_n$, and let $\mathbf{y}$, $\mathbf{v}$ and $\mathbf{u}$ denote their n-tuples of linearly independent solutions defined as above. For an arbitrary nonzero constant vector $\mathbf{c} = (c_1, \dots, c_n)^T$, the solution $\mathbf{c}^T\mathbf{y}(x)$ of the equation P_n has the zero at x_0 if and only if the hyperplane*

$$H(\mathbf{c}) \equiv c_1\xi_1 + \cdots + c_n\xi_n = 0 \quad \textit{in} \quad \mathbb{E}_n$$

intersects the curve $\mathbf{y}$ at the point of the parameter x_0.

Moreover, the solution $\mathbf{c}^T\mathbf{v}(x)$ of the equation T_n has the zero at x_0 if and only if the great circle $H(\mathbf{c}) \cap \mathbb{S}_{n-1}$ intersects the curve $\mathbf{v}$ at the point of the parameter x_0. And the solution $\mathbf{c}^T\mathbf{u}(s)$ of the equation R_n has the zero at $s_0 = g^{-1}(x_0)$ if and only if the great circle $H(\mathbf{c}) \cap \mathbb{S}_{n-1}$ intersects the curve $\mathbf{u}$ at the point of the parameter s_0.

In each of the above cases, the order of contact corresponds to the multiplicity of zero.

3. Classification of ω-limit behavior

We have seen that a class of globally equivalent equations from $\mathcal{L}_n$ is characterized by curve $\mathbf{v} \in \mathbb{S}_{n-1}$, having coordinates in C^n with the nonvanishing wronskian. Since the sphere $\mathbb{S}_{n-1}$ is compact, the ω-limit set of $\mathbf{v}$, denoted by $\omega(\mathbf{v})$, is nonempty, closed and connected, see e.g. [10]. Exactly one from the following cases occurs:

a_1**:** $\omega(\mathbf{v})$ is a point $\mathbf{p} \in \mathbb{S}_{n-1}$, i.e. a connected subset of the intersection of a 1-dimensional subspace with $\mathbb{S}_{n-1}$;

a_2**:** $\omega(\mathbf{v}) \subseteq (\mathbb{S}_{n-1} \cap \mathbb{E}_2)$, where $\mathbb{E}_2$ is a 2-dimensional subspace of $\mathbb{E}_n$, and the case a_1 is not valid;

...

a_i**:** $\omega(\mathbf{v}) \subseteq (\mathbb{S}_{n-1} \cap \mathbb{E}_i)$, where $\mathbb{E}_i$ is an i-dimensional subspace of $\mathbb{E}_n$, and neither from the above cases is valid;

...

a_{n-1}**:** $\omega(\mathbf{v}) \subseteq (\mathbb{S}_{n-1} \cap \mathbb{E}_{n-1})$, and neither from the above cases holds;

a_n**:** neither from the above cases is valid.

We will consider also the following subcases of the cases a_i for $i = 1, \dots, n$:

a_i^0**:** if the case a_i is valid and $\omega(\mathbf{v}) \subseteq \mathbb{S}_{n-1}^0$, where $\mathbb{S}_{n-1}^0$ is an *open hemisphere* of $\mathbb{S}_{n-1}$.

Evidently the case a_1 coincides with a_1^0.

4. Main result

Theorem. *Consider an equation P_n from $\mathcal{L}_n$; let T_n and R_n be equations defined as in §2, and $\mathbf{y}$, $\mathbf{v}$ and $\mathbf{u}$ denote their n-tuples of linearly independent solutions. Let $\omega(\mathbf{v})$ and $\omega(\mathbf{u})$ be the ω-limit sets of $\mathbf{v}$ and $\mathbf{u}$, respectively. If, for some i, the case a_i is valid for $\mathbf{v}$ (or for $\mathbf{u}$), then the same case holds for every equation globally equivalent to P_n. Moreover, if the subcase a_i^0 is valid for some i, then the same subcase is true for every equation globally equivalent to P_n.*

Proof. Suppose first that the case a_i is valid for $P_n \in \mathcal{L}_n$. First it means that $\omega(\mathbf{v}) \subseteq \mathbb{S}_{n-1} \cap \mathbb{E}_i$ for $\mathbf{v} := \mathbf{y}/\|\mathbf{y}\|$. Then for each $\mathbf{z}$,

$$\mathbf{z}(t) := Af(t) \cdot \mathbf{y}(h(t)),$$

obtained by a global transformation $(1'')$, we have

$$\omega(\mathbf{z}/\|\mathbf{z}\|) = \omega(Af \cdot \mathbf{y}(h)/\|Af \cdot \mathbf{y}(h)\|) \subseteq S_{n-1} \cap (A\mathbb{E}_i),$$

where $A\mathbb{E}_i$ is again an i-dimensional subspace of $\mathbb{E}_n$. Moreover, if $\omega(\mathbf{z}/|\mathbf{z}|) \subseteq S_{n-1} \cap (A\mathbb{E}_j)$ for some $j < i$, we would get the contradiction to our supposition. Hence the case a_i is valid for every equation from $\mathcal{L}_n$ globally equivalent to P_n.

Now suppose that the subcase a_i^0 is valid for P_n, that means that $\omega(\mathbf{v}) \subseteq \mathbb{S}_{n-1}^0 \cap \mathbb{E}_i$ for $\mathbf{v} := \mathbf{y}/\|\mathbf{y}\|$. Then for each $\mathbf{z}$, $\mathbf{z}(t) := Af(t) \cdot \mathbf{y}(h(t))$, we have $\omega(\mathbf{z}/\|\mathbf{z}\|) \subseteq \hat{S}_{n-1}^0 \cap (A\mathbb{E}_i)$, where $\hat{S}_{n-1}^0 = \{\mathbf{s};\ \mathbf{s} = A\mathbf{r}/\|A\mathbf{r}\|,\ \mathbf{r} \in S_{n-1}^0\}$ is again an open hemisphere in $\mathbb{E}_n$ and $A\mathbb{E}_i$ is an i-dimensional subspace of $\mathbb{E}_n$. Hence the case a_1^0 is valid for every equation from $\mathcal{L}_n$ globally equivalent to P_n. ■

Remark 1. This theorem also shows that we may speak about the above cases and subcases with respect to a given equation and not only with respect to a particular n-tuple of its solutions, because, due to an arbitrary matrix A in $(1'')$, these cases and subcases are characterized by the properties which are invariant with respect to a choice of an n-tuple of linearly independent solutions of the considered equation.

5. Consequences

Oscillation or nonoscillation will be always considered with respect to the right end of the definition interval of a considered equation.

Corollary 1. (*Oscillatory behavior of solutions). If the case a_1 is valid for $P_n \in \mathcal{L}_n$, then there do not exist n linearly independent oscillatory solutions (for $t \to b_-$) of P_n. Moreover, there exist n linearly independent nonoscillatory solutions of P_n as* $(t \to b_-)$.

Proof. Let P_n be a given equation, and $\mathbf{y}$ denote an n-tuple of its linearly independent solutions. Suppose that there exist n linearly independent oscillatory solutions of P_n. Then, due to Proposition 1, there are n great circles on $\mathbb{S}_{n-1}$, not containing a common point, each of them being intersected by $\mathbf{v} = \mathbf{y}/\|\mathbf{y}\|$, or equivalently, by $\mathbf{u}$ (see notation in §2) at points with infinitely many parameters to the right end of the interval of definition. Hence on each of these great circles there is at least one point belonging to $\omega(\mathbf{v})$ $(\omega(\mathbf{u}))$. Under our assumption, the case a_1 is valid for P_n, i.e. $\omega(\mathbf{v})$ is a single point, say $\mathbf{p}$ on $\mathbb{S}_{n-1}$. Thus this point must be common to n considered circles, which is a contradiction to the linear independence of the solutions. Hence there do not exist n linearly independent oscillatory solutions of P_n.

Now choose n independent vectors $\mathbf{c}_1, \dots, \mathbf{c}_n$ in $\mathbb{E}_n$ such that the hyperplanes $H(\mathbf{c}_i)$, $i = 1, \dots, n$ do not go through the point $\mathbf{p}$. Then each solution $\mathbf{c}_i^T \cdot \mathbf{y}(x)$ is nonoscillatory. In fact, if $\mathbf{c}_i^T \cdot \mathbf{y}(x)$ were oscillatory, then $\mathbf{y}/\|\mathbf{y}\| \cap H(\mathbf{c}_i)$ would be an infinite sequence on the great circle $\mathbb{S}_{n-1} \cap H(\mathbf{c}_i)$ that should have an accumulation point in $\omega(\mathbf{y}/\|\mathbf{y}\|) = \mathbf{p}$, contrary to our choice of the hyperplanes. ■

Corollary 2. (*Asymptotic behavior of solutions*). *If the case a_1 is valid for equation P_n from $\mathcal{L}_n$, then P_n admits an n-tuple $\mathbf{y}^* = (y_1^*, \dots, y_n^*)^T$ of linearly independent solutions such that*

$$\lim_{x \to b_-} \frac{y_1^*}{\sqrt{(y_1^*)^2 + \cdots + (y_n^*)^2}} = 1$$

and

$$\lim_{x \to b_-} \frac{y_i^*}{\sqrt{(y_1^*)^2 + \cdots + (y_n^*)^2}} = 0 \quad \textit{for} \quad i = 2, \dots, n.$$

Proof. In the case a_1 we have $\lim_{x \to b_-} \mathbf{y}(x)/\|\mathbf{y}(x)\| = \mathbf{p}$, $\mathbf{p}$ being a point on $\mathbb{S}_{n-1}$. Choose an n-tuple of orthonormal vectors $\mathbf{c}_1, \dots, \mathbf{c}_n$, where $\mathbf{c}_1 := \mathbf{p}$, otherwise arbitrary. Denote by C the orthogonal matrix $(\mathbf{c}_1, \dots, \mathbf{c}_n)$. Define $y_i^* := \mathbf{c}_i^T \cdot \mathbf{y}$, i.e. $\mathbf{y}^* = C^T \cdot \mathbf{y}$. Then

$$\lim_{x \to b_-} y_1^*/\|\mathbf{y}^*\| = \lim_{x \to b_-} \frac{\mathbf{c}_1^T \mathbf{y}}{\|\mathbf{y}^T C C^T \mathbf{y}\|} = \mathbf{c}_1^T \cdot \lim_{x \to b_-} \mathbf{y}/\|\mathbf{y}\| = \\ \mathbf{c}_1^T \cdot \mathbf{p} = \mathbf{c}_1^T \cdot \mathbf{c}_1 = 1$$

and for $i = 2, \dots, n$,

$$\lim_{x \to b_-} y_i^*/\|\mathbf{y}^*\| = \lim_{x \to b_-} \frac{\mathbf{c}_i^T \mathbf{y}}{|\mathbf{y}^T C C^T \mathbf{y}|} = \mathbf{c}_i^T \cdot \lim_{x \to b_-} \mathbf{y}/\|\mathbf{y}\| = \\ \mathbf{c}_i^T \cdot \mathbf{p} = \mathbf{c}_i^T \cdot \mathbf{c}_1 = 0. \quad \blacksquare$$

Corollary 3. *If the second order equation*

$$y'' + p_1(x)y' + p_0(x)y = 0 \quad on \quad I = (a, b), \quad -\infty \le a < b \le \infty \tag{2}$$

is nonoscillatory (for $x \to b_-$), then the case a_1 is valid for (2). *If the equation* (2) *is oscillatory (for $x \to b_-$), then the case a_2 holds for* (2). *The subcase a_2^0 cannot occur.*

Proof. For two linearly independent solutions y_1, y_2 of equation (2), $\mathbf{y} = (y_1, y_2)^T$, the curve $\mathbf{v} = \mathbf{y}/\|\mathbf{y}\|$ is an arc on the unit circle $\mathbb{S}_1$ in the plane $\mathbb{E}_2$. Due to Proposition 1, if equation (2) is oscillatory for $x \to b_-$, then this arc $\mathbf{v}$ infinitely many times encircles the origin (without turning points, see [13]), and hence $\omega(\mathbf{v})$ is exactly $\mathbb{S}_1$. If equation (2) is nonoscillatory for $x \to b_-$, then the arc $\mathbf{v}$ ends by approaching a point on $\mathbb{S}_1$, exactly its ω-limit set, and the case a_1 holds for (2). $\blacksquare$

Corollary 4. *If the case a_j is valid for an equation P_n for some $j > 1$, then there exist n linearly independent oscillatory solutions of P_n.*

Proof. In the case a_j for some $j > 1$, the set $\omega(\mathbf{v})$ contains two different points on $\mathbb{S}_{n-1}$, say $\mathbf{p}_1$ and $\mathbf{p}_2$. Evidently, there exist n hyperplanes $H(\mathbf{c}_i)$, $i = 1, \dots, n$, in $\mathbb{E}_n$ with linearly independent vectors $\mathbf{c}_1, \dots, \mathbf{c}_n$, each of them separating points $\mathbf{p}_1$ and $\mathbf{p}_2$ into opposite *open* halfspaces of $\mathbb{E}_n$, i.e. $\mathbf{c}_i^T\mathbf{p}_1 > 0$ and $\mathbf{c}_i^T\mathbf{p}_2 < 0$ for each $i = 1, \dots, n$. Hence, due to Proposition 1, each solution $\mathbf{c}_i^T\mathbf{y}(x)$ oscillates for $x \to b_-$, because the curve $\mathbf{v}$ intersects infinitely many times the hyperplane $H(\mathbf{c}_i)$ as $x \to b_-$. ■

Remark 1. As an immediate consequence of this corollary we may state:

If equation L_n does not admit n linearly independent oscillatory solutions, then the case a_1 is valid for it. In particular, if each solution of equation P_n is nonoscillatory, then the case a_1 takes place for L_n.

Corollary 5. *If, for some $i = 1, \dots, n$, the case a_i^0 is valid for equation P_n, then there exist n linearly independent nonoscillatory solutions of P_n.*

Proof. Let $\mathbf{y}$ denote an n-tuple of linearly independent solutions of P_n. Under our assumption, $\omega(\mathbf{y}/\|\mathbf{y}\|)$ lies inside an open hemisphere of $\mathbb{S}_{n-1}$ determined by a hyperplane $H(\mathbf{p})$. Evidently $\mathbf{p}^T\mathbf{y}(x)$ is a nonoscillatory solution. Moreover, $\omega(\mathbf{y}/\|\mathbf{y}\|)$ is closed, and hence there exists an neighbourhood N of the point $\mathbf{p} \in \mathbb{S}_{n-1}$ such that $H(\mathbf{q}) \cap \omega(\mathbf{y}/\|\mathbf{y}\|) = \varnothing$ for each $\mathbf{q} \in N$. If we take n linearly independent vectors (points) $\mathbf{q}_1, \dots, \mathbf{q}_n$ from N, then

$$y_i := \mathbf{q}_i^T\mathbf{y}, \quad i = 1, \dots, n,$$

are required nonoscillatory solutions. In fact, if one of these solutions were oscillatory, then, again due to Proposition 1, the corresponding hyperplane would intersect the curve $\mathbf{y}$ (or equivalently $\mathbf{y}/\|\mathbf{y}\|$) infinitely many times. Hence this hyperplane would contain at least one point in $\omega(\mathbf{y}/\|\mathbf{y}\|)$, contrary to our choice of the above hyperplanes. ■

Remark 2. Comparing Corollaries 4 and 5 we see that in the case a_i^0 with $i > 1$ for L_n, this equation admits both an n-tuple of oscillatory solutions and, at the same time, another n-tuple of nonoscillatory solutions.

Remark 3. Also other (e.g. topological) properties of $\omega(\mathbf{v})$ that are invariant with respect to the centroaffine transformations can be considered for introducing other, more detailed classifications of the classes of equivalent linear differential equations from $\mathcal{L}_n$.

6. Examples

1. The differential equation

$$y^{(n)} = 0 \quad \text{on} \quad (0, \infty)$$

has n linearly independent solutions: $x^{n-1}, x^{n-2}, \dots, 1$. For this equation the case a_1 holds, no solution is oscillatory and

$$\lim_{x\to\infty} \frac{x^{n-1}}{\sqrt{\sum_{j=0}^{n-1} x^{2j}}} = 1,$$

$$\lim_{x\to\infty} \frac{x^{n-2}}{\sqrt{\sum_{j=0}^{n-1} x^{2j}}} = 0, \dots, \lim_{x\to\infty} \frac{1}{\sqrt{\sum_{j=0}^{n-1} x^{2j}}} = 0,$$

in accordance with Corollary 2 and Remark 1.

2. The equation

$$y''' + 2y'' + 2y' = 0 \quad \text{on} \quad (0, \infty)$$

admits the solutions: 1, $e^{-x}\sin x$, $e^{-x}\cos x$. For this equation the case a_1 is valid. There are two linearly independent oscillatory solutions as $x \to \infty$, there are no three linearly independent oscillatory solutions. This equation admits three linearly independent nonoscillatory solutions, and

$$\lim_{x\to\infty} \frac{1}{\sqrt{1+e^{-2x}}} = 1, \quad \lim_{x\to\infty} \frac{e^{-x}\sin x}{\sqrt{1+e^{-2x}}} = 0, \quad \lim_{x\to\infty} \frac{e^{-x}\cos x}{\sqrt{1+e^{-2x}}} = 0,$$

as Corollaries 1 and 2 state.

3. However, the equation

$$y''' - 2y'' + 2y' = 0 \quad \text{on} \quad (0, \infty)$$

admits the solutions: 1, $e^{x}\sin x$, $e^{x}\cos x$; the corresponding ω-limit set is a great circle on the sphere $\mathbb{S}_2$ in $\mathbb{E}_3$ and hence the case a_2 is valid for it. However, the subcase a_2^0 does not take place. Except of the constant solutions, each other solution is oscillatory (as $x \to \infty$), see Corollaries 4,5 and Remark 2.

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