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ON SOME TWO-POINT BOUNDARY VALUE PROBLEMS FOR TWO-DIMENSIONAL SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. Sufficient conditions for the solvability of two-point boundary value problems for the system $x_i' = f_i(t, x_1, x_2)$ $(i = 1, 2)$ are given, where f_1 and $f_2 : [a_1, a_2] \times R^2 \to R$ are continuous functions.

1. Statement of the problems and formulation of the main results

Consider the system of ordinary differential equations

$$x_i' = f_i(t, x_1, x_2) \quad (i = 1, 2) \tag{1.1}$$

with boundary conditions

$$\lambda_{i1} x_1(a_i) + \lambda_{i2} x_2(a_i) + g_i(x_1, x_2) = 0 \quad (i = 1, 2) \tag{1.2}$$

or

$$\lambda_{i1} x_1(a_i) + \lambda_{i2} x_2(a_i) + h_i(x_1(a_i), x_2(a_i)) = 0 \quad (i = 1, 2), \tag{1.3}$$

where $-\infty < a_1 < a_2 < +\infty$, $\lambda_{ij} \in R$ $(i, j = 1, 2)$, the functions f_i : $[a_1, a_2] \times R^2 \to R$, $h_i : R^2 \to R$ $(i = 1, 2)$ are continuous and g_i : $C([a_1, a_2]; R^2) \to R$ $(i = 1, 2)$ are the continuous functionals.

The problems of the forms (1.1),(1.2) and (1.1),(1.3) have been studied earlier in [1-10]. In the present paper new criteria for solvability of these problems are established which have the nature of one-sided restrictions imposed on f_1 and f_2.

We use the following notation:

R is the set of all real numbers; $R_+ = [0, +\infty[$,

$D = [a_1, a_2] \times R^2$,

$D_1 = [a_1, a_2] \times (R \backslash \{0\}) \times R$; $D_2 = [a_1, a_2] \times R \times (R \backslash \{0\})$,

$C(A, B)$ is the set of continuous maps from A to B.

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A solution of the system (1.1) is sought in the class of continuously differentiable vector-functions $(x_1, x_2) : [a_1, a_2] \to R^2$.

1.1. Problem (1.1), (1.2). We shall study the problem (1.1),(1.2) in the case when

$$(-1)^i \sigma \cdot \lambda_{i1}\lambda_{i2} > 0 \quad (i = 1, 2),$$

and the functionals g_1 and g_2 satisfy the inequality

$$|g_1(x, y)| + |g_2(x, y)| \le l,$$

on $C([a_1, a_2]; R^2)$, where $\sigma \in \{-1, 1\}$ and $l \in R_+$.

Theorem 1.1. *Suppose that*

$$\sigma[f_1(t, x, y) - p_{11}(t, x, y)x - p_{12}(t, x, y)y] \operatorname{sgn} y \ge -q_0 \quad \textit{for} \quad (t, x, y) \in D_2, \tag{1.4}$$

$$\sigma[f_2(t, x, y) - p_{21}(t, x, y)x - p_{22}(t, x, y)y] \operatorname{sgn} x \ge -q_0 \quad \textit{for} \quad (t, x, y) \in D_1, \tag{1.5}$$

$$\sigma f_1(t, 0, y) \operatorname{sgn} y \ge 0 \quad \textit{for} \quad a_1 \le t \le a_2, \quad |y| \ge r_0, \tag{1.6}$$

$$\sigma f_2(t, x, 0) \operatorname{sgn} x \ge 0 \quad \textit{for} \quad a_1 \le t \le a_2, \quad |x| \ge r_0, \tag{1.7}$$

where $p_{11} : D_2 \to R$, $p_{22} : D_1 \to R$, p_{12} and $p_{21} : D \to R$ are continuous bounded functions and q_0, r_0 are positive constants. Then the problem (1.1.), (1.2) *has at least one solution.*

Corollary 1.1. *Let the inequalities*

$$\sigma f_1(t, x, y) \operatorname{sgn} y \ge p_0(|y| - |x|) - q_0, \tag{1.8}$$

$$\sigma f_2(t, x, y) \operatorname{sgn} x \ge p_0(|x| - |y|) - q_0, \tag{1.9}$$

hold on D, where p_0 and q_0 are positive constants. Then the problem (1.1), (1.2) *is solvable.*

Theorem 1.2. *Suppose that*

$$\sigma[f_1(t, x, y) - p_{11}(t, x, y)x] \operatorname{sgn} y \ge 0 \quad \textit{for} \quad a_1 \le t \le a_2, \ \ \mu xy > 0, \tag{1.10}$$

$$\sigma[f_1(t, x, y) - p_{11}(t, x, y)x - p_{12}(t, x, y)y] \operatorname{sgn} y \ge -q_0 \quad \textit{for} \quad a_1 \le t \le a_2, \ \ \mu xy < 0, \tag{1.11}$$

$$\sigma[f_2(t, x, y) - p_{22}(t, x, y)y] \operatorname{sgn} x \ge -q(x) \quad \textit{for} \quad a_1 \le t \le a_2, \ \ \mu xy > 0, \tag{1.12}$$

$$\sigma[f_2(t, x, y) - p_{21}(t, x, y)x - p_{22}(t, x, y)y] \operatorname{sgn} x \ge -q_0 \quad \textit{for} \quad a_1 \le t \le a_2, \ \ \mu xy < 0 \tag{1.13}$$

and the inequality (1.7) *holds, where* $p_{11} : D_2 \to R$, $p_{22} : D_1 \to R$, p_{12}, $p_{21} : D \to R$ *are continuous bounded functions,* $q \in C(R; R_+)$, $\mu \in \{-1, 1\}$ *and* q_0, r_0 *are positive constants. Then the problem* (1.1), (1.2) *has at least one solution.*

Theorem 1.3. *Suppose that*

$$\begin{gathered}\sigma[f_1(t,x,y) - p_{11}(t,x,y)x] \operatorname{sgn} y \geq -q_0 \\ \text{for} \quad a_1 \leq t \leq a_2, \quad \mu xy < 0,\end{gathered} \tag{1.14}$$

$$\begin{gathered}\sigma[f_2(t,x,y) - p_{22}(t,x,y)y] \operatorname{sgn} x \geq -q(x) \\ \text{for} \quad (t,x,y) \in D,\end{gathered} \tag{1.15}$$

and the inequalities (1.7) *and* (1.10) *hold, where* $p_{11} : D_2 \to R$, $p_{22} : D_1 \to R$ *are continuous bounded functions,* $q \in C(R, R_+)$, $\mu \in \{-1, 1\}$ *and* r_0, q_0 *are positive constants. Then the problem* (1.1), (1.2) *has at least one solution.*

1.2. The Problem (1.1), (1.3). We shall study the problem (1.1),(1.3) in the case when

$$(-1)^i \sigma \lambda_{i1} \lambda_{i2} \geq 0, \ \ |\lambda_{i1}| + |\lambda_{i2}| \neq 0 \quad (i = 1, 2)$$

and

$$\sup\{|h_i(x,y)| : (-1)^i \sigma xy > 0\} < +\infty \ \ (i = 1, 2),$$

where $\sigma \in \{-1, 1\}$.

Theorem 1.4. *Let the inequalities* (1.4)–(1.7) *hold, where* $p_{11} : D_2 \to R$, $p_{22} : D_1 \to R$, p_{12}, $p_{21} : D \to R$ *are continuous bounded functions and* q_0, r_0 *are positive constants. Then the problem* (1.1), (1.3) *has at least one solution.*

Corollary 1.2. *Let the inequalities* (1.8) *and* (1.9) *hold on* D, *where* p_0 *and* q_0 *are positive constants. Then the problem* (1.1), (1.3) *has one solution.*

Consider as an example the boundary value problem

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + g_{11}(t)x_2^{2k_1+1} + \\ &\quad + g_{12}(t)|x_1|^{n_1} x_2^{2m_1+1} + q_1(t), \\ x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + g_{21}(t)x_1^{2k_2+1} + \\ &\quad + g_{22}(t)|x_2|^{n_2} x_1^{2m_2+1} + q_2(t), \end{aligned} \tag{1.16}$$

$$x_2(a_1) = h_1(x_1(a_1)), \quad x_2(a_2) = h_2(x_1(a_2)), \tag{1.17}$$

where $n_i,\ k_i,\ m_i \in \{1,2,3,\dots\}$ $(i=1,2)$, $p_{ij},\ g_{ij},\ q_i \in C([a_1,a_2];R)$ $(i,j=1,2)$, $h_i \in C(R;R)$ $(i=1,2)$. It follows from Corollary 1.2 that if for some $\sigma \in \{-1,1\}$ and $r \in R_+$ the inequalities

$$\sigma g_{i1}(t) > 0, \ \ \sigma g_{i2}(t) \ge 0 \ \ \text{for} \ \ a_1 \le t \le a_2 \ \ (i=1,2)$$

and

$$(-1)^i h_i(x) \operatorname{sgn} x \le 0 \ \ \text{for} \ \ |x| \ge r \ \ (i=1,2) \tag{1.18}$$

hold, then the problem (1.16),(1.17) has at least one solution. Therefore, the problem (1.16),(1.17) is solvable in the resonance case, i.e. in the case when the corresponding homogeneous problem

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 \\ x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 \end{aligned}, \qquad x_2(a_1)=0, \ \ x_2(a_2)=0$$

has a nontrivial solution.

Theorem 1.5. *Let the inequalities* (1.7), *and* (1.10)-(1.13) *hold, where* $p_{11}: D_2 \to R$, $p_{22}: D_1 \to R$, $p_{12},\ p_{21}: D \to R$ *are continuous bounded functions,* $q \in C(R,R_+)$, $\mu \in \{-1,1\}$ *and* $r_0,\ q_0$ *are positive constants. Then the problem* (1.1), (1.3) *has at least one solution.*

Theorem 1.6. *Let the inequalities* (1.7), (1.10), (1.14) *and* (1.15) *hold, where* $p_{11}: D_2 \to R$, $p_{22}: D_1 \to R$ *are continuous bounded functions,* $q \in C(R;R_+)$, $\mu \in \{-1,1\}$ *and* $r_0,\ q_0$ *are positive constants. Then the problem* (1.1), (1.3) *has at least one solution.*

Consider as an example the system

$$\begin{aligned} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 - \sigma\mu(|x_1x_2| - \mu x_1x_2)^n x_1^{2k+1} &+ \\ + \Big[\frac{|x_1x_2| - \mu x_1x_2}{|x_1x_2|+1}\Big]^m q(t)&, \\ x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + f_1(x_1)\cdot f_2(x_2)&, \end{aligned} \tag{1.19}$$

where $\mu \in \{-1,1\}$, $m,\ n,\ k \in \{1,2,3,\dots\}$, $p_{ij},\ q \in C([a_1,a_2];R)$ $(i,j=1,2)$, $f_1: R \to R$ is a continuous function and $f_2: R \to R$ is a continuous bounded function. It follows from Theorem 1.6 that if for some $\sigma \in \{-1,1\}$ and $r \in R_+$ the inequalities (1.18) and

$$\sigma p_{12}(t) \ge 0, \ \ \sigma p_{21}(t) \ge 0 \ \ \text{for} \ \ a_1 \le t \le a_2$$

hold, then the problem (1.19),(1.17) is solvable.

2. Some Auxiliary Statements

In this section we shall give some lemmas on a priori estimates of the solutions of the system

$$x_i' = p_{i1}(t, x_1, x_2)x_1 + p_{i2}(t, x_1, x_2)x_2 + q_i(t, x_1, x_2) \quad (i = 1, 2), \tag{2.1}$$

where $q_1 : D_2 \to R$, $q_2 : D_1 \to R$ are continuous functions and $p_{11} : D_2 \to R$, $p_{22} : D_1 \to R$, p_{12}, $p_{21} : D \to R$ are continuous functions bounded by a positive number p_0.

Lemma 2.1. *Suppose that*

$$q_1(t, x, y) \operatorname{sgn} y \geq -q_0, \quad q_2(t, x, y) \operatorname{sgn} x \geq -q_0 \quad \textit{for} \quad (t, x, y) \in D, \tag{2.2}$$

$$q_1(t, 0, y) \operatorname{sgn} y > -p_{12}(t, 0, y)|y| \quad \textit{for} \quad a_1 \leq t \leq a_2, \quad |x| \geq r_0, \tag{2.3}$$

$$q_2(t, x, 0) \operatorname{sgn} x > -p_{21}(t, x, 0)|x| \quad \textit{for} \quad a_1 \leq t \leq a_2, \quad |y| \geq r_0, \tag{2.4}$$

where r_0 and q_0 are positive constants. Suppose, moreover, that an absolutely continuous vector-function $(x_1, x_2) : [a_1, a_2] \to R$ satisfies the system (2.1) *almost everywhere and the conditions*

$$\textit{either} \quad (-1)^i x_1(a_i)x_2(a_i) \leq 0 \quad \textit{or} \quad |x_1(a_i)| + |x_2(a_i)| \leq c \quad (i = 1, 2). \tag{2.5}$$

Then the estimate

$$|x_1(t)| + |x_2(t)| \leq (c + r_0 + 2q_0(a_2 - a_1)) \exp[4p_0(a_2 - a_1)] \quad \textit{for} \quad a_1 \leq t \leq a_2 \tag{2.6}$$

holds.

Proof. Let $t_0 \in]a_1, a_2[$. Suppose that $(-1)^k x_1(t_0)x_2(t_0) > 0$, where $k \in \{1, 2\}$. Then either

$$(-1)^k x_1(t)x_2(t) > 0 \quad \text{for} \quad \min\{t_0, a_k\} \leq t \leq \max\{t_0, a_k\}, \quad |x_1(a_k)| + |x_2(a_k)| \leq c$$

or $t_1 \in [\min\{t_0, a_k\}, \max\{t_0, a_k\}]$ can be found such that

$$(-1)^k x_1(t)x_2(t) > 0 \quad \text{for} \quad \min\{t_0, t_1\} < t < \max\{t_0, t_1\}, \quad x_1(t_1)x_2(t_1) = 0. \tag{2.7}$$

In the case when (2.7) holds, the inequality

$$x_1'(t_1)x_2(t_1)+x_2'(t_1)x_1(t_1)\le 0$$

together with (2.3) and (2.4) implies

$$|x_1(t_1)|+|x_2(t_1)|\le r_0.$$

Therefore, if $(-1)^k x_1(t_0)x_2(t_0)>0$, then $t_1\in[\min\{t_0,a_k\},\max\{t_0,a_k\}]$ can be found such that

$$\begin{gathered}(-1)^k x_1(t)x_2(t)>0 \quad \text{for} \quad \min\{t_0,t_1\}<t<\max\{t_0,t_1\},\\ |x_1(t_1)|+|x_2(t_1)|\le c+r_0.\end{gathered} \tag{2.8}$$

Integrating the sum $x_1'(t)+(-1)^k x_2'(t)$ from t_1 to t and taking into consideration (2.1),(2.2) and (2.8) we easily see that

$$\begin{gathered}|x_1(t)|+|x_2(t)|\le c+r_0+2q_0(a_2-a_1)-(-1)^k 4p_0\int_{t_1}^{t}[|x_1(\tau)|+|x_2(\tau)|]d\tau\\ \text{for} \quad \min\{t_0,t_1\}<t<\max\{t_0,t_1\}.\end{gathered}$$

Applying the Gronwall-Bellman lemma, we obtain that the estimate (2.6) holds for $t=t_0$.

Suppose now that $x_1(t_0)x_2(t_0)=0$. Then either a sequence $(t_n)_{n=1}^{+\infty}$, $t_n\in]a_1,a_2[\ n\in\{1,2,3,\dots\}$ can be found such that

$$\lim_{n\to+\infty}t_n=t_0,\quad x_1(t_n)x_2(t_n)\ne 0 \quad n\in\{1,2,\dots\} \tag{2.9}$$

or for some $\varepsilon\in]0,\min(b-t_0,t_0-a)[$

$$x_1(t)x_2(t)=0 \quad \text{for} \quad t_0-\varepsilon<t<t_0+\varepsilon. \tag{2.10}$$

If (2.9) is true, then as it already was shown above, the estimate (2.6) holds for $t=t_n\ n\in\{1,2,\dots\}$. Hence (2.6) holds for $t=t_0$ also. And if (2.10) is true, then according to (2.3) and (2.4) we obtain from the equality

$$x_1'(t_0)x_2(t_0)+x_2'(t_0)x_1(t_0)=0$$

that $|x_1(t_0)|+|x_2(t_0)|\le r_0$. $\square$

Lemma 2.2. *Suppose that*

$$q_1(t,x,y)\operatorname{sgn} y \geq -p_{12}(t,x,y)|y| \quad \text{for} \quad a_1 < t < a_2, \;\; xy > 0, \tag{2.11}$$

$$q_1(t,x,y)\operatorname{sgn} y \geq -q_0 \quad \text{for} \quad a_1 < t < a_2, \;\; xy < 0, \tag{2.12}$$

$$q_1(a_1,0,y)\operatorname{sgn} y > p_{12}(a_1,0,y)|y| \quad \text{for} \quad |y| \geq r_0, \tag{2.13}$$

$$q_2(t,x,y)\operatorname{sgn} x \geq -q(x) - p_{21}(t,x,y)|x| \quad \text{for} \quad a_1 < t < a_2, \;\; xy > 0, \tag{2.14}$$

$$q_2(t,x,y)\operatorname{sgn} x \geq -q_0 \quad \text{for} \quad a_1 < t < a_2, \;\; xy < 0 \tag{2.15}$$

and (2.4) *holds, where* $q \in C(R;R_+)$ *and* r_0 *and* q_0 *are positive constants. Then for any absolutely continuous vector-function* $(x_1,x_2):[a_1,a_2] \to R$ *satisfying the system* (2.1) *and the conditions* (2.5), *the estimate*

$$|x_1(t)| + |x_2(t)| \leq 2(c + r_0 + 2q_0(a_2 - a_1) + \max\{q(x) : |x| \leq \leq (c + r_0)\exp[p_0(a_2 - a_1)]\})\exp[4p_0(a_2 - a_1)] \quad \text{for} \quad a_1 \leq t \leq a_2 \tag{2.16}$$

holds.

Proof. Let $t_0 \in]a_1,a_2[$. Suppose first $x_1(t_0)x_2(t_0) > 0$. Then either

$$x_1(t)x_2(t) > 0 \quad \text{for} \quad t_0 < t \leq a_2 \quad |x_1(a_2)| + |x_2(a_2)| \leq c$$

or $t_1 \in]t_0,a_2[$ can be found such that

$$x_1(t)x_2(t) > 0 \quad \text{for} \quad t_0 < t < t_1, \;\; x_1(t_1)x_2(t_1) = 0. \tag{2.17}$$

If (2.17) is true, then according to (2.11) from (2.1) we have

$$|x_1(t)|' \geq -p_0|x_1(t)| \quad \text{for} \quad t_0 < t < t_1. \tag{2.18}$$

Therefore if $x_1(t_1) = 0$, then $x_1(t) \equiv 0$ for $t_0 < t < t_1$ which contradicts (2.17). So $x_2(t_1) = 0$, and hence $x_2'(t_1)\operatorname{sgn} x_1(t_1) \leq 0$. From this according to (2.4) we see that $|x_1(t_1)| \leq r_0$.

Thus if $x_1(t_0)x_2(t_0) > 0$, then $t_1 \in]t_0,a_2]$ can be found such that

$$x_1(t)x_2(t) > 0 \quad \text{for} \quad t_0 < t < t_1, \;\; |x_1(t_1)| + |x_2(t_1)| \leq c + r_0.$$

By virtue of the above-said and from (2.18) we easily find that

$$|x_1(t)| \leq (c + r_0)\exp[p_0(a_2 - a_1)] \quad \text{for} \quad t_0 \leq t \leq t_1. \tag{2.19}$$

According to (2.11),(2.14) and (2.19) the second of the equalities (2.1) implies

$$|x_2'(t)| \geq -p_0|x_2(t)| - q(x_1(t)) \quad \text{for} \quad t_0 < t < t_1$$

and

$$|x_2(t)| \le (c + r_0 + \max\{q(x) : |x| \le (c + r_0)\} \exp[p_0(a_2 - a_1)]) \times \\ \times \exp[p_0(a_2 - a_1)] \quad \text{for} \quad t_0 \le t \le t_1.$$

Therefore the estimate (2.16) holds for $t = t_0$.

Suppose now that $x_1(t_0)x_2(t_0) < 0$. Then either

$$x_1(t)x_2(t) \le 0, \ \ x_2(t) \ne 0 \quad \text{for} \quad a_1 < t < t_0, \ \ |x_1(a_1)| + |x_2(a_1)| \le c,$$

or

$$x_1(t)x_2(t) \le 0, \ \ x_2(t) \ne 0 \quad \text{for} \quad a_1 < t < t_0, \ \ x_1(a_1) = 0, \tag{2.20}$$

or $t_1 \in [a_1, t_0[$ can be found such that

$$x_1(t)x_2(t) \le 0, \ \ x_2(t) \ne 0 \quad \text{for} \quad t_1 < t < t_0, \ \ x_2(t_1) = 0. \tag{2.21}$$

If (2.20) ((2.21)) is true, then according to (2.13) ((2.4)) we obtain from the inequality $x_1'(a_1) \operatorname{sgn} x_2(a_1) \le 0$ ($x_2'(t_1) \operatorname{sgn} x_1(t_1) \le 0$) that $|x_2(a_1)| \le r_0$ ($|x_1(t_1)| \le r_0$).

Thus if $x_1(t_0)x_2(t_0) < 0$, then $t_1 \in [a_1, t_0[$ can be found such that

$$x_1(t)x_2(t) \le 0, \quad x_2(t) \ne 0 \quad \text{for} \quad t_1 < t < t_0, \\ |x_1(t_1)| + |x_2(t_1)| \le c + r_0. \tag{2.22}$$

Integrating the difference of the equalities (2.1) from t_1 to t, taking into consideration (2.12),(2.15) and applying the Gronwall-Bellman lemma, we see that the estimate (2.16) holds for $t = t_0$.

Consider, at least, the case when $x_1(t_0)x_2(t_0) = 0$. Then either a sequence $(t_n)_{n=1}^{+\infty}$, $t_n \in]a_1, a_2[$, $n \in \{1, 2, 3, \dots\}$ can be found such that (2.9) holds or for some $\varepsilon \in]0, \min(b - t_0, t_0 - a)[$ (2.10) is valid. Suppose that (2.10) is true. Then either $x_1(t_0) = x_2(t_0) = 0$ or

$$x_1(t_0) \ne 0, \quad x_2(t_0) = 0 \tag{2.23}$$

or

$$x_2(t_0) \ne 0, \quad x_1(t_0) = 0. \tag{2.24}$$

Let (2.23) be fulfilled. Then $\varepsilon_1 \in]0, \varepsilon[$ can be found such that

$$x_1(t) \ne 0, \quad x_2(t) = 0 \quad \text{for} \quad t_0 - \varepsilon_1 < t < t_0.$$

According to (2.4) from the equality $x_2'(t_0) \operatorname{sgn} x_1(t_0) = 0$ we have that $|x_1(t_0)| \le r_0$. Therefore, the estimate (2.16) is true for $t = t_0$.

Let (2.24) be fulfilled. Then $\varepsilon_1 \in]0, \varepsilon]$ can be found such that

$$x_2(t) \ne 0, \quad x_1(t) = 0 \quad \text{for} \quad t_0 - \varepsilon_1 < t < t_0.$$

Put

$$\alpha = \inf\{\tau \in]a_1, t_0[: x_1(t) \equiv 0, \quad x_2(t) \neq 0 \quad \text{for} \quad \tau < t < t_0\}.$$

If $\alpha = a_1$, then according to (2.13) from the equality $x_1'(a_1) \operatorname{sgn} x_2(a_1) = 0$ we find that $|x_2(a_1)| \leq r_0$. And if $\alpha > a_1$, then either

$$x_1(\alpha) = x_2(\alpha) = 0$$

or $\varepsilon_0 \in]0, \alpha - a_1[$ can be found such that

$$x_1(t)x_2(t) < 0 \quad \text{for} \quad \alpha - \varepsilon_0 \leq t < \alpha.$$

Since $x_1(\alpha-\varepsilon_0)x_2(\alpha-\varepsilon_0) < 0$, as it was already shown above, $t_1 \in [a_1, \alpha-\varepsilon_0[$ can be found such that (2.22) holds.

Thus, if (2.24) is valid, then $t_1 \in [a_1, t_0[$ can be found such that (2.22) is true.

Integrating the difference of the equalities (2.1) from t_1 to t, taking into consideration (2.12),(2.15) and applying the Gronwall-Bellman lemma, we see that the estimate (2.16) is true for $t = t_0$. □

The proof of the following lemma is quite analogous.

Lemma 2.3. *Suppose that*

$$q_1(t,x,y) \operatorname{sgn} y \geq -p_{12}(t,x,y)|y| - q_0 \quad \textit{for} \quad a_1 < t < a_2, \quad xy < 0,$$
$$q_2(t,x,y) \operatorname{sgn} x \geq -q(x) - p_{21}(t,x,y)|x| \quad \textit{for} \quad (t,x,y) \in D,$$

and the conditions (2.4), (2.11) *and* (2.13) *hold, where* $q \in C(R; R_+)$ *and* r_0, q_0 *are positive constants. Then for any absolutely continuous vector-function* $(x_1, x_2) : [a_1, a_2] \to R$ *satisfying the system* (2.1) *and conditions* (2.5)*, the estimate* (2.16) *holds.*

3. Proof of the Main Results

We shall carry out the proof only in the case $\sigma = \mu = 1$, since the general case by the change of variables

$$\overline{x}_1(t) = -\sigma\mu x_1\left(\sigma\mu t + \frac{1-\sigma\mu}{2}\right),$$
$$\overline{x}_2(t) = -\sigma x_2\left(\sigma\mu t + \frac{1-\sigma\mu}{2}\right)$$

can be reduced to this one.

Proof of Theorem 1.1. Assume first that instead of (1.6) and (1.7) the conditions

$$f_1(t,0,y) \operatorname{sgn} y > 0 \quad \text{for} \quad a_1 \leq t \leq a_2, \quad |y| \geq r_0,$$
$$f_2(t,x,0) \operatorname{sgn} x > 0 \quad \text{for} \quad a_1 \leq t \leq a_2, \quad |x| \geq r_0,$$

are fulfilled.

Put

$$\eta = \sum_{i,j=1}^{2} |\lambda_{ij}|^{-1}, \quad p_0 = \sup\{|p_{ij}(t,x,y)| : i,j = 1,2,\ (t,x,y) \in D\},$$
$$r_1 = 1 + (\eta l + r_0 + 2q_0(a_2 - a_1)) \exp[4p_0(a_2 - a_1)], \tag{3.1}$$

$$\chi(\tau) = \begin{cases} 1 & \text{for} \quad 0 \le \tau \le r_1 \\ 2 - \frac{\tau}{r_1} & \text{for} \quad r_1 < \tau < 2r_1 \\ 0 & \text{for} \quad \tau \ge 2r_1 \end{cases}, \tag{3.2}$$

$$q_i(t,x,y) = \chi(|x| + |y|)[f_i(t,x,y) - p_{i1}(t,x,y)x - p_{i2}(t,x,y)y]$$
$$(i = 1,2) \quad \text{for} \quad (t,x,y) \in D, \tag{3.3}$$
$$\widetilde{q}_1(t,x,y) = \chi(|x| + |y|)[f_1(t,x,y) - y] \quad \text{for} \quad (t,x,y) \in D,$$
$$\widetilde{q}_2(t,x,y) = \chi(|x| + |y|)[f_2(t,x,y) - x] \quad \text{for} \quad (t,x,y) \in D,$$
$$\widetilde{g}_i(x,y) = \chi(\|x\|_C + \|y\|_C) g_i(x,y) \quad (i = 1,2)$$
$$\text{for} \quad x,y \in C([a_1,a_2];R), \tag{3.4}$$

where $\|p\|_C = \max\{|p(t)| : t \in [a_1,a_2]\}$, and consider the boundary value problem

$$x_1' = x_2 + \widetilde{q}_1(t,x_1,x_2),$$
$$x_2' = x_1 + \widetilde{q}_2(t,x_1,x_2), \tag{3.5}$$
$$\lambda_{i1}x_1(a_i) + \lambda_{i2}x_2(a_i) + \widetilde{g}_i(x_1,x_2) = 0 \quad (i = 1,2). \tag{3.6}$$

According to Theorem 2.1 from [2], the problem (3.5),(3.6) has at least one solution (x_1,x_2). It is easy to see that (x_1,x_2) is a solution of the system

$$x_i' = \widetilde{p}_{i1}(t,x_1,x_2)x_1 + \widetilde{p}_{i2}(t,x_1,x_2)x_2 + q_i(t,x_1,x_2) \quad (i = 1,2), \tag{3.7}$$

where

$$\widetilde{p}_{ij}(t,x,y) = 1 + (p_{ij}(t,x,y) - 1)\chi(|x| + |y|) \quad (i,j = 1,2,\ i \ne j),$$
$$\widetilde{p}_{ii}(t,x,y) = p_{ii}(t,x,y)\chi(|x| + |y|) \quad (i = 1,2). \tag{3.8}$$

In view of (3.2),(3.4) and (3.6) we have

$$\sum_{i=1}^{2} |\lambda_{i1}x_1(a_i)| + |\lambda_{i2}x_2(a_i)| \le l.$$

from which we get that the solution (x_1,x_2) of the system (3.7) satisfies the conditions

$$\text{either} \quad x_1(a_1)x_2(a_1) > 0 \quad \text{or} \quad |x_1(a_1)| + |x_2(a_1)| \le \eta l$$

and

$$\text{either } \ x_1(a_2)x_2(a_2) < 0 \ \text{ or } \ |x_1(a_2)| + |x_2(a_2)| \le \eta l.$$

According to Lemma 2.1 and (3.1) we have

$$|x_1(t)| + |x_2(t)| \le r_1 \quad \text{for} \quad a_1 \le t \le a_2. \tag{3.9}$$

This estimate together with (3.2)-(3.4) and (3.6)-(3.8) implies that (x_1, x_2) is a solution of the problem (3.1),(3.2). Moreover, (3.9) holds.

Consider now the case when (1.6) and (1.7) are fulfilled. According to what has been proved above, for any natural n the system of the differential equations

$$x_1' = f_1(t, x_1, x_2) + \frac{x_2}{n(1 + |x_2|)}$$
$$x_2' = f_2(t, x_1, x_2) + \frac{x_1}{n(1 + |x_1|)}$$

has the solution (x_{1n}, x_{2n}) satisfying the boundary conditions (1.2) and the inequality

$$|x_{1n}(t)| + |x_{2n}(t)| \le r_1 \quad \text{for} \quad a_1 \le t \le a_2.$$

It is clear that the sequences of functions $(x_{in})_{n=1}^{+\infty}$ $(i = 1, 2)$ are uniformly bounded and equicontinuous on $[a_1, a_2]$. Therefore, without loss of generality, we can assume that they are uniformly convergent. Putting

$$x_i(t) = \lim_{n \to +\infty} x_{in}(t) \quad \text{for} \quad a_1 \le t \le a_2 \quad (i = 1, 2)$$

it is easy to see that (x_1, x_2) is a solution of the problem (1.1),(1.2). ■

The proofs of the other theorems are quite analogous to the one of Theorem 1.1. The difference is that instead of Lemma 2.1 one has to apply Lemma 2.2 in proving Theorems 1.2, 1.5 and Lemma 2.3 in proving Theorems 1.3, 1.6.

Applying Theorems 1.1 and 1.4 in the case when

$$p_{11}(t, x, y) = \begin{cases} p_{11}(t)\,\mathrm{sgn}(xy) & \text{for} \quad x \ne 0 \\ p_{11}(t) & \text{for} \quad x = 0 \end{cases}$$
$$p_{22}(t, x, y) = \begin{cases} p_{22}(t)\,\mathrm{sgn}(xy) & \text{for} \quad y \ne 0 \\ p_{22}(t) & \text{for} \quad y = 0 \end{cases}$$

one can easily be convinced in the validity of Corollaries 1.1 and 1.2.

References

1. I.T. Kiguradze, On a boundary value problem for a system of two differential equations. (Russian) *Trudy Tbiliss. Univ. Mat. Mekh. Astron.* **1**(**137**)A(1971), 77-87.

2. I.T. Kiguradze, The boundary value problems for systems of ordinary differential equations. (Russian) *Current problems in mathematics. Newest results,* v. 30 (Russian), 3-103, *Itogi nauki i tekhniki, Akad. Nauk SSSR, Vsesoyuzn. Inst. Nauchn. i Tekhn. Inform., Moscow,* 1987.

3. N.P. Lezhava, On the solvability of a nonlinear problem for a system of two differential equations. (Russian) *Bull. Acad. Sci. Georgian SSR* **68**(1972), No. 3, 545-547.

4. G.N. Mil'shtein, On a boundary value problem for a system of two differential equations. (Russian) *Differentsial'nie Uravneniya* **1**(1961), No. 12, 1628-1639.

5. A.I. Perov, On a boundary value problem for a system of two differential equations. (Russian) *Dokl. Akad. Nauk SSSR* **144**(1962), No. 3, 493-496.

6. B.L. Shekhter, On one boundary value problem for two-dimensional discontinuous differential systems. (Russian) *Some problems of ordinary differential equations theory* (Russian), *Proceedings of I.N. Vekua Institute of Applied Mathematics* v. 8, 79-161. *Tbilisi University Press, Tbilisi,* 1980.

7. B.L. Shekhter, On singular boundary value problems for two-dimensional differential systems. *Arch. Math.* **19**(1983), No. 1, 19-41.

8. N.I. Vasil'ev, Some boundary value problems for a system of two first order differential equations, I. (Russian) *Latv. Mat. Ezhegodnik* **5**(1969), 11-24.

9. N.I. Vasil'ev, Some boundary value problems for a system of two first order differential equations, II. (Russian) *Latv. Mat. Ezhegodnik* **6**(1969), 31-39.

10. P. Waltman, Existence and uniqueness of solutions of boundary value problems for two-dimensional systems of nonlinear differential equations. *Trans. Amer. Math. Soc.* **153**(1971), No. 1, 223-234.

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