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ON SOME PROPERTIES OF MULTIPLE CONJUGATE TRIGONOMETRIC SERIES

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ABSTRACT. We have obtained the estimate in the terms of partial and mixed moduli of continuity of deviation of Cesáro $(C,\boldsymbol{\alpha})$ means ($\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)$, $\alpha_i\in\mathbb{R}$, $\alpha_i>-1$, $i=\overline{1,n}$) of the sequence of rectangular partial sums of n-multiple ($n>1$) conjugate trigonometric series from n-multiple truncated conjugate function. This estimate implies the result on the m_λ-convergence ($\lambda\geq 1$) of $(C,\boldsymbol{\alpha})$ means ($\alpha_i>0$, $i=\overline{1,n}$), provided that the essential conditions are imposed on the partial moduli of continuity. Finally, it is shown, that the m_λ-convergence cannot be replaced by ordinary convergence.

1. Let $f\in L([-\pi;\pi]^n)$, $n\in\mathbb{N}$, $n>1$, be a function, 2π-periodic in each variable, $\sigma_n[f]$ its n-multiple trigonometric Fourier series, and $\bar{\sigma}_n[f]$ its conjugate series with respect to n variables (see, e.g., [1]).

We set

$$\mathbf{m}=(m_1,\dots,m_n)\ \ (m_i\in\mathbb{N},\ i=\overline{1,n});$$
$$\mathbf{x}=(x_1,\dots,x_n)\ \ (x_i\in\mathbb{R},\ i=\overline{1,n});$$
$$\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)\ \ (\alpha_i\in\mathbb{R},\ \alpha_i>-1,\ i=\overline{1,n});\ \ \mathbf{M}=\{1,2,\dots,n\}.$$

By M_j we denote the set of all subsets of M with j elements, by $M^{(A)}$ a set $M\backslash A$ ($A\subset M$), and by $M_j^{(A)}$ the set of all subsets of $M^{(A)}$ with j elements.

For $B=\{i_i,\dots,i_k\}\subset M$ we define $m(B)=\{1/m_{i_1},\dots,1/m_{i_k}\}$ and the

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truncated conjugate function with respect to the corresponding variables

$$\bar{f}_{m(B)}(\mathbf{x}) =$$
$$= \frac{1}{(-2\pi)^k} \int_{1/m_{i_1}}^{\pi} \dots \int_{1/m_{i_k}}^{\pi} \Big(\Delta_{k,s_{i_k}} \Big(\Delta_{k-1,s_{i_{k-1}}} \big(\dots \big(\Delta_{1,s_{i_1}}(f;\mathbf{x}) \big) \dots \big) \Big) \Big) \times$$
$$\times \prod_{j=1}^{k} \operatorname{ctg} \frac{s_{i_j}}{2} ds_{i_1} \dots ds_{i_k}, \quad \bar{f}_{\mathbf{m}}(\mathbf{x}) = \bar{f}_{m(M)}(\mathbf{x}),$$

where

$$\Delta_{i,h}(f,\mathbf{x}) = f(x_1,\dots,x_{i-1},x_i+h,x_{i+1},\dots,x_n) -$$
$$- f(x_1,\dots,x_{i-1},x_i-h,x_{i+1},\dots,x_n), \quad i=\overline{1,n}.$$

For $f \in L^q([-\pi;\pi]^n)$, $1 \le q \le +\infty$ $(L^\infty = C)$, we consider its mixed modulus of continuity

$$\omega_B\big(m(B);f\big)_{L^q} =$$
$$= \sup_{|h_1|<1/m_{i_1},\dots,|h_k|<1/m_{i_k}} \left\| \Delta_k^{h_k} \big(\Delta_{k-1}^{h_{k-1}} \big(\dots \big(\Delta_1^{h_1}(f;\mathbf{x}) \big) \dots \big) \big) \right\|_{L^q},$$

where $\Delta_i^h(f;\mathbf{x}) = f(x_1,\dots,x_{i-1},x_i+h,x_{i+1},\dots,x_n) - f(x_1,\dots,x_n)$.

Let $\bar{\sigma}_{\mathbf{m}}^{\boldsymbol{\alpha}}(\mathbf{x};f)$ be Cesáro means of $\bar{\sigma}_n[f]$.

In the sequel by $A, B, A_1, B_1, C(\alpha), C(\beta), C(\alpha,\beta)$, etc. we will denote, in general, different positive constants.

Finally, we set

$$\lambda(k,\beta) = \begin{cases} k^{-\beta}, & -1<\beta<0, \\ \ln(k+1), & \beta=0, \\ 1, & \beta>0, \ (k=1,2,\dots). \end{cases}$$

In the present paper we give the estimate of the deviation of n-multiple Cesáro means of the sequence of rectangular partial sums of $\bar{\sigma}_n[f]$ from $\bar{f}_{\mathbf{m}}$ in the norm of L^q, $q \in [1;+\infty]$ $(L^\infty = C)$, in terms of partial and mixed moduli of continuity of f. This result generalizes the corresponding result of L.Zhizhiashvili (see [1]).

From this estimate ensues the result on the Cesáro summability of the sequence of rectangular partial sums of $\bar{\sigma}_n[f]$ and then the correctness of this result is shown.

2. The following is true:

Theorem 1. *If* $f \in L^q([-\pi;\pi]^n)$, $q \in [1;+\infty]$ $(L^\infty = C)$, *then*

$$\|\bar{\sigma}_{\mathbf{m}}^{\boldsymbol{\alpha}}(\mathbf{x};f) - \bar{f}_{\mathbf{m}}(\mathbf{x})\|_{L^q} \le C(\boldsymbol{\alpha}) \left\{ \sum_{k=1}^{n-1} \sum_{B\in M_k} \omega_B(m(B);f)_{L^q} \times \right.$$

$$\times \prod_{\substack{j=1 \\ \{i_1,\dots,i_k\}\in M_k}}^{k} \lambda(m_{i_j},\alpha_{i_j}) + \sum_{i=1}^{n-1}\sum_{k=1}^{n-i}\sum_{A\in M_i}\sum_{B\in M_k^{(A)}} \omega_B(m(B);\bar{f}_{m(A)})_{L^q} \times$$

$$\left. \times \prod_{\substack{j=1 \\ \{t_1,\dots,t_k\}\in M_k^{(A)}}}^{k} \lambda(m_{t_j},\alpha_{t_j}) + \prod_{i=1}^{n} \lambda(m_i,\alpha_i)\omega_M(m(M);f)_{L^q} \right\}. \qquad (1)$$

Proof. For simplicity, we will prove the theorem in the case $n = 2$ which is typical. We will use the method of L.Zhizhiashvili ([1], p. 160-191), this method proving to be true for $n \ge 3$.

Let $\omega(\delta_1,\delta_2;f)_{L^p}$ be the mixed modulus of continuity of the function $f(x,y)$, $x,y \in \mathbb{R}$, with respect to two variables. By $\omega_i(\delta;f)_{L^q}$ $(i \in \mathbb{N},\ \delta \in \mathbb{R}^+)$, as usual, we denote the modulus of continuity of f in L^q with respect to the corresponding variable. Let $m,n \in \mathbb{N}$, $\alpha,\beta \in \mathbb{R}$, $\alpha,\beta > -1$, $\bar{f}_{m,n}(x,y)$ be the two-dimensional truncated conjugate function and $\bar{\sigma}_{mn}^{\alpha,\beta}(x,y;f)$ be the Cesáro means of the double conjugate series $\bar{\sigma}_2[f]$.

We have ([1], p.187-188):

$$\bar{\sigma}_{mn}^{\alpha,\beta}(x,y;f) = 1/\pi^2 \int_0^{\pi/m}\int_0^{\pi/n} \psi_{x,y}(s,t;f)\bar{K}_m^\alpha(s)\bar{K}_n^\beta(t)\,dt\,ds +$$

$$+ 1/(2\pi^2)\int_0^{\pi/m}\int_{\pi/n}^{\pi} \psi_{x,y}(s,t;f)\operatorname{ctg}\frac{t}{2}\bar{K}_m^\alpha(s)\,dt\,ds +$$

$$+ 1/\pi^2 \int_0^{\pi/m}\int_{\pi/n}^{\pi} \psi_{x,y}(s,t;f)\bar{K}_m^\alpha(s)H_{n,1}^\beta(t)\,dt\,ds +$$

$$+ 1/\pi^2 \int_0^{\pi/m}\int_{\pi/n}^{\pi} \psi_{x,y}(s,t;f)\bar{K}_m^\alpha(s)H_{n,2}^\beta(t)\,dt\,ds +$$

$$+ 1/(2\pi^2)\int_{\pi/m}^{\pi}\int_0^{\pi/n} \psi_{x,y}(s,t;f)\operatorname{ctg}\frac{s}{2}\bar{K}_n^\beta(t)\,dt\,ds +$$

$$+ 1/\pi^2 \int_{\pi/m}^{\pi}\int_0^{\pi/n} \psi_{x,y}(s,t;f)H_{m,1}^\alpha(s)\bar{K}_n^\beta(t)\,dt\,ds +$$

$$+ 1/\pi^2 \int_{\pi/m}^{\pi}\int_0^{\pi/n} \psi_{x,y}(s,t;f)H_{m,2}^\alpha(s)\bar{K}_n^\beta(t)\,dt\,ds +$$

$$+ 1/(4\pi^2)\int_{\pi/m}^{\pi}\int_{\pi/n}^{\pi} \psi_{x,y}(s,t;f)\operatorname{ctg}\frac{s}{2}\operatorname{ctg}\frac{t}{2}\,dt\,ds +$$

$$
\begin{aligned}
&+1/(2\pi^2)\int_{\pi/m}^{\pi}\int_{\pi/n}^{\pi}\psi_{x,y}(s,t;f)\operatorname{ctg}\frac{s}{2}H_{n,1}^{\beta}(t)\,dt\,ds+\\
&+1/(2\pi^2)\int_{\pi/m}^{\pi}\int_{\pi/n}^{\pi}\psi_{x,y}(s,t;f)\operatorname{ctg}\frac{s}{2}H_{n,2}^{\beta}(t)\,dt\,ds+\\
&+1/(2\pi^2)\int_{\pi/m}^{\pi}\int_{\pi/n}^{\pi}\psi_{x,y}(s,t;f)\operatorname{ctg}\frac{t}{2}H_{m,1}^{\alpha}(s)\,dt\,ds+\\
&+1/\pi^2\int_{\pi/m}^{\pi}\int_{\pi/n}^{\pi}\psi_{x,y}(s,t;f)H_{m,1}^{\alpha}(s)H_{n,1}^{\beta}(t)\,dt\,ds+\\
&+1/\pi^2\int_{\pi/m}^{\pi}\int_{\pi/n}^{\pi}\psi_{x,y}(s,t;f)H_{m,1}^{\alpha}(s)H_{n,2}^{\beta}(t)\,dt\,ds+\\
&+1/(2\pi^2)\int_{\pi/m}^{\pi}\int_{\pi/n}^{\pi}\psi_{x,y}(s,t;f)\operatorname{ctg}\frac{t}{2}H_{m,2}^{\alpha}(s)\,dt\,ds+\\
&+1/\pi^2\int_{\pi/m}^{\pi}\int_{\pi/n}^{\pi}\psi_{x,y}(s,t;f)H_{m,2}^{\alpha}(s)H_{n,1}^{\beta}(t)\,dt\,ds+\\
&+1/\pi^2\int_{\pi/m}^{\pi}\int_{\pi/n}^{\pi}\psi_{x,y}(s,t;f)H_{m,2}^{\alpha}(s)H_{n,2}^{\beta}(t)\,dt\,ds=\\
&=\sum_{j=1}^{16}P_{mn}^{(j)}(x,y;f),
\end{aligned}
\tag{2}
$$

where

$$
\begin{aligned}
\psi_{x,y}(s,t;f)=f(x+s,y+t)-f(x-s,y+t)-\\
-f(x+s,y-t)+f(x-s,y-t)
\end{aligned}
\tag{3}
$$

and $H_{m,1}^{\alpha}(s)$, $H_{m,2}^{\alpha}(s)$ are the summands of the conjugate Fejér kernel $\bar{K}_m^{\alpha}(s)$ (see [2], pp.157-160)

$$
\bar{K}_m^{\alpha}(s)=\frac{1}{2}\operatorname{ctg}\frac{s}{2}+H_{m,1}^{\alpha}(s)+H_{m,2}^{\alpha}(s),
\tag{4}
$$

$$
H_{m,1}^{\alpha}(s)=-\frac{\cos\left((m+\frac{1}{2}+\frac{\alpha}{2})s-\frac{\pi\alpha}{2}\right)}{A_m^{\alpha}(2\sin\frac{s}{2})^{1+\alpha}}.
\tag{5}
$$

Besides,

$$
|\bar{K}_m^{\alpha}(s)|\le C(\alpha)m,\quad |s|\le\pi;
\tag{6}
$$

$$
|H_{m,2}^{\alpha}(s)|\le\frac{C(\alpha)}{m^2s^3},\quad \pi/m\le|s|\le\pi,\ \ m>1.
\tag{7}
$$

The estimate (7) is more precise than the estimate (2.1.14) in [1] and it can be proved by arguments analogous to those in [2], pp.157-160.

From (2) we obtain

$$\bar{\sigma}_{mn}^{\alpha,\beta}(x,y;f)-\bar{f}_{mn}(x,y)=\bar{\sigma}_{mn}^{\alpha,\beta}(x,y;f)-P_{mn}^{(8)}(x,y;f)= \\ =\sum_{k=1}^{16}{}' P_{mn}^{(k)}(x,y;f), \tag{8}$$

where $'$ indicates that the eighth member is omitted (the replacement of $1/m$ $(1/n)$ in $\bar{f}_{mn}(x,y)$ by π/m (π/n) does not matter).

In the sequel we will use the inequality (see [3], p.179)

$$\Big\{\int_a^b\Big|\int_c^d f(x,y)\,dy\Big|^q dx\Big\}^{1/q}\le\int_c^d\Big\{\int_a^b|f(x,y)|^q\,dx\Big\}^{1/q}dy, \\ 1\le q<+\infty. \tag{9}$$

Taking into account (6) and (9), we obtain

$$\|P_{mn}^{(1)}(x,y;f)\|_{L^q}\le C(\alpha,\beta)\big\{\omega_1(1/m;f)_{L^q}+\omega_2(1/n;f)_{L^q}\big\}. \tag{10}$$

It is easy to see, that

$$P_{mn}^{(2)}(x,y;f)=-\frac{1}{\pi}\int_0^{\pi/m}\big[\bar{f}_n^{(2)}(x+s,y)-\bar{f}_n^{(2)}(x-s,y)\big]\bar{K}_m^{\alpha}(s)\,ds,$$

$$P_{mn}^{(5)}(x,y;f)=-\frac{1}{\pi}\int_0^{\pi/n}\big[\bar{f}_m^{(1)}(x,y+t)-\bar{f}_m^{(1)}(x,y-t)\big]\bar{K}_n^{\beta}(t)\,dt,$$

where

$$\bar{f}_m^{(1)}(x,y)=-\frac{1}{2\pi}\int_{\pi/m}^{\pi}[f(x+s,y)-f(x-s,y)]\operatorname{ctg}\frac{s}{2}\,ds, \tag{11}$$

$$\bar{f}_n^{(2)}(x,y)=-\frac{1}{2\pi}\int_{\pi/n}^{\pi}[f(x,y+t)-f(x,y-t)]\operatorname{ctg}\frac{t}{2}\,dt. \tag{12}$$

Hence, using again (6) and (9), we obtain:

$$\|P_{mn}^{(2)}(x,y;f)\|_{L^q}\le C(\alpha,\beta)\omega_1(1/m;\bar{f}_n^{(2)})_{L^q}; \tag{13}$$

$$\|P_{mn}^{(5)}(x,y;f)\|_{L^q}\le C(\alpha,\beta)\omega_2(1/n;\bar{f}_m^{(1)})_{L^q}. \tag{14}$$

Now, it is easy to see that for estimation of $P_{mn}^{(3)}(x,y;f)$ it suffices to estimate the integral

$$I(m,n)=n^{-\beta}\int_0^{\pi/m}\int_{\pi/n}^{\pi}\psi_{x,y}(s,t;f)\bar{K}_m^{\alpha}(s)\omega_\beta(t)\cos nt\,dt\,ds,$$

where

$$\omega_\beta(t)=\frac{\cos\frac{1+\beta}{2}t}{(\sin\frac{t}{2})^{1+\beta}}.$$

We have

$$2I(m,n) = n^{-\beta}\Big\{\int_0^{\pi/m}\int_{\pi/n}^{\pi}\big[\psi_{x,y}(s,t;f)-\psi_{x,y}(s,t+\pi/n;f)\big]\times$$
$$\times\bar{K}_m^{\alpha}(s)\omega_\beta(t)\cos nt\,dt\,ds+\int_0^{\pi/m}\int_{\pi/n}^{\pi}\psi_{x,y}(s,t+\pi/n;f)\big[\omega_\beta(t)-$$
$$-\omega_\beta(t+\pi/n)\big]\bar{K}_m^{\alpha}(s)\cos nt\,dt\,ds+$$
$$+\int_0^{\pi/m}\int_{\pi-\pi/n}^{\pi}\psi_{x,y}(s,t+\pi/n;f)\bar{K}_m^{\alpha}(s)\omega_\beta(t+\pi/n)\cos nt\,dt\,ds-$$
$$-\int_0^{\pi/m}\int_0^{\pi/n}\psi_{x,y}(s,t+\pi/n;f)\bar{K}_m^{\alpha}(s)\omega_\beta(t+\pi/n)\cos nt\,dt\,ds\Big\}.$$

Now we note that (see [1], p.56, (2.1.18))

$$|\omega_\beta(t)-\omega_\beta(t+\pi/n)| \le C(\beta)/(nt^{2+\beta}), \quad \frac{\pi}{n}\le t\le\pi. \tag{15}$$

(6) and (15) yield $\|I(m,n)\|_{L^q} \le C(\alpha,\beta)\lambda(n,\beta)\omega_2(1/n;f)_{L^q}$ and hence

$$\|P_{mn}^{(3)}(x,y;f)\|_{L^q} \le C(\alpha,\beta)\lambda(n,\beta)\omega_2(1/n;f)_{L^q}. \tag{16}$$

Analogously

$$\|P_{mn}^{(6)}(x,y;f)\|_{L^q} \le C(\alpha,\beta)\lambda(m,\alpha)\omega_1(1/m;f)_{L^q}. \tag{17}$$

Furthermore, using (6),(7) and (9), we obtain

$$\begin{gathered}\|P_{mn}^{(k)}(x,y;f)\|_{L^q} \le C(\alpha,\beta)\big\{\omega_1(1/m;f)_{L^q}+\omega_2(1/n;f)_{L^q}\big\}\\ (k=4,7,16).\end{gathered} \tag{18}$$

Analogously, taking into account that

$$P_{mn}^{(9)}(x,y;f) = -\frac{1}{\pi}\int_{\pi/n}^{\pi}\big[\bar{f}_m^{(1)}(x,y+t)-\bar{f}_m^{(1)}(x,y-t)\big]H_{n,1}^{\beta}(t)\,dt,$$
$$P_{mn}^{(11)}(x,y;f) = -\frac{1}{\pi}\int_{\pi/m}^{\pi}\big[\bar{f}_n^{(2)}(x+s,y)-\bar{f}_n^{(2)}(x-s,y)\big]H_{m,1}^{\alpha}(s)\,ds,$$

we can prove

$$\|P_{mn}^{(9)}(x,y;f)\|_{L^q} \le C(\alpha,\beta)\lambda(n,\beta)\omega_2(1/n;\bar{f}_m^{(1)})_{L^q}; \tag{19}$$
$$\|P_{mn}^{(11)}(x,y;f)\|_{L^q} \le C(\alpha,\beta)\lambda(m,\alpha)\omega_1(1/m;\bar{f}_n^{(2)})_{L^q}. \tag{20}$$

Using the same arguments and applying (6), (7) and (9), we can prove

$$\|P_{mn}^{(10)}(x,y;f)\|_{L^q} \le C(\alpha,\beta)\omega_2(1/n;\bar{f}_m^{(1)})_{L^q}; \tag{21}$$

$$\|P_{mn}^{(14)}(x,y;f)\|_{L^q} \le C(\alpha,\beta)\omega_1(1/m;\bar{f}_n^{(2)})_{L^q}. \tag{22}$$

Now we observe that the following lemma holds true (see [1], p.160, Lemma 10):

Lemma 1. *Let* $f \in L^q([-\pi;\pi]^2)$, $1 \le q \le +\infty$, *and*

$$\begin{aligned}\phi_{x,y}(s,t;f) &= f(x+s,y+t)+f(x-s,y+t)+f(x+s,y-t)+\\ &+f(x-s,y-t)-4f(x,y).\end{aligned} \tag{23}$$

Then

$$\begin{aligned}&\left\|m^{-\alpha}n^{-\beta}\int_{\pi/m}^{\pi}\int_{\pi/n}^{\pi}\phi_{x,y}(s,t;f)\omega_\alpha(s)\omega_\beta(t)\ \begin{matrix}\sin\\ \cos\end{matrix}\ ms\ \begin{matrix}\sin\\ \cos\end{matrix}\ nt\,dt\,ds\right\|_{L^q} =\\ &= O\big\{\lambda(m,\alpha)\lambda(n,\beta)\omega(1/m,1/n;f)_{L^q}+\lambda(m,\alpha)\omega_1(1/m;f)_{L^q}+\\ &+\lambda(n,\beta)\omega_2(1/n;f)_{L^q}\big\}.\end{aligned}$$

There is another lemma in [1] (see p.171, Lemma 11), which can be corrected by means of (7) as follows:

Lemma 2. *For* $f \in L^q([-\pi;\pi]^2)$, $1 \le q \le +\infty$, *we have*

$$\begin{aligned}&\left\|m^{-\alpha}\int_{\pi/m}^{\pi}\int_{\pi/n}^{\pi}\phi_{x,y}(s,t;f)\omega_\alpha(s)H_{n,2}^{\beta}(t)\ \begin{matrix}\sin\\ \cos\end{matrix}\ ms\,dt\,ds\right\|_{L^q} =\\ &= O\big\{\lambda(m,\alpha)\omega_1(1/m;f)_{L^q}+\omega_2(1/n;f)_{L^q}\big\};\\ &\left\|n^{-\beta}\int_{\pi/m}^{\pi}\int_{\pi/n}^{\pi}\phi_{x,y}(s,t;f)\omega_\beta(t)H_{m,2}^{\alpha}(s)\ \begin{matrix}\sin\\ \cos\end{matrix}\ nt\,dt\,ds\right\|_{L^q} =\\ &= O\big\{\lambda(n,\beta)\omega_2(1/n;f)_{L^q}+\omega_1(1/m;f)_{L^q}\big\}.\end{aligned}$$

The lemmas and (3), (5), (7) and (23) yield

$$\begin{aligned}&\big\|P_{mn}^{(12)}(x,y;f)+P_{mn}^{(13)}(x,y;f)+P_{mn}^{(15)}(x,y;f)\big\|_{L^q} \le\\ &\le C(\alpha,\beta)\big\{\lambda(m,\alpha)\lambda(n,\beta)\omega(1/m,1/n;f)_{L^q}+\lambda(m,\alpha)\omega_1(1/m;f)_{L^q}+\\ &+\lambda(n,\beta)\omega_2(1/n;f)_{L^q}\big\}.\end{aligned} \tag{24}$$

Finally, (2)–(24) yield

$$\begin{aligned}&\big\|\bar{\sigma}_{mn}^{\alpha,\beta}(x,y;f)-\bar{f}_{mn}(x,y)\big\|_{L^q} \le\\ &\le C(\alpha,\beta)\big\{\lambda(m,\alpha)\lambda(n,\beta)\omega(1/m,1/n;f)_{L^q}+\lambda(m,\alpha)\omega_1(1/m;f)_{L^q}+\\ &+\lambda(n,\beta)\omega_2(1/n;f)_{L^q}+\lambda(m,\alpha)\omega_1(1/m;\bar{f}_n^{(2)})_{L^q}+\\ &+\lambda(n,\beta)\omega_2(1/n;\bar{f}_m^{(1)})_{L^q}\big\},\end{aligned} \tag{25}$$

which is the formula (1) in the case $n=2$. □

Corollary. *If $f \in C([-\pi;\pi]^n)$ $(n \geq 2)$ and*

$$\omega_i(\delta; f)_C = o\big(1/\ln^{n-1}(1/\delta)\big) \quad (i = \overline{1,n}) \tag{26}$$

as $\delta \to 0+$, then $\lim\limits_{\mathbf{m}_\lambda \to \infty} \|\bar{\sigma}^{\boldsymbol{\alpha}}_{\mathbf{m}}(\mathbf{x}; f) - \bar{f}_{\mathbf{m}}(\mathbf{x})\|_C = 0$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\alpha_i > 0$ $(i = \overline{1,n})$, $\lambda \geq 1$.

Now we will prove that the m_λ-summability in the corollary is essential. Namely, the following result holds true:

Theorem 2. *There exists a function $f \in C([-\pi;\pi]^n)$ which satisfies (26) and*

$$\overline{\lim_{\mathbf{m} \to \infty}} |\bar{\sigma}^{\boldsymbol{\alpha}}_{\mathbf{m}}(\mathbf{0}; f) - \bar{f}_{\mathbf{m}}(\mathbf{0})| = +\infty, \tag{27}$$

$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\alpha_i > 0$ $(i = \overline{1,n})$, $\mathbf{0} = (0, 0, \dots, 0)$.

Proof. We will prove the theorem for $n = 2$, this case being quite typical. First, let $\alpha, \beta \in (0;1)$. We set

$$m_k = 2^{2^{2^k}}, \quad n_k = m_k^{m_k^{m_k^{m_k}}}, \quad (k \in \mathbb{N}); \tag{28}$$

$$\ln^{(2)} x = \ln\ln x, \quad \ln^{(k)} x = \ln\ln^{(k-1)} x, \quad k \in \mathbb{N}, \quad k \geq 3;$$

$$g_k(x) = \begin{cases} \frac{1}{\ln(1/(x - \pi/(6m_k)))\ln^{(2)}(1/(x - \pi/(6m_k)))}, & \frac{\pi}{6m_k} < x \leq \frac{11\pi}{60m_k}, \\ \frac{1}{\ln(1/(\pi/(5m_k) - x))\ln^{(2)}(1/(\pi/(5m_k) - x))}, & \frac{11\pi}{60m_k} \leq x < \frac{\pi}{5m_k}, \\ 0, & x \in (0;\pi) \setminus \left(\frac{\pi}{6m_k}; \frac{\pi}{5m_k}\right). \end{cases}$$

Furthermore,

$$h(x) = \sum_{k=1}^{\infty} g_k(x);$$

$$p(y) = \begin{cases} \frac{1}{\ln(2\pi/y)\ln^{(2)}(2\pi/y)}, & y \in (0; \frac{\pi}{2}], \\ \frac{1}{\ln(2\pi/(\pi - y))\ln^{(2)}(2\pi/(\pi - y))}, & y \in [\frac{\pi}{2}; \pi); \end{cases}$$

$$f(x,y) = \begin{cases} h(x)p(y), & (x,y) \in (0;\pi)^2, \\ 0, & (x,y) \in [-\pi;\pi]^2 \setminus (0;\pi)^2. \end{cases}$$

Finally, outside the square $[-\pi;\pi]^2$, we extend the function f by periodicity with the period 2π in each variable. It is easy to see that f satisfies (26).

From now on we set $m = 2m_k$, $n = n_k + 1$. We have

$$\bar{\sigma}^{\alpha,\beta}_{mn}(0,0;f) - \bar{f}_{mn}(0,0) = 1/\pi^2 \int_0^{\pi}\int_0^{\pi} f(x,y)\bar{K}^{\alpha}_m(x)\bar{K}^{\beta}_n(y)\,dy\,dx -$$
$$-1/(4\pi^2)\int_{1/m}^{\pi}\int_{1/n}^{\pi} f(x,y)\operatorname{ctg}\frac{x}{2}\operatorname{ctg}\frac{y}{2}\,dy\,dx =$$
$$= 1/\pi^2\int_0^{1/m}\int_0^{1/n} f(x,y)\bar{K}^{\alpha}_m(x)\bar{K}^{\beta}_n(y)\,dy\,dx +$$
$$+1/\pi^2\int_0^{1/m}\int_{1/n}^{\pi} f(x,y)\bar{K}^{\alpha}_m(x)\bar{K}^{\beta}_n(y)\,dy\,dx +$$
$$+1/\pi^2\int_{1/m}^{\pi}\int_0^{1/n} f(x,y)\bar{K}^{\alpha}_m(x)\bar{K}^{\beta}_n(y)\,dy\,dx +$$
$$+1/\pi^2\int_{1/m}^{\pi}\int_{1/n}^{\pi} f(x,y)\big(\bar{K}^{\alpha}_m(x)\bar{K}^{\beta}_n(y) - 1/4\operatorname{ctg}\frac{x}{2}\operatorname{ctg}\frac{y}{2}\big)\,dy\,dx =$$
$$= \sum_{j=1}^{4} R_j(m,n). \tag{29}$$

Obviously,

$$R_1(m,n) = o(1) \quad (m,n\to\infty). \tag{30}$$

Then,

$$R_2(m,n) = 1/\pi^2\int_0^{1/m}\int_{1/n}^{\pi} f(x,y)\bar{K}^{\alpha}_m(x)\,\frac{1}{2}\operatorname{ctg}\frac{y}{2}\,dy\,dx -$$
$$- 1/\pi^2\int_0^{1/m}\int_{1/n}^{\pi} f(x,y)\bar{K}^{\alpha}_m(x)H^{\beta}_n(y)\,dy\,dx =$$
$$= R_2'(m,n) + R_2''(m,n), \tag{31}$$

where

$$H^{\alpha}_n(t) = \frac{1}{A^{\alpha}_n}\sum_{\nu=0}^{n} A^{\alpha-1}_{\nu}\,\frac{\cos(\nu+1/2)t}{2\sin(t/2)}. \tag{32}$$

The following estimates hold true (see [2], (5.12)):

$$|\bar{K}^{\alpha}_n(t)| \le n, \quad |t|\le\pi; \tag{33}$$
$$|H^{\alpha}_n(t)| \le C(\alpha)n^{-\alpha}t^{-(\alpha+1)}, \quad 1/n\le|t|\le\pi. \tag{34}$$

Now we have

$$\int_{1/n}^{\pi} p(y)\,\frac{1}{2}\operatorname{ctg}\frac{y}{2}\,dy = \int_{1/n}^{\pi/2} p(y)\,\frac{1}{2}\operatorname{ctg}\frac{y}{2}\,dy + \\ + \int_{\pi/2}^{\pi} p(y)\,\frac{1}{2}\operatorname{ctg}\frac{y}{2}\,dy = U_1(n) + U_2(n). \tag{35}$$

Now,

$$U_1(n) = -C_1 \int_{1/n}^{\pi/2} \frac{d(2\pi y)}{2\pi y \ln(2\pi y) \ln|\ln(2\pi y)|} \leq \\ \leq C_2 \ln^{(3)} n \leq C_3 m_k \ln m_k; \tag{36}$$

$$|U_2(n)| \leq M. \tag{37}$$

(31) and (35)–(37) yield

$$|R_2'(m,n)| \leq \frac{C}{\ln m_{k+1} \ln^{(2)} m_{k+1}}\, m_k \ln m_k. \tag{38}$$

As to $R_2''(m,n)$, we have

$$|R_2''(m,n)| \leq \frac{C(\beta)}{\ln m_{k+1} \ln^{(2)} m_{k+1}} \int_0^{1/m} \int_{1/n}^{\pi} \frac{m}{n^\beta y^{\beta+1}}\,dy\,dx \leq \\ \leq \frac{C(\beta)}{\ln m_{k+1} \ln^{(2)} m_{k+1}}. \tag{39}$$

(31), (38) and (39) yield

$$|R_2(m,n)| \leq \frac{C(\beta)}{\ln m_{m+1} \ln^{(2)} m_{k+1}} m_k \ln m_k. \tag{40}$$

Analogously

$$R_3(m,n) = R_3'(m,n) + R_3''(m,n). \tag{41}$$

Now,

$$|R_3'(m,n)| \leq \frac{C}{\ln n \ln^{(2)} n}\, \ln m, \tag{42}$$

$$|R_3''(m,n)| \leq \frac{C(\alpha)}{\ln n \ln^{(2)} n}. \tag{43}$$

(41)–(43) yield

$$|R_3(m,n)| = o(1) \quad (m,n \to \infty). \tag{44}$$

Now let us consider $R_4(m,n)$. We break it into 4 parts as follows

$$R_4(m,n) = \Big(\int_{1/m}^{1/m^\tau} \int_{1/n}^{1/n^\tau} + \int_{1/m}^{1/m^\tau} \int_{1/n^\tau}^{\pi} + \int_{1/m^\tau}^{\pi} \int_{1/n}^{1/n^\tau} + \\ + \int_{1/m^\tau}^{\pi} \int_{1/n^\tau}^{\pi} \Big) 1/\pi^2 f(x,y) \big(\bar{K}_m^\alpha(x) \bar{K}_n^\beta(y) - \\ - \frac{1}{4} \operatorname{ctg} \frac{x}{2} \operatorname{ctg} \frac{y}{2}\big)\, dy\, dx = \sum_{j=1}^{4} I_j(m,n), \tag{45}$$

where $1/2 \le \tau < 1$.

We have

$$|I_4(m,n)| \le \\ \le C(\alpha,\beta) \int_{1/m^\tau}^{\pi} \int_{1/n^\tau}^{\pi} \Big(\frac{1}{m^\alpha x^{\alpha+1} y} + \frac{1}{n^\beta y^{\beta+1} x} + \frac{1}{m^\alpha n^\beta x^{\alpha+1} y^{\beta+1}} \Big)\, dy\, dx.$$

It is easy to see that

$$|I_4(m,n)| = o(1) \quad (m,n \to \infty). \tag{46}$$

Now we estimate $I_2(m,n)$.

$$I_2(m,n) = \sum_{k=1}^{3} Q_k(m,n). \tag{47}$$

$$|Q_2(m,n)| = \Big| 1/\pi^2 \int_{1/m}^{1/m^\tau} \int_{1/n^\tau}^{\pi} f(x,y) \frac{-H_n^\beta(y)}{2\operatorname{tg}(x/2)}\, dy\, dx \Big| \le \\ \le C(\beta) \max |f| \ln m \frac{n^{\beta\tau}}{n^\beta}$$

and

$$|Q_2(m,n)| = o(1) \quad (m,n \to \infty); \tag{48}$$

$$|Q_3(m,n)| = \Big| 1/\pi^2 \int_{1/m}^{1/m^\tau} \int_{1/n^\tau}^{\pi} f(x,y) H_m^\alpha(x) H_n^\beta(y)\, dy\, dx \Big| = o(1) \\ (m,n \to \infty). \tag{49}$$

Next we will show that $|Q_1(m,n)| \to +\infty$ as $m,n \to \infty$.

We have $\frac{1}{m} < \frac{\pi}{6m_k} < \frac{\pi}{5m_k} < \frac{1}{m^\tau}$. Therefore

$$Q_1(m,n) = 1/\pi^2 \int_{\pi/6m_k}^{\pi/5m_k} h(x)(-H_m^\alpha(x))\, dx \times$$

$$\times \int_{1/n^\tau}^{\pi} p(y) \frac{1}{2\operatorname{tg}(y/2)}\, dy = Q_1^{(1)}(m) Q_1^{(2)}(n). \tag{50}$$

Since $m = 2m_k$, we have $\cos(i+1/2)x > 0$ for $i = 0, 1, \dots, m$ and $x \in [\frac{\pi}{6m_k}; \frac{\pi}{5m_k}]$. Hence $(-H_m^\alpha(x)) < 0$ (see (32)). Therefore we have

$$\begin{aligned}|Q_1^{(1)}(m)| = 1/\pi^2 \int_{\pi/6m_k}^{\pi/5m_k} H_m^\alpha(x) h(x)\, dx \geq \\ \geq \frac{C}{A_m^\alpha} \int_{21\pi/120m_k}^{23\pi/120m_k} \sum_{i=0}^{m} A_i^{\alpha-1} \frac{\cos\frac{9\pi}{20}}{x} h(x)\, dx \geq \\ \geq \frac{C}{A_m^\alpha} A_m^\alpha \frac{1}{\ln m_k \ln^{(2)} m_k} \int_{21\pi/120m_k}^{23\pi/120m_k} \frac{dx}{x} \geq \frac{C}{\ln m_k \ln^{(2)} m_k}.\end{aligned} \tag{51}$$

As to $Q_1^{(2)}(n)$, we have (analogously to (36))

$$Q_1^{(2)}(n) \geq C m_k \ln m_k. \tag{52}$$

(50)–(52) imply

$$|Q_1(m,n)| \geq \frac{C}{\ln m_k \ln^{(2)} m_k} m_k \ln m_k. \tag{53}$$

(28), (40) and (53) yield

$$|Q_1(m,n)| - R_2(m,n) \to +\infty \tag{54}$$

as $m, n \to \infty$ (for $m = 2m_k$, $n = n_k + 1$).

From (47)–(49) and (54) we obtain

$$|I_2(m,n)| - R_2(m,n) \to +\infty \quad (m, n \to \infty). \tag{55}$$

Now let us consider $I_3(m,n)$. As in the case of $I_2(m,n)$, we have

$$I_3(m,n) = \sum_{\nu=1}^{3} J_\nu(m,n). \tag{56}$$

Then

$$\begin{aligned}|J_1(m,n)| = 1/\pi^2 \Big| \int_{1/m^\tau}^{\pi} h(x)(-H_m^\alpha(x))\, dx \int_{1/n}^{1/n^\tau} \frac{p(y)}{2\operatorname{tg}(y/2)}\, dy \Big| \leq \\ \leq C(\alpha) \frac{m^{\alpha\tau}}{m^\alpha} (\ln^{(3)} n^\tau - \ln^{(3)} n) = \frac{C(\alpha)}{m^{\alpha(1-\tau)}} \ln \frac{\ln^{(2)} n^\tau}{\ln^{(2)} n};\end{aligned} \tag{57}$$

$$
\begin{aligned}
|J_2(m,n)| &= \left|1/\pi^2 \int_{1/m^\tau}^{\pi} \frac{h(x)}{2\,\mathrm{tg}(x/2)}\,dx \int_{1/n}^{1/n^\tau} p(y)(-H_n^\beta(y))\,dy\right| \leq \\
&\leq \frac{C(\beta)\ln m}{\ln n \ln^{(2)} n} \int_{1/n}^{1/n^\tau} \frac{dy}{n^\beta y^{\beta+1}}; \qquad (58)\\
|J_3(m,n)| &= \left|1/\pi^2 \int_{1/m^\tau}^{\pi} \int_{1/n}^{1/n^\tau} h(x)p(y)H_m^\alpha(x)H_n^\beta(y)\,dy\,dx\right| \leq \\
&\leq C(\alpha,\beta)\frac{m^{\alpha\tau}}{m^\alpha}\frac{n^\beta}{n^\beta}. \qquad (59)
\end{aligned}
$$

(56)–(59) yield

$$
I_3(m,n) = o(1) \quad (m,n\to\infty). \qquad (60)
$$

Now we consider $I_1(m,n)$. As above,

$$
I_1(m,n) = \sum_{j=1}^{3} T_j(m,n). \qquad (61)
$$

We have

$$
\begin{aligned}
|T_1(m,n)| &= 1/\pi^2\left|\int_{1/m}^{1/m^\tau}\int_{1/n}^{1/n^\tau} h(x)p(y)(-H_m^\alpha(x))\frac{1}{2}\,\mathrm{ctg}\,\frac{y}{2}\,dy\,dx\right| \leq \\
&\leq \frac{C_1(\alpha)}{\ln m_k \ln^{(2)} m_k}\,\frac{C_2(\alpha)}{\ln n_k \ln^{(2)} n_k}\,\frac{m^\alpha}{m^\alpha}\,\ln n; \qquad (62)\\
|T_2(m,n)| &= 1/\pi^2\left|\int_{1/m}^{1/m^\tau}\int_{1/n}^{1/n^\tau} h(x)p(y)(-H_n^\beta(y))\frac{1}{2}\,\mathrm{ctg}\,\frac{x}{2}\,dy\,dx\right| \leq \\
&\leq \frac{C_1(\beta)}{\ln m_k \ln^{(2)} m_k}\,\frac{C_2(\beta)}{\ln n_k \ln^{(2)} n_k}\,\frac{n^\beta}{n^\beta}\,\ln m; \qquad (63)\\
|T_3(m,n)| &= 1/\pi^2\left|\int_{1/m}^{1/m^\tau}\int_{1/n}^{1/n^\tau} h(x)p(y)H_m^\alpha(x)H_n^\beta(y)\,dy\,dx\right| \leq \\
&\leq \frac{C_1(\alpha,\beta)}{\ln m_k \ln^{(2)} m_k}\,\frac{C_2(\alpha,\beta)}{\ln n_k \ln^{(2)} n_k}\,\frac{m^\alpha n^\beta}{m^\alpha n^\beta}. \qquad (64)
\end{aligned}
$$

(61)–(64) yield

$$
I_1(m,n) = o(1) \quad (m,n\to\infty) \qquad (65)
$$

(we remind once more that $m = 2m_k$, $n = n_k + 1$).

Finally, (29), (30), (44)–(46), (55), (60) and (65) prove the Theorem 2 in the case $n = 2$ and $\alpha,\beta \in (0;1)$.

For $\alpha = 1$ we have

$$H_n^1(t) = \frac{\sin(n+1)t}{(n+1)(2\sin\frac{t}{2})^2} \tag{66}$$

and an estimate analogous to (34) holds true.

Now we consider the case when $\alpha > 1$. Using the method represented in [4], p.507, we obtain

$$H_n^\alpha(t) = \frac{1}{A_n^\alpha 2\sin\frac{t}{2}}\mathrm{Re}\Big\{\frac{e^{i(n+1/2)t}}{(1-e^{-it})^\alpha} - e^{-\frac{1}{2}it}\sum_{m=1}^{d} A_n^{\alpha-m}(1-e^{-it})^{-m} - \\ - \sum_{m=n+1}^{\infty} A_m^{\alpha-d-1}e^{-i(m-n-1/2)t}(1-e^{-it})^{-d}\Big\}, \tag{67}$$

where $d = [\alpha]$.

Taking the real part we obtain that the first term of the finite sum is 0. Therefore if $[\alpha] = 1$ we apply once more the Abel transformation to the infinite sum in (67) and obtain

$$H_n^\alpha(t) = \frac{\cos((n+\frac{1}{2}+\frac{\alpha}{2})t - \frac{\pi\alpha}{2})}{A_n^\alpha(2\sin\frac{t}{2})^{1+\alpha}} - \frac{(1-\alpha)\alpha\cos(t/2)}{8(n+1)(n+\alpha)(\sin\frac{t}{2})^3} - \\ - \frac{1}{A_n^\alpha}\sum_{m=1}^{\infty} A_{m+n+1}^{\alpha-3}\frac{\sin(m-1)t}{(2\sin\frac{t}{2})^3}. \tag{68}$$

Then, again, we have an estimate analogous to (34), which enables us to fulfil the proof. Namely,

$$|H_n^\alpha(t)| \le \frac{C_1(\alpha)}{n^\alpha t^{\alpha+1}} + \frac{C_2(\alpha)}{n^2 t^3}. \tag{69}$$

If $[\alpha] = 2$ without further transformation we obtain

$$H_n^\alpha(t) = \frac{\cos((n+\frac{1}{2}+\frac{\alpha}{2})t - \frac{\pi\alpha}{2})}{A_n^\alpha(2\sin\frac{t}{2})^{1+\alpha}} + \frac{(\alpha-1)\alpha\cos(t/2)}{8(n+\alpha-1)(n+\alpha)(\sin\frac{t}{2})^3} + \\ + \frac{1}{A_n^\alpha}\sum_{m=1}^{\infty} A_{m+n}^{\alpha-3}\frac{\cos(m-3/2)t}{(2\sin\frac{t}{2})^3} \tag{70}$$

and, again, (69) holds true.

Analogous equations may be obtained if $[\alpha] \ge 3$.

Now, when $\alpha = 1$ and $\beta = 1$, we use (66). If $\alpha = 1$ and $\beta < 1$ (or vice versa), we use (34) and (66). If $\alpha = 1$ and $\beta > 1$ (or vice versa), we use (66) and (67) (for the corresponding d). If $\alpha > 1$ and $\beta < 1$ (or vice versa), we use again (67) (for the corresponding d) and (34).

In the n-dimensional case we define f as follows. We set

$$m_k = 2^{2^{2^k}}, \quad (k \in \mathbb{N});$$

$$g_k(x) = \begin{cases} \frac{1}{\ln^{n-1}(1/(x-\pi/(6m_k)))\ln^{(n)}(1/(x-\pi/(6m_k)))}, & \frac{\pi}{6m_k} < x \le \frac{11\pi}{60m_k}, \\ \frac{1}{\ln^{n-1}(1/(\pi/(5m_k)-x))\ln^{(n)}(1/(\pi/(5m_k)-x))}, & \frac{11\pi}{60m_k} \le x < \frac{\pi}{5m_k}, \\ 0, & x \in (0;\pi)\setminus\left(\frac{\pi}{6m_k};\frac{\pi}{5m_k}\right), \end{cases}$$

Again

$$h(x) = \sum_{k=1}^{\infty} g_k(x).$$

Then,

$$p(x_2,\dots,x_n) = \\ = \frac{1}{\ln^{n-1}\frac{(2\pi)^{n-1}}{\prod_{i=2}^{n}(\pi/2-|\pi/2-x_i|)}\ln^{(n)}\frac{(2\pi)^{n-1}}{\prod_{i=2}^{n}(\pi/2-|\pi/2-x_i|)}} \tag{71}$$

for $(x_2,\dots,x_n) \in (0;\pi)^{n-1}$.

And, finally

$$f(x_1,\dots,x_n) = \begin{cases} h(x_1)p(x_2,\dots,x_n), & (x_1,\dots,x_n) \in (0;\pi)^n, \\ 0, & (x_1,\dots,x_n) \in [-\pi;\pi]^n\setminus(0;\pi)^n. \end{cases}$$

Outside $[-\pi;\pi]^n$ we extend the function f by periodicity with the period 2π in each variable.

We observe, that functions of the $p(x_2,\dots,x_n)$-type were for the first time introduced and applied in the works of L.Zhizhiashvili (see [1], [5]). □

Remark 1. *For the function $f(x_1,\dots,x_n)$ a stronger condition than* (26) *holds true, namely,*

$$\omega_i(\delta;f)_C \le \frac{C(f,n)}{\ln^{n-1}(1/\delta)\ln^{(n)}(1/\delta)}, \quad i = \overline{1,n}.$$

Remark 2. *Results analogous to Theorem* 1*, the corollary and Theorem* 2 *hold true for the n-multiple Abel–Poisson summability method.*

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