

ON SOME PROPERTIES OF MULTIPLE CONJUGATE  
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ABSTRACT. We have obtained the estimate in the terms of partial and mixed moduli of continuity of deviation of Cesàro  $(C, \alpha)$  means  $(\alpha = (\alpha_1, \dots, \alpha_n), \alpha_i \in \mathbb{R}, \alpha_i > -1, i = \overline{1, n})$  of the sequence of rectangular partial sums of  $n$ -multiple  $(n > 1)$  conjugate trigonometric series from  $n$ -multiple truncated conjugate function. This estimate implies the result on the  $m_\lambda$ -convergence  $(\lambda \geq 1)$  of  $(C, \alpha)$  means  $(\alpha_i > 0, i = \overline{1, n})$ , provided that the essential conditions are imposed on the partial moduli of continuity. Finally, it is shown, that the  $m_\lambda$ -convergence cannot be replaced by ordinary convergence.

1. Let  $f \in L([-\pi; \pi]^n)$ ,  $n \in \mathbb{N}$ ,  $n > 1$ , be a function,  $2\pi$ -periodic in each variable,  $\sigma_n[f]$  its  $n$ -multiple trigonometric Fourier series, and  $\bar{\sigma}_n[f]$  its conjugate series with respect to  $n$  variables (see, e.g., [1]).

We set

$$\begin{aligned} \mathbf{m} &= (m_1, \dots, m_n) \quad (m_i \in \mathbb{N}, i = \overline{1, n}); \\ \mathbf{x} &= (x_1, \dots, x_n) \quad (x_i \in \mathbb{R}, i = \overline{1, n}); \\ \boldsymbol{\alpha} &= (\alpha_1, \dots, \alpha_n) \quad (\alpha_i \in \mathbb{R}, \alpha_i > -1, i = \overline{1, n}); \quad \mathbf{M} = \{1, 2, \dots, n\}. \end{aligned}$$

By  $M_j$  we denote the set of all subsets of  $M$  with  $j$  elements, by  $M^{(A)}$  a set  $M \setminus A$  ( $A \subset M$ ), and by  $M_j^{(A)}$  the set of all subsets of  $M^{(A)}$  with  $j$  elements.

For  $B = \{i_1, \dots, i_k\} \subset M$  we define  $m(B) = \{1/m_{i_1}, \dots, 1/m_{i_k}\}$  and the

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truncated conjugate function with respect to the corresponding variables

$$\begin{aligned} & \bar{f}_{m(B)}(\mathbf{x}) = \\ &= \frac{1}{(-2\pi)^k} \int_{1/m_{i_1}}^{\pi} \cdots \int_{1/m_{i_k}}^{\pi} \left( \Delta_{k,s_{i_k}} \left( \Delta_{k-1,s_{i_{k-1}}} \left( \cdots \left( \Delta_{1,s_{i_1}}(f; \mathbf{x}) \right) \cdots \right) \right) \right) \times \\ & \quad \times \prod_{j=1}^k \operatorname{ctg} \frac{s_{i_j}}{2} ds_{i_1} \cdots ds_{i_k}, \quad \bar{f}_{\mathbf{m}}(\mathbf{x}) = \bar{f}_{m(M)}(\mathbf{x}), \end{aligned}$$

where

$$\begin{aligned} \Delta_{i,h}(f, \mathbf{x}) &= f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - \\ & \quad - f(x_1, \dots, x_{i-1}, x_i - h, x_{i+1}, \dots, x_n), \quad i = \overline{1, n}. \end{aligned}$$

For  $f \in L^q([-\pi; \pi]^n)$ ,  $1 \leq q \leq +\infty$  ( $L^\infty = C$ ), we consider its mixed modulus of continuity

$$\begin{aligned} & \omega_B(m(B); f)_{L^q} = \\ &= \sup_{|h_1| < 1/m_{i_1}, \dots, |h_k| < 1/m_{i_k}} \left\| \Delta_k^{h_k} \left( \Delta_{k-1}^{h_{k-1}} \left( \cdots \left( \Delta_1^{h_1}(f; \mathbf{x}) \right) \cdots \right) \right) \right\|_{L^q}, \end{aligned}$$

where  $\Delta_i^h(f; \mathbf{x}) = f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_n)$ .

Let  $\bar{\sigma}_{\mathbf{m}}^\alpha(\mathbf{x}; f)$  be Cesáro means of  $\bar{\sigma}_n[f]$ .

In the sequel by  $A, B, A_1, B_1, C(\alpha), C(\beta), C(\alpha, \beta)$ , etc. we will denote, in general, different positive constants.

Finally, we set

$$\lambda(k, \beta) = \begin{cases} k^{-\beta}, & -1 < \beta < 0, \\ \ln(k + 1), & \beta = 0, \\ 1, & \beta > 0, \quad (k = 1, 2, \dots). \end{cases}$$

In the present paper we give the estimate of the deviation of  $n$ -multiple Cesáro means of the sequence of rectangular partial sums of  $\bar{\sigma}_n[f]$  from  $\bar{f}_{\mathbf{m}}$  in the norm of  $L^q$ ,  $q \in [1; +\infty]$  ( $L^\infty = C$ ), in terms of partial and mixed moduli of continuity of  $f$ . This result generalizes the corresponding result of L.Zhizhiashvili (see [1]).

From this estimate ensues the result on the Cesáro summability of the sequence of rectangular partial sums of  $\bar{\sigma}_n[f]$  and then the correctness of this result is shown.

**2.** The following is true:

**Theorem 1.** *If  $f \in L^q([-\pi; \pi]^n)$ ,  $q \in [1; +\infty]$  ( $L^\infty = C$ ), then*

$$\begin{aligned} & \|\bar{\sigma}_{\mathbf{m}}^\alpha(\mathbf{x}; f) - \bar{f}_{\mathbf{m}}(\mathbf{x})\|_{L^q} \leq C(\alpha) \left\{ \sum_{k=1}^{n-1} \sum_{B \in M_k} \omega_B(m(B); f)_{L^q} \times \right. \\ & \times \prod_{\substack{j=1 \\ \{i_1, \dots, i_k\} \in M_k}}^k \lambda(m_{i_j}, \alpha_{i_j}) + \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \sum_{A \in M_i} \sum_{B \in M_k^{(A)}} \omega_B(m(B); \bar{f}_{m(A)})_{L^q} \times \\ & \times \left. \prod_{\substack{j=1 \\ \{t_1, \dots, t_k\} \in M_k^{(A)}}}^k \lambda(m_{t_j}, \alpha_{t_j}) + \prod_{i=1}^n \lambda(m_i, \alpha_i) \omega_M(m(M); f)_{L^q} \right\}. \quad (1) \end{aligned}$$

*Proof.* For simplicity, we will prove the theorem in the case  $n = 2$  which is typical. We will use the method of L.Zhizhiashvili ([1], p. 160-191), this method proving to be true for  $n \geq 3$ .

Let  $\omega(\delta_1, \delta_2; f)_{L^p}$  be the mixed modulus of continuity of the function  $f(x, y)$ ,  $x, y \in \mathbb{R}$ , with respect to two variables. By  $\omega_i(\delta; f)_{L^q}$  ( $i \in \mathbb{N}$ ,  $\delta \in \mathbb{R}^+$ ), as usual, we denote the modulus of continuity of  $f$  in  $L^q$  with respect to the corresponding variable. Let  $m, n \in \mathbb{N}$ ,  $\alpha, \beta \in \mathbb{R}$ ,  $\alpha, \beta > -1$ ,  $\bar{f}_{m,n}(x, y)$  be the two-dimensional truncated conjugate function and  $\bar{\sigma}_{mn}^{\alpha, \beta}(x, y; f)$  be the Cesàro means of the double conjugate series  $\bar{\sigma}_2[f]$ .

We have ([1], p.187-188):

$$\begin{aligned} \bar{\sigma}_{mn}^{\alpha, \beta}(x, y; f) &= 1/\pi^2 \int_0^{\pi/m} \int_0^{\pi/n} \psi_{x,y}(s, t; f) \bar{K}_m^\alpha(s) \bar{K}_n^\beta(t) dt ds + \\ &+ 1/(2\pi^2) \int_0^{\pi/m} \int_{\pi/n}^\pi \psi_{x,y}(s, t; f) \operatorname{ctg} \frac{t}{2} \bar{K}_m^\alpha(s) dt ds + \\ &+ 1/\pi^2 \int_0^{\pi/m} \int_{\pi/n}^\pi \psi_{x,y}(s, t; f) \bar{K}_m^\alpha(s) H_{n,1}^\beta(t) dt ds + \\ &+ 1/\pi^2 \int_0^{\pi/m} \int_{\pi/n}^\pi \psi_{x,y}(s, t; f) \bar{K}_m^\alpha(s) H_{n,2}^\beta(t) dt ds + \\ &+ 1/(2\pi^2) \int_{\pi/m}^\pi \int_0^{\pi/n} \psi_{x,y}(s, t; f) \operatorname{ctg} \frac{s}{2} \bar{K}_n^\beta(t) dt ds + \\ &+ 1/\pi^2 \int_{\pi/m}^\pi \int_0^{\pi/n} \psi_{x,y}(s, t; f) H_{m,1}^\alpha(s) \bar{K}_n^\beta(t) dt ds + \\ &+ 1/\pi^2 \int_{\pi/m}^\pi \int_0^{\pi/n} \psi_{x,y}(s, t; f) H_{m,2}^\alpha(s) \bar{K}_n^\beta(t) dt ds + \\ &+ 1/(4\pi^2) \int_{\pi/m}^\pi \int_{\pi/n}^\pi \psi_{x,y}(s, t; f) \operatorname{ctg} \frac{s}{2} \operatorname{ctg} \frac{t}{2} dt ds + \end{aligned}$$

$$\begin{aligned}
& + 1/(2\pi^2) \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \psi_{x,y}(s, t; f) \operatorname{ctg} \frac{s}{2} H_{n,1}^{\beta}(t) dt ds + \\
& + 1/(2\pi^2) \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \psi_{x,y}(s, t; f) \operatorname{ctg} \frac{s}{2} H_{n,2}^{\beta}(t) dt ds + \\
& + 1/(2\pi^2) \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \psi_{x,y}(s, t; f) \operatorname{ctg} \frac{t}{2} H_{m,1}^{\alpha}(s) dt ds + \\
& + 1/\pi^2 \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \psi_{x,y}(s, t; f) H_{m,1}^{\alpha}(s) H_{n,1}^{\beta}(t) dt ds + \\
& + 1/\pi^2 \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \psi_{x,y}(s, t; f) H_{m,1}^{\alpha}(s) H_{n,2}^{\beta}(t) dt ds + \\
& + 1/(2\pi^2) \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \psi_{x,y}(s, t; f) \operatorname{ctg} \frac{t}{2} H_{m,2}^{\alpha}(s) dt ds + \\
& + 1/\pi^2 \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \psi_{x,y}(s, t; f) H_{m,2}^{\alpha}(s) H_{n,1}^{\beta}(t) dt ds + \\
& + 1/\pi^2 \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \psi_{x,y}(s, t; f) H_{m,2}^{\alpha}(s) H_{n,2}^{\beta}(t) dt ds = \\
& = \sum_{j=1}^{16} P_{mn}^{(j)}(x, y; f), \tag{2}
\end{aligned}$$

where

$$\begin{aligned}
\psi_{x,y}(s, t; f) &= f(x + s, y + t) - f(x - s, y + t) - \\
& - f(x + s, y - t) + f(x - s, y - t) \tag{3}
\end{aligned}$$

and  $H_{m,1}^{\alpha}(s)$ ,  $H_{m,2}^{\alpha}(s)$  are the summands of the conjugate Fejér kernel  $\bar{K}_m^{\alpha}(s)$  (see [2], pp.157-160)

$$\bar{K}_m^{\alpha}(s) = \frac{1}{2} \operatorname{ctg} \frac{s}{2} + H_{m,1}^{\alpha}(s) + H_{m,2}^{\alpha}(s), \tag{4}$$

$$H_{m,1}^{\alpha}(s) = -\frac{\cos\left(\left(m + \frac{1}{2} + \frac{\alpha}{2}\right)s - \frac{\pi\alpha}{2}\right)}{A_m^{\alpha} (2 \sin \frac{s}{2})^{1+\alpha}}. \tag{5}$$

Besides,

$$|\bar{K}_m^{\alpha}(s)| \leq C(\alpha)m, \quad |s| \leq \pi; \tag{6}$$

$$|H_{m,2}^{\alpha}(s)| \leq \frac{C(\alpha)}{m^2 s^3}, \quad \pi/m \leq |s| \leq \pi, \quad m > 1. \tag{7}$$

The estimate (7) is more precise than the estimate (2.1.14) in [1] and it can be proved by arguments analogous to those in [2], pp.157-160.

From (2) we obtain

$$\begin{aligned} \bar{\sigma}_{mn}^{\alpha,\beta}(x, y; f) - \bar{f}_{mn}(x, y) &= \bar{\sigma}_{mn}^{\alpha,\beta}(x, y; f) - P_{mn}^{(8)}(x, y; f) = \\ &= \sum_{k=1}^{16}{}' P_{mn}^{(k)}(x, y; f), \end{aligned} \tag{8}$$

where ' indicates that the eighth member is omitted (the replacement of  $1/m$  ( $1/n$ ) in  $\bar{f}_{mn}(x, y)$  by  $\pi/m$  ( $\pi/n$ ) does not matter).

In the sequel we will use the inequality (see [3], p.179)

$$\left\{ \int_a^b \left| \int_c^d f(x, y) dy \right|^q dx \right\}^{1/q} \leq \int_c^d \left\{ \int_a^b |f(x, y)|^q dx \right\}^{1/q} dy, \tag{9}$$

$1 \leq q < +\infty.$

Taking into account (6) and (9), we obtain

$$\|P_{mn}^{(1)}(x, y; f)\|_{L^q} \leq C(\alpha, \beta) \{ \omega_1(1/m; f)_{L^q} + \omega_2(1/n; f)_{L^q} \}. \tag{10}$$

It is easy to see, that

$$\begin{aligned} P_{mn}^{(2)}(x, y; f) &= -\frac{1}{\pi} \int_0^{\pi/m} [\bar{f}_n^{(2)}(x+s, y) - \bar{f}_n^{(2)}(x-s, y)] \bar{K}_m^\alpha(s) ds, \\ P_{mn}^{(5)}(x, y; f) &= -\frac{1}{\pi} \int_0^{\pi/n} [\bar{f}_m^{(1)}(x, y+t) - \bar{f}_m^{(1)}(x, y-t)] \bar{K}_n^\beta(t) dt, \end{aligned}$$

where

$$\bar{f}_m^{(1)}(x, y) = -\frac{1}{2\pi} \int_{\pi/m}^\pi [f(x+s, y) - f(x-s, y)] \operatorname{ctg} \frac{s}{2} ds, \tag{11}$$

$$\bar{f}_n^{(2)}(x, y) = -\frac{1}{2\pi} \int_{\pi/n}^\pi [f(x, y+t) - f(x, y-t)] \operatorname{ctg} \frac{t}{2} dt. \tag{12}$$

Hence, using again (6) and (9), we obtain:

$$\|P_{mn}^{(2)}(x, y; f)\|_{L^q} \leq C(\alpha, \beta) \omega_1(1/m; \bar{f}_n^{(2)})_{L^q}; \tag{13}$$

$$\|P_{mn}^{(5)}(x, y; f)\|_{L^q} \leq C(\alpha, \beta) \omega_2(1/n; \bar{f}_m^{(1)})_{L^q}. \tag{14}$$

Now, it is easy to see that for estimation of  $P_{mn}^{(3)}(x, y; f)$  it suffices to estimate the integral

$$I(m, n) = n^{-\beta} \int_0^{\pi/m} \int_{\pi/n}^\pi \psi_{x,y}(s, t; f) \bar{K}_m^\alpha(s) \omega_\beta(t) \cos nt dt ds,$$

where

$$\omega_\beta(t) = \frac{\cos \frac{1+\beta}{2} t}{(\sin \frac{t}{2})^{1+\beta}}.$$

We have

$$\begin{aligned}
 2I(m, n) = & n^{-\beta} \left\{ \int_0^{\pi/m} \int_{\pi/n}^{\pi} [\psi_{x,y}(s, t; f) - \psi_{x,y}(s, t + \pi/n; f)] \times \right. \\
 & \times \bar{K}_m^\alpha(s) \omega_\beta(t) \cos nt \, dt \, ds + \int_0^{\pi/m} \int_{\pi/n}^{\pi} \psi_{x,y}(s, t + \pi/n; f) [\omega_\beta(t) - \\
 & \quad \left. - \omega_\beta(t + \pi/n)] \bar{K}_m^\alpha(s) \cos nt \, dt \, ds + \right. \\
 & + \int_0^{\pi/m} \int_{\pi - \pi/n}^{\pi} \psi_{x,y}(s, t + \pi/n; f) \bar{K}_m^\alpha(s) \omega_\beta(t + \pi/n) \cos nt \, dt \, ds - \\
 & \left. - \int_0^{\pi/m} \int_0^{\pi/n} \psi_{x,y}(s, t + \pi/n; f) \bar{K}_m^\alpha(s) \omega_\beta(t + \pi/n) \cos nt \, dt \, ds \right\}.
 \end{aligned}$$

Now we note that (see [1], p.56, (2.1.18))

$$|\omega_\beta(t) - \omega_\beta(t + \pi/n)| \leq C(\beta)/(nt^{2+\beta}), \quad \frac{\pi}{n} \leq t \leq \pi. \tag{15}$$

(6) and (15) yield  $\|I(m, n)\|_{L^q} \leq C(\alpha, \beta)\lambda(n, \beta)\omega_2(1/n; f)_{L^q}$  and hence

$$\|P_{mn}^{(3)}(x, y; f)\|_{L^q} \leq C(\alpha, \beta)\lambda(n, \beta)\omega_2(1/n; f)_{L^q}. \tag{16}$$

Analogously

$$\|P_{mn}^{(6)}(x, y; f)\|_{L^q} \leq C(\alpha, \beta)\lambda(m, \alpha)\omega_1(1/m; f)_{L^q}. \tag{17}$$

Furthermore, using (6),(7) and (9), we obtain

$$\begin{aligned}
 \|P_{mn}^{(k)}(x, y; f)\|_{L^q} \leq & C(\alpha, \beta) \{ \omega_1(1/m; f)_{L^q} + \omega_2(1/n; f)_{L^q} \} \\
 & (k = 4, 7, 16).
 \end{aligned} \tag{18}$$

Analogously, taking into account that

$$\begin{aligned}
 P_{mn}^{(9)}(x, y; f) = & -\frac{1}{\pi} \int_{\pi/n}^{\pi} [\bar{f}_m^{(1)}(x, y + t) - \bar{f}_m^{(1)}(x, y - t)] H_{n,1}^\beta(t) \, dt, \\
 P_{mn}^{(11)}(x, y; f) = & -\frac{1}{\pi} \int_{\pi/m}^{\pi} [\bar{f}_n^{(2)}(x + s, y) - \bar{f}_n^{(2)}(x - s, y)] H_{m,1}^\alpha(s) \, ds,
 \end{aligned}$$

we can prove

$$\|P_{mn}^{(9)}(x, y; f)\|_{L^q} \leq C(\alpha, \beta)\lambda(n, \beta)\omega_2(1/n; \bar{f}_m^{(1)})_{L^q}; \tag{19}$$

$$\|P_{mn}^{(11)}(x, y; f)\|_{L^q} \leq C(\alpha, \beta)\lambda(m, \alpha)\omega_1(1/m; \bar{f}_n^{(2)})_{L^q}. \tag{20}$$

Using the same arguments and applying (6), (7) and (9), we can prove

$$\|P_{mn}^{(10)}(x, y; f)\|_{L^q} \leq C(\alpha, \beta)\omega_2(1/n; \bar{f}_m^{(1)})_{L^q}; \tag{21}$$

$$\|P_{mn}^{(14)}(x, y; f)\|_{L^q} \leq C(\alpha, \beta)\omega_1(1/m; \bar{f}_n^{(2)})_{L^q}. \tag{22}$$

Now we observe that the following lemma holds true (see [1], p.160, Lemma 10):

**Lemma 1.** *Let  $f \in L^q([-\pi; \pi]^2)$ ,  $1 \leq q \leq +\infty$ , and*

$$\begin{aligned} \phi_{x,y}(s, t; f) &= f(x + s, y + t) + f(x - s, y + t) + f(x + s, y - t) + \\ &+ f(x - s, y - t) - 4f(x, y). \end{aligned} \tag{23}$$

Then

$$\begin{aligned} &\left\| m^{-\alpha} n^{-\beta} \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \phi_{x,y}(s, t; f) \omega_{\alpha}(s) \omega_{\beta}(t) \frac{\sin ms}{\cos} \frac{\sin nt}{\cos} dt ds \right\|_{L^q} = \\ &= O\{\lambda(m, \alpha)\lambda(n, \beta)\omega(1/m, 1/n; f)_{L^q} + \lambda(m, \alpha)\omega_1(1/m; f)_{L^q} + \\ &\quad + \lambda(n, \beta)\omega_2(1/n; f)_{L^q}\}. \end{aligned}$$

There is another lemma in [1] (see p.171, Lemma 11), which can be corrected by means of (7) as follows:

**Lemma 2.** *For  $f \in L^q([-\pi; \pi]^2)$ ,  $1 \leq q \leq +\infty$ , we have*

$$\begin{aligned} &\left\| m^{-\alpha} \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \phi_{x,y}(s, t; f) \omega_{\alpha}(s) H_{n,2}^{\beta}(t) \frac{\sin ms}{\cos} dt ds \right\|_{L^q} = \\ &= O\{\lambda(m, \alpha)\omega_1(1/m; f)_{L^q} + \omega_2(1/n; f)_{L^q}\}; \\ &\left\| n^{-\beta} \int_{\pi/m}^{\pi} \int_{\pi/n}^{\pi} \phi_{x,y}(s, t; f) \omega_{\beta}(t) H_{m,2}^{\alpha}(s) \frac{\sin nt}{\cos} dt ds \right\|_{L^q} = \\ &= O\{\lambda(n, \beta)\omega_2(1/n; f)_{L^q} + \omega_1(1/m; f)_{L^q}\}. \end{aligned}$$

The lemmas and (3), (5), (7) and (23) yield

$$\begin{aligned} &\|P_{mn}^{(12)}(x, y; f) + P_{mn}^{(13)}(x, y; f) + P_{mn}^{(15)}(x, y; f)\|_{L^q} \leq \\ &\leq C(\alpha, \beta)\{\lambda(m, \alpha)\lambda(n, \beta)\omega(1/m, 1/n; f)_{L^q} + \lambda(m, \alpha)\omega_1(1/m; f)_{L^q} + \\ &\quad + \lambda(n, \beta)\omega_2(1/n; f)_{L^q}\}. \end{aligned} \tag{24}$$

Finally, (2)–(24) yield

$$\begin{aligned} &\|\bar{\sigma}_{mn}^{\alpha, \beta}(x, y; f) - \bar{f}_{mn}(x, y)\|_{L^q} \leq \\ &\leq C(\alpha, \beta)\{\lambda(m, \alpha)\lambda(n, \beta)\omega(1/m, 1/n; f)_{L^q} + \lambda(m, \alpha)\omega_1(1/m; f)_{L^q} + \\ &\quad + \lambda(n, \beta)\omega_2(1/n; f)_{L^q} + \lambda(m, \alpha)\omega_1(1/m; \bar{f}_n^{(2)})_{L^q} + \\ &\quad + \lambda(n, \beta)\omega_2(1/n; \bar{f}_m^{(1)})_{L^q}\}, \end{aligned} \tag{25}$$

which is the formula (1) in the case  $n = 2$ .  $\square$

**Corollary.** *If  $f \in C([-\pi; \pi]^n)$  ( $n \geq 2$ ) and*

$$\omega_i(\delta; f)_C = o(1/\ln^{n-1}(1/\delta)) \quad (i = \overline{1, n}) \quad (26)$$

*as  $\delta \rightarrow 0+$ , then  $\lim_{\mathbf{m}_\lambda \rightarrow \infty} \|\bar{\sigma}_{\mathbf{m}}^\alpha(\mathbf{x}; f) - \bar{f}_{\mathbf{m}}(\mathbf{x})\|_C = 0$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $\alpha_i > 0$  ( $i = \overline{1, n}$ ),  $\lambda \geq 1$ .*

Now we will prove that the  $m_\lambda$ -summability in the corollary is essential. Namely, the following result holds true:

**Theorem 2.** *There exists a function  $f \in C([-\pi; \pi]^n)$  which satisfies (26) and*

$$\overline{\lim}_{\mathbf{m} \rightarrow \infty} |\bar{\sigma}_{\mathbf{m}}^\alpha(\mathbf{0}; f) - \bar{f}_{\mathbf{m}}(\mathbf{0})| = +\infty, \quad (27)$$

$\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $\alpha_i > 0$  ( $i = \overline{1, n}$ ),  $\mathbf{0} = (0, 0, \dots, 0)$ .

*Proof.* We will prove the theorem for  $n = 2$ , this case being quite typical. First, let  $\alpha, \beta \in (0; 1)$ . We set

$$\begin{aligned} m_k &= 2^{2^{2^k}}, \quad n_k = m_k^{m_k^{m_k}}, \quad (k \in \mathbb{N}); \\ \ln^{(2)} x &= \ln \ln x, \quad \ln^{(k)} x = \ln \ln^{(k-1)} x, \quad k \in \mathbb{N}, \quad k \geq 3; \\ g_k(x) &= \begin{cases} \frac{1}{\ln(1/(x-\pi/(6m_k))) \ln^{(2)}(1/(x-\pi/(6m_k)))}, & \frac{\pi}{6m_k} < x \leq \frac{11\pi}{60m_k}, \\ \frac{1}{\ln(1/(\pi/(5m_k)-x)) \ln^{(2)}(1/(\pi/(5m_k)-x))}, & \frac{11\pi}{60m_k} \leq x < \frac{\pi}{5m_k}, \\ 0, & x \in (0; \pi) \setminus \left(\frac{\pi}{6m_k}; \frac{\pi}{5m_k}\right). \end{cases} \end{aligned} \quad (28)$$

Furthermore,

$$\begin{aligned} h(x) &= \sum_{k=1}^{\infty} g_k(x); \\ p(y) &= \begin{cases} \frac{1}{\ln(2\pi/y) \ln^{(2)}(2\pi/y)}, & y \in (0; \frac{\pi}{2}], \\ \frac{1}{\ln(2\pi/(\pi-y)) \ln^{(2)}(2\pi/(\pi-y))}, & y \in [\frac{\pi}{2}; \pi); \end{cases} \\ f(x, y) &= \begin{cases} h(x)p(y), & (x, y) \in (0; \pi)^2, \\ 0, & (x, y) \in [-\pi; \pi]^2 \setminus (0; \pi)^2. \end{cases} \end{aligned}$$

Finally, outside the square  $[-\pi; \pi]^2$ , we extend the function  $f$  by periodicity with the period  $2\pi$  in each variable. It is easy to see that  $f$  satisfies (26).



From now on we set  $m = 2m_k$ ,  $n = n_k + 1$ . We have

$$\begin{aligned}
 \bar{\sigma}_{mn}^{\alpha,\beta}(0, 0; f) - \bar{f}_{mn}(0, 0) &= 1/\pi^2 \int_0^\pi \int_0^\pi f(x, y) \bar{K}_m^\alpha(x) \bar{K}_n^\beta(y) dy dx - \\
 &\quad - 1/(4\pi^2) \int_{1/m}^\pi \int_{1/n}^\pi f(x, y) \operatorname{ctg} \frac{x}{2} \operatorname{ctg} \frac{y}{2} dy dx = \\
 &= 1/\pi^2 \int_0^{1/m} \int_0^{1/n} f(x, y) \bar{K}_m^\alpha(x) \bar{K}_n^\beta(y) dy dx + \\
 &\quad + 1/\pi^2 \int_0^{1/m} \int_{1/n}^\pi f(x, y) \bar{K}_m^\alpha(x) \bar{K}_n^\beta(y) dy dx + \\
 &\quad + 1/\pi^2 \int_{1/m}^\pi \int_0^{1/n} f(x, y) \bar{K}_m^\alpha(x) \bar{K}_n^\beta(y) dy dx + \\
 &\quad + 1/\pi^2 \int_{1/m}^\pi \int_{1/n}^\pi f(x, y) (\bar{K}_m^\alpha(x) \bar{K}_n^\beta(y) - 1/4 \operatorname{ctg} \frac{x}{2} \operatorname{ctg} \frac{y}{2}) dy dx = \\
 &\quad = \sum_{j=1}^4 R_j(m, n). \tag{29}
 \end{aligned}$$

Obviously,

$$R_1(m, n) = o(1) \quad (m, n \rightarrow \infty). \tag{30}$$

Then,

$$\begin{aligned}
 R_2(m, n) &= 1/\pi^2 \int_0^{1/m} \int_{1/n}^\pi f(x, y) \bar{K}_m^\alpha(x) \frac{1}{2} \operatorname{ctg} \frac{y}{2} dy dx - \\
 &\quad - 1/\pi^2 \int_0^{1/m} \int_{1/n}^\pi f(x, y) \bar{K}_m^\alpha(x) H_n^\beta(y) dy dx = \\
 &= R_2'(m, n) + R_2''(m, n), \tag{31}
 \end{aligned}$$

where

$$H_n^\alpha(t) = \frac{1}{A_n^\alpha} \sum_{\nu=0}^n A_\nu^{\alpha-1} \frac{\cos(\nu + 1/2)t}{2 \sin(t/2)}. \tag{32}$$

The following estimates hold true (see [2], (5.12)):

$$|\bar{K}_n^\alpha(t)| \leq n, \quad |t| \leq \pi; \tag{33}$$

$$|H_n^\alpha(t)| \leq C(\alpha) n^{-\alpha} t^{-(\alpha+1)}, \quad 1/n \leq |t| \leq \pi. \tag{34}$$

Now we have

$$\begin{aligned} \int_{1/n}^{\pi} p(y) \frac{1}{2} \operatorname{ctg} \frac{y}{2} dy &= \int_{1/n}^{\pi/2} p(y) \frac{1}{2} \operatorname{ctg} \frac{y}{2} dy + \\ &+ \int_{\pi/2}^{\pi} p(y) \frac{1}{2} \operatorname{ctg} \frac{y}{2} dy = U_1(n) + U_2(n). \end{aligned} \quad (35)$$

Now,

$$\begin{aligned} U_1(n) &= -C_1 \int_{1/n}^{\pi/2} \frac{d(2\pi y)}{2\pi y \ln(2\pi y) \ln |\ln(2\pi y)|} \leq \\ &\leq C_2 \ln^{(3)} n \leq C_3 m_k \ln m_k; \end{aligned} \quad (36)$$

$$|U_2(n)| \leq M. \quad (37)$$

(31) and (35)–(37) yield

$$|R'_2(m, n)| \leq \frac{C}{\ln m_{k+1} \ln^{(2)} m_{k+1}} m_k \ln m_k. \quad (38)$$

As to  $R''_2(m, n)$ , we have

$$\begin{aligned} |R''_2(m, n)| &\leq \frac{C(\beta)}{\ln m_{k+1} \ln^{(2)} m_{k+1}} \int_0^{1/m} \int_{1/n}^{\pi} \frac{m}{n^{\beta} y^{\beta+1}} dy dx \leq \\ &\leq \frac{C(\beta)}{\ln m_{k+1} \ln^{(2)} m_{k+1}}. \end{aligned} \quad (39)$$

(31), (38) and (39) yield

$$|R_2(m, n)| \leq \frac{C(\beta)}{\ln m_{m+1} \ln^{(2)} m_{k+1}} m_k \ln m_k. \quad (40)$$

Analogously

$$R_3(m, n) = R'_3(m, n) + R''_3(m, n). \quad (41)$$

Now,

$$|R'_3(m, n)| \leq \frac{C}{\ln n \ln^{(2)} n} \ln m, \quad (42)$$

$$|R''_3(m, n)| \leq \frac{C(\alpha)}{\ln n \ln^{(2)} n}. \quad (43)$$

(41)–(43) yield

$$|R_3(m, n)| = o(1) \quad (m, n \rightarrow \infty). \quad (44)$$

Now let us consider  $R_4(m, n)$ . We break it into 4 parts as follows

$$\begin{aligned}
 R_4(m, n) = & \left( \int_{1/m}^{1/m^\tau} \int_{1/n}^{1/n^\tau} + \int_{1/m}^{1/m^\tau} \int_{1/n^\tau}^\pi + \int_{1/m^\tau}^\pi \int_{1/n}^{1/n^\tau} + \right. \\
 & \left. + \int_{1/m^\tau}^\pi \int_{1/n^\tau}^\pi \right) 1/\pi^2 f(x, y) (\bar{K}_m^\alpha(x) \bar{K}_n^\beta(y) - \\
 & - \frac{1}{4} \operatorname{ctg} \frac{x}{2} \operatorname{ctg} \frac{y}{2}) dy dx = \sum_{j=1}^4 I_j(m, n), \tag{45}
 \end{aligned}$$

where  $1/2 \leq \tau < 1$ .

We have

$$\begin{aligned}
 & |I_4(m, n)| \leq \\
 \leq & C(\alpha, \beta) \int_{1/m^\tau}^\pi \int_{1/n^\tau}^\pi \left( \frac{1}{m^\alpha x^{\alpha+1} y} + \frac{1}{n^\beta y^{\beta+1} x} + \frac{1}{m^\alpha n^\beta x^{\alpha+1} y^{\beta+1}} \right) dy dx.
 \end{aligned}$$

It is easy to see that

$$|I_4(m, n)| = o(1) \quad (m, n \rightarrow \infty). \tag{46}$$

Now we estimate  $I_2(m, n)$ .

$$\begin{aligned}
 I_2(m, n) &= \sum_{k=1}^3 Q_k(m, n). \tag{47} \\
 |Q_2(m, n)| &= \left| 1/\pi^2 \int_{1/m}^{1/m^\tau} \int_{1/n^\tau}^\pi f(x, y) \frac{-H_n^\beta(y)}{2 \operatorname{tg}(x/2)} dy dx \right| \leq \\
 &\leq C(\beta) \max |f| \ln m \frac{n^{\beta\tau}}{n^\beta}
 \end{aligned}$$

and

$$|Q_2(m, n)| = o(1) \quad (m, n \rightarrow \infty); \tag{48}$$

$$\begin{aligned}
 |Q_3(m, n)| &= \left| 1/\pi^2 \int_{1/m}^{1/m^\tau} \int_{1/n^\tau}^\pi f(x, y) H_m^\alpha(x) H_n^\beta(y) dy dx \right| = o(1) \tag{49} \\
 &(m, n \rightarrow \infty).
 \end{aligned}$$

Next we will show that  $|Q_1(m, n)| \rightarrow +\infty$  as  $m, n \rightarrow \infty$ .

We have  $\frac{1}{m} < \frac{\pi}{6m_k} < \frac{\pi}{5m_k} < \frac{1}{m^\tau}$ . Therefore

$$Q_1(m, n) = 1/\pi^2 \int_{\pi/6m_k}^{\pi/5m_k} h(x) (-H_m^\alpha(x)) dx \times$$

$$\times \int_{1/n^\tau}^\pi p(y) \frac{1}{2 \operatorname{tg}(y/2)} dy = Q_1^{(1)}(m) Q_1^{(2)}(n). \tag{50}$$

Since  $m = 2m_k$ , we have  $\cos(i + 1/2)x > 0$  for  $i = 0, 1, \dots, m$  and  $x \in [\frac{\pi}{6m_k}; \frac{\pi}{5m_k}]$ . Hence  $(-H_m^\alpha(x)) < 0$  (see (32)). Therefore we have

$$\begin{aligned} |Q_1^{(1)}(m)| &= 1/\pi^2 \int_{\pi/6m_k}^{\pi/5m_k} H_m^\alpha(x) h(x) dx \geq \\ &\geq \frac{C}{A_m^\alpha} \int_{21\pi/120m_k}^{23\pi/120m_k} \sum_{i=0}^m A_i^{\alpha-1} \frac{\cos \frac{9\pi}{20}}{x} h(x) dx \geq \\ &\geq \frac{C}{A_m^\alpha} A_m^\alpha \frac{1}{\ln m_k \ln^{(2)} m_k} \int_{21\pi/120m_k}^{23\pi/120m_k} \frac{dx}{x} \geq \frac{C}{\ln m_k \ln^{(2)} m_k}. \end{aligned} \tag{51}$$

As to  $Q_1^{(2)}(n)$ , we have (analogously to (36))

$$Q_1^{(2)}(n) \geq C m_k \ln m_k. \tag{52}$$

(50)–(52) imply

$$|Q_1(m, n)| \geq \frac{C}{\ln m_k \ln^{(2)} m_k} m_k \ln m_k. \tag{53}$$

(28), (40) and (53) yield

$$|Q_1(m, n)| - R_2(m, n) \rightarrow +\infty \tag{54}$$

as  $m, n \rightarrow \infty$  (for  $m = 2m_k, n = n_k + 1$ ).

From (47)–(49) and (54) we obtain

$$|I_2(m, n)| - R_2(m, n) \rightarrow +\infty \quad (m, n \rightarrow \infty). \tag{55}$$

Now let us consider  $I_3(m, n)$ . As in the case of  $I_2(m, n)$ , we have

$$I_3(m, n) = \sum_{\nu=1}^3 J_\nu(m, n). \tag{56}$$

Then

$$\begin{aligned} |J_1(m, n)| &= 1/\pi^2 \left| \int_{1/m^\tau}^\pi h(x) (-H_m^\alpha(x)) dx \int_{1/n}^{1/n^\tau} \frac{p(y)}{2 \operatorname{tg}(y/2)} dy \right| \leq \\ &\leq C(\alpha) \frac{m^{\alpha\tau}}{m^\alpha} (\ln^{(3)} n^\tau - \ln^{(3)} n) = \frac{C(\alpha)}{m^{\alpha(1-\tau)}} \ln \frac{\ln^{(2)} n^\tau}{\ln^{(2)} n}; \end{aligned} \tag{57}$$

$$\begin{aligned}
|J_2(m, n)| &= \left| 1/\pi^2 \int_{1/m^\tau}^\pi \frac{h(x)}{2 \operatorname{tg}(x/2)} dx \int_{1/n}^{1/n^\tau} p(y)(-H_n^\beta(y)) dy \right| \leq \\
&\leq \frac{C(\beta) \ln m}{\ln n \ln^{(2)} n} \int_{1/n}^{1/n^\tau} \frac{dy}{n^\beta y^{\beta+1}}; \tag{58}
\end{aligned}$$

$$\begin{aligned}
|J_3(m, n)| &= \left| 1/\pi^2 \int_{1/m^\tau}^\pi \int_{1/n}^{1/n^\tau} h(x)p(y)H_m^\alpha(x)H_n^\beta(y) dy dx \right| \leq \\
&\leq C(\alpha, \beta) \frac{m^{\alpha\tau} n^\beta}{m^\alpha n^\beta}. \tag{59}
\end{aligned}$$

(56)–(59) yield

$$I_3(m, n) = o(1) \quad (m, n \rightarrow \infty). \tag{60}$$

Now we consider  $I_1(m, n)$ . As above,

$$I_1(m, n) = \sum_{j=1}^3 T_j(m, n). \tag{61}$$

We have

$$\begin{aligned}
|T_1(m, n)| &= 1/\pi^2 \left| \int_{1/m}^{1/m^\tau} \int_{1/n}^{1/n^\tau} h(x)p(y)(-H_m^\alpha(x)) \frac{1}{2} \operatorname{ctg} \frac{y}{2} dy dx \right| \leq \\
&\leq \frac{C_1(\alpha)}{\ln m_k \ln^{(2)} m_k} \frac{C_2(\alpha)}{\ln n_k \ln^{(2)} n_k} \frac{m^\alpha}{m^\alpha} \ln n; \tag{62}
\end{aligned}$$

$$\begin{aligned}
|T_2(m, n)| &= 1/\pi^2 \left| \int_{1/m}^{1/m^\tau} \int_{1/n}^{1/n^\tau} h(x)p(y)(-H_n^\beta(y)) \frac{1}{2} \operatorname{ctg} \frac{x}{2} dy dx \right| \leq \\
&\leq \frac{C_1(\beta)}{\ln m_k \ln^{(2)} m_k} \frac{C_2(\beta)}{\ln n_k \ln^{(2)} n_k} \frac{n^\beta}{n^\beta} \ln m; \tag{63}
\end{aligned}$$

$$\begin{aligned}
|T_3(m, n)| &= 1/\pi^2 \left| \int_{1/m}^{1/m^\tau} \int_{1/n}^{1/n^\tau} h(x)p(y)H_m^\alpha(x)H_n^\beta(y) dy dx \right| \leq \\
&\leq \frac{C_1(\alpha, \beta)}{\ln m_k \ln^{(2)} m_k} \frac{C_2(\alpha, \beta)}{\ln n_k \ln^{(2)} n_k} \frac{m^\alpha n^\beta}{m^\alpha n^\beta}. \tag{64}
\end{aligned}$$

(61)–(64) yield

$$I_1(m, n) = o(1) \quad (m, n \rightarrow \infty) \tag{65}$$

(we remind once more that  $m = 2m_k$ ,  $n = n_k + 1$ ).

Finally, (29), (30), (44)–(46), (55), (60) and (65) prove the Theorem 2 in the case  $n = 2$  and  $\alpha, \beta \in (0; 1)$ .

For  $\alpha = 1$  we have

$$H_n^1(t) = \frac{\sin(n+1)t}{(n+1)(2\sin\frac{t}{2})^2} \quad (66)$$

and an estimate analogous to (34) holds true.

Now we consider the case when  $\alpha > 1$ . Using the method represented in [4], p.507, we obtain

$$\begin{aligned} H_n^\alpha(t) = & \frac{1}{A_n^\alpha 2 \sin \frac{t}{2}} \operatorname{Re} \left\{ \frac{e^{i(n+1/2)t}}{(1-e^{-it})^\alpha} - e^{-\frac{1}{2}it} \sum_{m=1}^d A_n^{\alpha-m} (1-e^{-it})^{-m} - \right. \\ & \left. - \sum_{m=n+1}^{\infty} A_m^{\alpha-d-1} e^{-i(m-n-1/2)t} (1-e^{-it})^{-d} \right\}, \end{aligned} \quad (67)$$

where  $d = [\alpha]$ .

Taking the real part we obtain that the first term of the finite sum is 0. Therefore if  $[\alpha] = 1$  we apply once more the Abel transformation to the infinite sum in (67) and obtain

$$\begin{aligned} H_n^\alpha(t) = & \frac{\cos((n + \frac{1}{2} + \frac{\alpha}{2})t - \frac{\pi\alpha}{2})}{A_n^\alpha (2\sin\frac{t}{2})^{1+\alpha}} - \frac{(1-\alpha)\alpha \cos(t/2)}{8(n+1)(n+\alpha)(\sin\frac{t}{2})^3} - \\ & - \frac{1}{A_n^\alpha} \sum_{m=1}^{\infty} A_{m+n+1}^{\alpha-3} \frac{\sin(m-1)t}{(2\sin\frac{t}{2})^3}. \end{aligned} \quad (68)$$

Then, again, we have an estimate analogous to (34), which enables us to fulfil the proof. Namely,

$$|H_n^\alpha(t)| \leq \frac{C_1(\alpha)}{n^\alpha t^{\alpha+1}} + \frac{C_2(\alpha)}{n^2 t^3}. \quad (69)$$

If  $[\alpha] = 2$  without further transformation we obtain

$$\begin{aligned} H_n^\alpha(t) = & \frac{\cos((n + \frac{1}{2} + \frac{\alpha}{2})t - \frac{\pi\alpha}{2})}{A_n^\alpha (2\sin\frac{t}{2})^{1+\alpha}} + \frac{(\alpha-1)\alpha \cos(t/2)}{8(n+\alpha-1)(n+\alpha)(\sin\frac{t}{2})^3} + \\ & + \frac{1}{A_n^\alpha} \sum_{m=1}^{\infty} A_{m+n}^{\alpha-3} \frac{\cos(m-3/2)t}{(2\sin\frac{t}{2})^3} \end{aligned} \quad (70)$$

and, again, (69) holds true.

Analogous equations may be obtained if  $[\alpha] \geq 3$ .

Now, when  $\alpha = 1$  and  $\beta = 1$ , we use (66). If  $\alpha = 1$  and  $\beta < 1$  (or vice versa), we use (34) and (66). If  $\alpha = 1$  and  $\beta > 1$  (or vice versa), we use (66) and (67) (for the corresponding  $d$ ). If  $\alpha > 1$  and  $\beta < 1$  (or vice versa), we use again (67) (for the corresponding  $d$ ) and (34).

In the  $n$ -dimensional case we define  $f$  as follows. We set

$$m_k = 2^{2^{2^k}}, \quad (k \in \mathbb{N});$$

$$g_k(x) = \begin{cases} \frac{1}{\ln^{n-1}(1/(x-\pi/(6m_k))) \ln^{(n)}(1/(x-\pi/(6m_k)))}, & \frac{\pi}{6m_k} < x \leq \frac{11\pi}{60m_k}, \\ \frac{1}{\ln^{n-1}(1/(\pi/(5m_k)-x)) \ln^{(n)}(1/(\pi/(5m_k)-x))}, & \frac{11\pi}{60m_k} \leq x < \frac{\pi}{5m_k}, \\ 0, & x \in (0; \pi) \setminus \left(\frac{\pi}{6m_k}; \frac{\pi}{5m_k}\right), \end{cases}$$

Again

$$h(x) = \sum_{k=1}^{\infty} g_k(x).$$

Then,

$$p(x_2, \dots, x_n) = \frac{1}{\ln^{n-1} \frac{(2\pi)^{n-1}}{\prod_{i=2}^n (\pi/2 - |\pi/2 - x_i|)} \ln^{(n)} \frac{(2\pi)^{n-1}}{\prod_{i=2}^n (\pi/2 - |\pi/2 - x_i|)}} \quad (71)$$

for  $(x_2, \dots, x_n) \in (0; \pi)^{n-1}$ .

And, finally

$$f(x_1, \dots, x_n) = \begin{cases} h(x_1)p(x_2, \dots, x_n), & (x_1, \dots, x_n) \in (0; \pi)^n, \\ 0, & (x_1, \dots, x_n) \in [-\pi; \pi]^n \setminus (0; \pi)^n. \end{cases}$$

Outside  $[-\pi; \pi]^n$  we extend the function  $f$  by periodicity with the period  $2\pi$  in each variable.

We observe, that functions of the  $p(x_2, \dots, x_n)$ -type were for the first time introduced and applied in the works of L.Zhizhiashvili (see [1], [5]).  $\square$

**Remark 1.** For the function  $f(x_1, \dots, x_n)$  a stronger condition than (26) holds true, namely,

$$\omega_i(\delta; f)_C \leq \frac{C(f, n)}{\ln^{n-1}(1/\delta) \ln^{(n)}(1/\delta)}, \quad i = \overline{1, n}.$$

**Remark 2.** Results analogous to Theorem 1, the corollary and Theorem 2 hold true for the  $n$ -multiple Abel–Poisson summability method.

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