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# TWO-DIMENSIONAL PROBLEMS OF STATIONARY FLOW OF A NONCOMPRESSIBLE VISCOUS FLUID IN THE CASE OF OZEEN'S LINEARIZATION

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Abstract. Two-dimensional boundary value problems of flow of a viscous micropolar fluid are investigated in the case of linearization by Ozeen's method.

**1. Basic Equations.** A system of equations of motion of a noncompressible micropolar fluid was obtained in 1964 by Condiff and Dahler [1] and, independently, in 1966 by Eringen. It is a generalization of the classical system of Navier–Stokes for micropolar fluids. In real life we observe such properties in fluids containing polymer particles as admixtures. When fluids of this kind flow along the body, surface friction is 30 to 35% less than in the case of flow of fluids without polymer admixtures [2]. It is impossible to predict such effects by the classical theory of Navier–Stokes, but a fairly good explanation can be found within the framework of the theory of micropolar fluids.

We consider a two-dimensional model of stationary flow of a micropolar fluid. A system of the basic equations then has the form

$$\begin{aligned}
&\operatorname{div}\widetilde{v}=0,\\
&(\mu+\alpha)\Delta\widetilde{v}_1+2\alpha\frac{\partial\widetilde{\omega}}{\partial x_2}-\frac{\partial p}{\partial x_1}+\rho F_1=\rho\sum_{k=1}^{2}\widetilde{v}_k\frac{\partial\widetilde{v}_1}{\partial x_k},\\
&(\mu+\alpha)\Delta\widetilde{v}_2-2\alpha\frac{\partial\widetilde{\omega}}{\partial x_1}-\frac{\partial p}{\partial x_2}+\rho F_2=\rho\sum_{k=1}^{2}\widetilde{v}_k\frac{\partial\widetilde{v}_2}{\partial x_k},\\
&\gamma\Delta\widetilde{\omega}-4\alpha\widetilde{\omega}+2\alpha\Big(\frac{\partial\widetilde{v}_2}{\partial x_1}-\frac{\partial\widetilde{v}_1}{\partial x_2}\Big)+\rho F_3=\mathcal{I}\sum_{k=1}^{2}\widetilde{v}_k\frac{\partial\widetilde{\omega}}{\partial x_k},
\end{aligned}\tag{1}$$

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where $\widetilde{v}=(\widetilde{v}_1,\widetilde{v}_2)$ is the velocity vector, $\widetilde{F}=(F_1,F_2)$ is the mass force, $p$ is the pressure, $\rho$ is the density; $\mu$, $\alpha$, $\gamma$, $\mathcal{I}$ are positive constants. In the two-dimensional case the microrotation and mass moment vectors have one component each and are denoted by $\widetilde{\omega}$ and $F_3$, respectively.

Since the system (1) is nonlinear, we come across certain difficulties during its investigation. On the other hand, to solve many problems of applied nature it is sufficient to consider a linearized variant of the system (1). The equations of Navier-Stokes can be linearized by two well-known methods: that of Stokes and that of Ozeen. When using Stokes' method of linearization, nonlinear terms are totally discarded. This method yields satisfactory results for small $\widetilde{v}$ and $\widetilde{\omega}$ (note that in this case nonlinear terms are small values of higher order). However, if the fluid flow velocity $\widetilde{v}$ is not a small value, this model leads to an essential error. In particular, the effects predicted by this method when a fluid flows along a solid body do not agree with experimental data.

A lesser error is obtained in the case of linearization by Ozeen's method consisting in the following: it is assumed that fluid flow differs but little from flow along the $x_1$-axis with the constant velocity $v_0$. Then we set

$$\widetilde{v}_k=v_0\delta_{k1}+v_k,\quad k=1,2,\quad \widetilde{\omega}=\omega,$$

where $v_k$, $k=1,2$, $\omega$ are small values; $\delta_{kj}$ is the Kronecker symbol.

On substituting these values in (1), we obtain an Ozeen-linearized system of equations of stationary flow of a micropolar fluid in the two-dimensional case:

$$\begin{aligned}
&\operatorname{div} v=0,\\
&(\mu+\alpha)\Delta v_1+2\alpha\frac{\partial\omega}{\partial x_2}-\frac{\partial p}{\partial x_1}+\rho F_1=\eta_1\frac{\partial v_1}{\partial x_1},\\
&(\mu+\alpha)\Delta v_2-2\alpha\frac{\partial\omega}{\partial x_1}-\frac{\partial p}{\partial x_2}+\rho F_2=\eta_1\frac{\partial v_2}{\partial x_1},\\
&\gamma\Delta\omega-4\alpha\omega+2\alpha\Big(\frac{\partial v_2}{\partial x_1}+\frac{\partial v_1}{\partial x_2}\Big)+\rho F_3=\eta_2\frac{\partial\omega}{\partial x_1}.
\end{aligned}\tag{2}$$

Here $\eta_1=\rho v_0$, $\eta_2=\mathcal{I}v_0$.

The system (2) can be rewritten in the matrix form if we introduce the notation

$$L(\partial_x)=\begin{Vmatrix}
(\mu+\alpha)\Delta & 0 & 2\alpha\frac{\partial}{\partial x_2}\\
0 & (\mu+\alpha)\Delta & -2\alpha\frac{\partial}{\partial x_1}\\
-2\alpha\frac{\partial}{\partial x_2} & 2\alpha\frac{\partial}{\partial x_1} & \gamma\Delta-4\alpha
\end{Vmatrix},$$

$$G(\partial_x) = \left\| \begin{matrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ 0 \end{matrix} \right\|, \qquad \eta = \left\| \begin{matrix} \eta_1 & 0 & 0 \\ 0 & \eta_1 & 0 \\ 0 & 0 & \eta_2 \end{matrix} \right\|,$$

$$(v_1, v_2, \omega) \equiv u = (u_1, u_2, u_3) = \left\| \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix} \right\|, \quad F = (F_1, F_2, F_3) = \left\| \begin{matrix} F_1 \\ F_2 \\ F_3 \end{matrix} \right\|.$$

Now the system (2) takes the form

$$\begin{aligned} &\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \\ &L(\partial_x)u - G(\partial_x)p + \rho F = \eta \frac{\partial u}{\partial x_1}. \end{aligned} \tag{3}$$

Alongside with the system (3), we will also consider its conjugate

$$\begin{aligned} &\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \\ &L(\partial_x)u - G(\partial_x)p + \rho F = -\eta \frac{\partial u}{\partial x_1}, \end{aligned} \tag{4}$$

which is obtained from (3) if we replace $\eta_1$ and $\eta_2$ by $-\eta_1$ and $-\eta_2$, respectively.

Let us formulate the boundary value problems for (2). Denote by $D^+$ a finite domain in the Euclidean two-dimensional space $\mathbb{R}^2$ bounded with a piecewise-smooth contour $S$. Let $D^- \equiv \mathbb{R}^2 \backslash (D^+ \cup S)$. Denote by $n(y) = (n_1(y), n_2(y))$ the unit normal at a point $y \in S$, external with respect to the domain $D^+$.

The pair $(u, p)$, where $u = (v_1, v_2, \omega)$, will be called regular in $D^+$ if $u \in \mathbb{C}^2(D^+) \cap \mathbb{C}^1(\bar{D}^+)$, $p \in \mathbb{C}^1(D^+) \cap \mathbb{C}(\bar{D}^+)$.

The pair $(u, p)$ will be called regular in $D^-$ if $u \in \mathbb{C}^2(D^-) \cap \mathbb{C}^1(\bar{D}^-)$, $p \in \mathbb{C}^1(D^-) \cap \mathbb{C}(\bar{D}^-)$ and the conditions

$$u(x) = O\big(|x|^{-\frac{1}{2}}\big), \quad p(x) = o(1) \tag{5}$$

are fulfilled in the neighbourhood of $|x| = \infty$.

The boundary value problems for the system (3) are formulate as follows:

**Problem (I)$^\pm$.** In the domain $D^\pm$ find a regular solution of the system (3) by the boundary condition

$$\lim_{D^\pm \ni x \to y \in S} u(x) = f(y). \tag{6}$$

**Problem (II)$^{\pm}$.** In the domain $D^{\pm}$ find a regular solution of the system (3) by the boundary condition

$$\lim_{D^{\pm}\ni x\to y\in S}\Big[P(\partial_x,n(y))u(x)-\frac{1}{2}n_1(y)\eta u(x)-N(y)p(x)\Big]=f(y), \tag{7}$$

where $f=(f_1,f_2,f_3)$ is a given vector on $S$,

$$P(\partial_x,n(y))=\|P_{ij}(\partial_x,n(y))\|_{3\times 3}, \tag{8}$$

$$P_{ij}(\partial_x,n(y))=(\mu+\alpha)n_j(y)\frac{\partial}{\partial x_i}+(\mu+\alpha)\delta_{ij}\sum_{k=1}^{2}n_k(y)\frac{\partial}{\partial x_k},\ \ i,j=1,2;$$
$$P_{i3}(\partial_x,n(y))=2\alpha(i-1)n_1(y)+2\alpha(i-2)n_2(y),\ \ i=1,2;$$
$$P_{3j}(\partial_x,n(y))=0,\ \ j=1,2;$$
$$P_{33}(\partial_x,n(y))=\gamma\sum_{k=1}^{2}n_k(y)\frac{\partial}{\partial x_k};$$
$$N_i(y)=(N_1(y),N_2(y),N_3(y)),\ \ N_i(y)=n_i(y),\ i=1,2;\ \ N_3(y)=0.$$

The boundary value problems $(\widetilde{\mathrm{I}})^{\pm}$ and $(\widetilde{\mathrm{II}})^{\pm}$ for the conjugate system (4) are formulated similarly. In that case the boundary condition of Problem (I)$^{\pm}$ coincides with (6), while the boundary condition of Problem (II)$^{\pm}$ is obtained from (7) if we replace $\eta_1$ and $\eta_2$ by $-\eta_1$ and $-\eta_2$, respectively.

*Remark.* Our previous thorough treatment of the boundary value problems of stationary flow of a micropolar fluid under Ozeen's linearization in the three-dimensional case is given in [3]. Since investigations of the two-dimensional problems are mostly the same, we will dwell on only the part differing from the three-dimensional case.

**2. On Fundamental solutions.** The fundamental solutions of the system (1) are found from the relations

$$\begin{aligned}
&\frac{\partial \overset{(m)}{v}_1}{\partial x_1}+\frac{\partial \overset{(m)}{v}_2}{\partial x_2}=0,\\
&(\mu+\alpha)\Delta \overset{(m)}{v}_1-\eta_1\frac{\partial \overset{(m)}{v}_1}{\partial x_1}+2\alpha\frac{\partial \overset{(m)}{\omega}}{\partial x_2}-\frac{\partial \overset{(m)}{p}}{\partial x_1}+\overset{(m)}{a}_1\,\delta(x)=0,\\
&(\mu+\alpha)\Delta \overset{(m)}{v}_2-\eta_1\frac{\partial \overset{(m)}{v}_2}{\partial x_1}-2\alpha\frac{\partial \overset{(m)}{\omega}}{\partial x_1}-\frac{\partial \overset{(m)}{p}}{\partial x_2}+\overset{(m)}{a}_2\,\delta(x)=0,\\
&\gamma\Delta \overset{(m)}{\omega}-4\alpha \overset{(m)}{\omega}-\eta_2\frac{\partial \overset{(m)}{\omega}}{\partial x_1}-2\alpha\frac{\partial \overset{(m)}{v}_1}{\partial x_2}+2\alpha\frac{\partial \overset{(m)}{v}_2}{\partial x_1}+\overset{(m)}{b}\,\delta(x)=0,
\end{aligned} \tag{9}$$

where $\delta(x)$ is the Dirac distribution,

$$\overset{(m)}{a}_k = 2\delta_{km}, \quad \overset{(m)}{b} = 0, \quad k, m = 1, 2; \quad \overset{(3)}{a}_k = 0, \quad k = 1, 2, \quad \overset{(3)}{b} = 2.$$

Assume that the fundamental solutions $(\overset{(m)}{v}_1, \overset{(m)}{v}_2, \overset{(m)}{\omega}, \overset{(m)}{p})$, $m = 1, 2, 3$, satisfy the conditions

$$\lim_{|x|\to\infty} \big( \overset{(m)}{v}_1(x), \overset{(m)}{v}_2(x), \overset{(m)}{\omega}(x), \overset{(m)}{p}(x) \big) = 0. \tag{10}$$

Then $\overset{(m)}{v}_1$, $\overset{(m)}{v}_2$, $\overset{(m)}{\omega}$ and $\overset{(m)}{p}$ are gradually increasing distributions in $\mathbb{R}^2$, and, on subjecting the system (9) to the Fourier transformation, we obtain

$$\begin{aligned}
&\xi_1 \widehat{\overset{(m)}{v}}_1 + \xi_2 \widehat{\overset{(m)}{v}}_2 = 0,\\
&i\xi_1 \widehat{\overset{(m)}{p}} - (\mu+\alpha)|\xi|^2 \widehat{\overset{(m)}{v}}_1 + i\eta_1\xi_1 \widehat{\overset{(m)}{v}}_1 - 2i\alpha\xi_2 \widehat{\overset{(m)}{\omega}} + \overset{(m)}{a}_1 = 0,\\
&i\xi_2 \widehat{\overset{(m)}{p}} - (\mu+\alpha)|\xi|^2 \widehat{\overset{(m)}{v}}_2 + i\eta_1\xi_1 \widehat{\overset{(m)}{v}}_2 + 2i\alpha\xi_1 \widehat{\overset{(m)}{\omega}} + \overset{(m)}{a}_2 = 0,\\
&-\gamma|\xi|^2 \widehat{\overset{(m)}{\omega}} - 4\alpha \widehat{\overset{(m)}{\omega}} + i\eta_2\xi_1 \widehat{\overset{(m)}{\omega}} + 2i\alpha\xi_2 \widehat{\overset{(m)}{v}}_1 - 2i\alpha\xi_1 \widehat{\overset{(m)}{v}}_2 + \overset{(m)}{b} = 0.
\end{aligned} \tag{11}$$

Here $|\xi| \equiv (\xi_1^2+\xi_2^2)^{1/2}$, $\widehat{f}$ denotes the Fourier transform of the gradually increasing distribution $f$:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^2} e^{i\xi\cdot x} f(x)\,dx, \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2, \quad |\xi| \equiv (\xi_1^2+\xi_2^2)^{1/2}.$$

Solving the system (11) with respect to $\widehat{\overset{(m)}{v}}_1$, $\widehat{\overset{(m)}{v}}_2$, $\widehat{\overset{(m)}{\omega}}$, $\widehat{\overset{(m)}{p}}$, we have

$$\begin{aligned}
\widehat{\overset{(m)}{p}} &= \frac{i}{|\xi|^2}\xi\cdot \overset{(m)}{a},\\
\widehat{\overset{(m)}{v}}_1 &= \frac{\gamma|\xi|^2 - i\eta_2\xi_1 + 4\alpha}{\Phi(\xi)} \overset{(m)}{a}_1 - \frac{2i\alpha\xi_2}{\Phi(\xi)} \overset{(m)}{b} -\\
&\quad - \frac{\gamma|\xi|^2 - i\eta_2\xi_1 + 4\alpha}{|\xi|^2\Phi(\xi)} \big(\xi\cdot \overset{(m)}{a}\big)\xi_1,\\
\widehat{\overset{(m)}{v}}_2 &= \frac{\gamma|\xi|^2 - i\eta_2\xi_1 + 4\alpha}{\Phi(\xi)} \overset{(m)}{a}_2 + \frac{2i\alpha\xi_1}{\Phi(\xi)} \overset{(m)}{b} -\\
&\quad - \frac{\gamma|\xi|^2 - i\eta_2\xi_1 + 4\alpha}{|\xi|^2\Phi(\xi)} \big(\xi\cdot \overset{(m)}{a}\big)\xi_2,\\
\widehat{\overset{(m)}{\omega}} &= \frac{(\mu+\alpha)|\xi|^2 - i\eta_1\xi_1}{\Phi(\xi)} \overset{(m)}{b} + \frac{2i\alpha}{\Phi(\xi)} \big( \overset{(m)}{a}_1 \xi_2 - \overset{(m)}{a}_2 \xi_1 \big), \quad m = 1, 2, 3,
\end{aligned} \tag{12}$$

where

$$\Phi(\xi)=\gamma(\mu+\alpha)|\xi|^4-i[\eta_1\gamma+\eta_2(\mu+\alpha)]\xi_1|\xi|^2-\eta_1\eta_2\xi_1^2+4\mu\alpha|\xi|^2-4i\alpha\eta_1\xi_1.$$

Denote by $\overset{(m)}{v}_1$, $\overset{(m)}{v}_2$, $\overset{(m)}{\omega}$, $\overset{(m)}{p}$, $m=1,2,3$, the inverse Fourier transforms of $\widehat{\overset{(m)}{v}}_1$, $\widehat{\overset{(m)}{v}}_2$, $\widehat{\overset{(m)}{\omega}}$, $\widehat{\overset{(m)}{p}}$, respectively, and by $\Gamma$ and $Q$ the matrices

$$\Gamma=\|\Gamma_{ik}\|_{3\times3},\quad \Gamma_{ik}=\overset{(k)}{v}_i,\quad i=1,2,\quad k=1,2,3;\quad \Gamma_{3k}=\overset{(k)}{\omega}; \tag{13}$$
$$Q=\|Q_k\|_{3\times1},\quad Q_k=\overset{(k)}{p},\quad k=1,2;\quad Q_3=0.$$

$\Gamma$ and $Q$ will be called the matrices of fundamental solutions of the system (1). Their Fourier transform is

$$\widehat{Q}(\xi)=\Big(\frac{2i\xi_1}{|\xi|^2},\frac{2i\xi_2}{|\xi|^2},0\Big), \tag{14}$$

$$\widehat{\Gamma}(\xi)=\left\|\begin{matrix} 2A(\xi)\Big(1-\frac{\xi_1^2}{|\xi|^2}\Big) & -2A(\xi)\frac{\xi_1\xi_2}{|\xi|^2} & -2B(\xi)\xi_2\\ -2A(\xi)\frac{\xi_1\xi_2}{|\xi|^2} & 2A(\xi)\Big(1-\frac{\xi_2^2}{|\xi|^2}\Big) & 2B(\xi)\xi_1\\ 2B(\xi)\xi_2 & -2B(\xi)\xi_1 & \frac{2}{\Phi(\xi)}\Big((\mu+\alpha)|\xi|^2-i\eta_1\xi_1\Big)\end{matrix}\right\|, \tag{15}$$

where

$$A(\xi)=\frac{\gamma|\xi|^2-i\eta_2\xi_1+4\alpha}{\Phi(\xi)},\quad B(\xi)=\frac{2i\alpha}{\Phi(\xi)}.$$

Calculating the inverse Fourier transform of (14), we get

$$Q(x)=\Big(\frac{1}{\pi}\,\frac{x_1}{|x|^2},\,\frac{1}{\pi}\,\frac{x_2}{|x|^2},\,0\Big). \tag{16}$$

Though the inverse transform of $\widehat{\Gamma}$ is not expressed in terms of elementary functions, we can nevertheless obtain asymptotic representations of the fundamental matrix $\Gamma$ in the neighbourhood of the points $|x|=0$ and $|x|=\infty$, which is convenient in investigating the boundary value problems.

Represent $\widehat{\Gamma}$ in the form

$$\widehat{\Gamma}(\xi)=\widehat{\Gamma}^{(0)}(\xi)+\widehat{\Gamma}^{(1)}(\xi),$$

where

$$\widehat{\Gamma}^{(0)}(\xi)=\left\|\begin{matrix}\frac{2\xi_1^2}{(\mu+\alpha)|\xi|^4} & -\frac{2\xi_1\xi_2}{(\mu+\alpha)|\xi|^4} & 0\\ -\frac{2\xi_1\xi_2}{(\mu+\alpha)|\xi|^4} & \frac{2\xi_2^2}{(\mu+\alpha)|\xi|^4} & 0\\ 0 & 0 & \frac{2}{\gamma|\xi|^2}\end{matrix}\right\| \tag{17}$$

and the elements of $\widehat{\Gamma}^{(1)}$ are written as

$$\widehat{\Gamma}^{(1)}_{ij}(\xi) = \sum_{3\le 2k-|\alpha|\le 4} a^{ij}_{k,\alpha}\frac{|\xi|^\alpha}{|\xi|^{2k}} + \varphi_{ij}(\xi), \quad i,j=1,2,3. \tag{18}$$

Here $a^{ij}_{k,\alpha}$ are some constants, $\varphi_{ij}$ are functions admitting, in the neighbourhood of $|x|=\infty$, the estimate $\varphi_{ij}(\xi)=O(|\xi|^{-5})$.

Taking (18) into account, it can be proved [3] that, $\Gamma^{(1)}_{ij}$ satisfies, in the neighbourhood of $|x|=0$, the conditions

$$\begin{aligned} \Gamma^{(1)}_{ij}(x) &= O(1), \quad i,j=1,2,3;\\ \partial^\alpha\Gamma^{(1)}_{ij}(x) &= O\big(|x|^{1-|\alpha|}\ln|x|\big), \quad i,j=1,2,3. \end{aligned} \tag{19}$$

Performing the inverse Fourier transform of (17), we obtain

$$\Gamma^{(0)}(x) = \begin{Vmatrix} -\frac{1}{4\pi(\mu+\alpha)}\big(\ln|x| + \frac{x_1^2}{|x|^2}\big) & -\frac{x_1x_2}{4\pi(\mu+\alpha)|x|^2} & 0\\ -\frac{x_1x_2}{4\pi(\mu+\alpha)|x|^2} & -\frac{1}{4\pi(\mu+\alpha)}\big(\ln|x| + \frac{x_2^2}{|x|^2}\big) & 0\\ 0 & 0 & -\frac{1}{\pi\gamma}\ln|x| \end{Vmatrix}. \tag{20}$$

Thus we have

**Theorem 1.** *The fundamental matrix $\Gamma$ is represented as*

$$\Gamma = \Gamma^{(0)} + \Gamma^{(1)}, \tag{21}$$

*where $\Gamma^{(0)}$ is defined from* (20)*, $\Gamma^{(1)} \in \mathbb{C}^\infty(\mathbb{R}\backslash\{0\})$ and satisfies, in the neighbourhood of the point $|x|=0$, the conditions* (19)*.*

To obtain the necessary representation of $\Gamma$ in the neighbourhood of $|x|=\infty$ we represent $\widehat{\Gamma}$ as the sum

$$\widehat{\Gamma}(\xi) = \widehat{\Gamma}^{(\infty)}(\xi) + \widehat{\Gamma}^{(2)}(\xi), \tag{22}$$

where

$$\widehat{\Gamma}^{(\infty)}(\xi) = \frac{2}{(\mu|\xi|^2 - i\eta_1\xi_1)}\begin{Vmatrix} \frac{\xi_2^2}{|\xi|^2} & -\frac{\xi_1\xi_2}{|\xi|^2} & -\frac{i\xi_2}{2}\\ -\frac{\xi_1\xi_2}{|\xi|^2} & \frac{\xi_1^2}{|\xi|^2} & \frac{i\xi_1}{2}\\ \frac{i\xi_2}{2} & -\frac{i\xi_1}{2} & \frac{i\eta_1\xi_1}{4\mu} \end{Vmatrix}. \tag{23}$$

Then, taking (15) into account, it can be proved that the components of $\widehat{\Gamma}^{(2)}$ satisfy the estimates

$$\begin{aligned}
&\big|\partial^\alpha\widehat{\Gamma}^{(2)}_{ij}(\xi)\big|\leq\frac{c}{(|\xi|^2+|\xi_1|)^{|\alpha|}}, \quad |\xi|\leq 1, \ \ |\alpha|\leq 2, \ \ i,j=1,2,\\
&\big|\partial^\alpha\widehat{\Gamma}^{(2)}_{3j}(\xi)\big|\leq\frac{c|\xi|}{(|\xi|^2+|\xi_1|)^{|\alpha|}}, \quad |\xi|\leq 1, \ \ |\alpha|\leq 2, \ \ j=1,2,3,\\
&\big|\partial^\alpha\widehat{\Gamma}^{(2)}_{i3}(\xi)\big|\leq\frac{c|\xi|}{(|\xi|^2+|\xi_1|)^{|\alpha|}}, \quad |\xi|\leq 1, \ \ |\alpha|\leq 2, \ \ i=1,2,3,
\end{aligned}\tag{24}$$

from which we obtain the estimates of $\Gamma^{(2)}$ in the neighbouhood of the point $|x|=\infty$ [3]:

$$\begin{aligned}
&\partial^\alpha\widehat{\Gamma}^{(2)}_{ij}(x)\leq o\big(|x|^{-1}\big), \quad |\alpha|\geq 0, \ \ i,j=1,2,3,\\
&\partial^\alpha\widehat{\Gamma}^{(2)}_{3j}(x)\leq o\big(|x|^{-2}\big), \quad |\alpha|\geq 1, \ \ j=1,2,3,\\
&\partial^\alpha\widehat{\Gamma}^{(2)}_{i3}(x)\leq o\big(|x|^{-2}\big), \quad |\alpha|\geq 1, \ \ i=1,2,3.
\end{aligned}\tag{25}$$

Let us now calculate the inverse Fourier transform of $\widehat{\Gamma}^{(\infty)}$. We obtain

$$\begin{aligned}
&\Gamma^{(\infty)}_{11}(x)=\frac{1}{2\pi\mu}\Big(K_0(m|x|)-\frac{x_1}{|x|}K_0'(m|x|)\Big)e^{mx_1}-\frac{1}{2\pi m\mu}\,\frac{x_1}{|x|^2},\\
&\Gamma^{(\infty)}_{12}(x)=\Gamma^{(\infty)}_{21}(x)=-\frac{1}{2\pi\mu}\,\frac{x_2}{|x|}K_0'(m|x|)e^{mx_1}-\frac{1}{2\pi m\mu}\,\frac{x_2}{|x|^2},\\
&\Gamma^{(\infty)}_{22}(x)=\frac{1}{2\pi\mu}\Big(K_0(m|x|)+\frac{x_1}{|x|}K_0'(m|x|)\Big)e^{mx_1}+\frac{1}{2\pi m\mu}\,\frac{x_1}{|x|^2},\\
&\Gamma^{(\infty)}_{13}(x)=-\Gamma^{(\infty)}_{31}(x)=\frac{m}{2\pi\mu}\,\frac{x_2}{|x|}K_0'(m|x|)e^{mx_1},\\
&\Gamma^{(\infty)}_{23}(x)=-\Gamma^{(\infty)}_{32}(x)=-\frac{m}{2\pi\mu}\Big(K_0(m|x|)+\frac{x_1}{|x|}K_0'(m|x|)\Big)e^{mx_1},\\
&\Gamma^{(\infty)}_{33}(x)=-\frac{\eta_1 m}{4\pi\mu^2}\Big(K_0(m|x|)+\frac{x_1}{|x|}K_0'(m|x|)\Big)e^{mx_1},
\end{aligned}\tag{26}$$

where $K_0$ is the MacDonald function of zero order:

$$K_0(t)=\int_0^\infty e^{-t\,\mathrm{ch}\,\eta}\,d\eta, \quad m=\frac{\eta_1}{2\mu}.$$

The equalities (22)–(26) imply the validity of

**Theorem 2.** *The fundamental matrix* $\Gamma$ *is represented in the form*

$$\Gamma=\Gamma^{(\infty)}+\Gamma^{(2)},\tag{27}$$

*where the components* $\Gamma^{(\infty)}$ *are written as* (26) *and* $\Gamma^{(2)}$ *admits the estimates* (25) *in the neighbourhood of the point* $|x|=\infty$.

It is not difficult to obtain the following asymptotic representation in the neighbourhood of the infinity of the MacDonald function and its derivative:

$$K_0(t) = \frac{\sqrt{\pi}}{\sqrt{2t}} e^{-t}\Big(1 - \frac{1}{8t}\Big) + O\Big(\frac{e^{-t}}{t^2\sqrt{t}}\Big),$$
$$K_0'(t) = -\frac{\sqrt{\pi}}{\sqrt{2t}} e^{-t}\Big(1 + \frac{3}{8t}\Big) + O\Big(\frac{e^{-t}}{t^2\sqrt{t}}\Big),$$
$$K_0''(t) = \frac{\sqrt{\pi}}{\sqrt{2t}} e^{-t}\Big(1 + \frac{7}{8t}\Big) + O\Big(\frac{e^{-t}}{t^2\sqrt{t}}\Big).$$

Hence we obtain the asymptotic representations of the components of the matrix $\Gamma^{(\infty)}$ in the neighbourhood of the point $|x| = \infty$:

$$\begin{aligned}
\Gamma_{11}^{(\infty)}(x) &= \frac{(|x| + x_1)e^{-m(|x|-x_1)}}{2\sqrt{2\pi m}|x|^{3/2}} - \frac{1}{2\pi m\mu}\,\frac{x_1}{|x|^2} + O\big(|x|^{-3/2}\big),\\
\Gamma_{12}^{(\infty)}(x) &= \Gamma_{21}^{(\infty)}(x) = \frac{x_2 e^{-m(|x|-x_1)}}{2\sqrt{2\pi m}\mu|x|^{3/2}} - \frac{1}{2\pi m\mu}\,\frac{x_2}{|x|^2} + O\big(|x|^{-3/2}\big),\\
\Gamma_{13}^{(\infty)}(x) &= -\Gamma_{31}^{(\infty)}(x) = -\frac{\sqrt{m}}{2\sqrt{2\pi}}\,\frac{x_2 e^{-m(|x|-x_1)}}{|x|^{3/2}} + O\big(|x|^{-3/2}\big),\\
\Gamma_{22}^{(\infty)}(x) &= \frac{(|x| - x_1)e^{-m(|x|-x_1)}}{2\sqrt{2\pi m}|x|^{3/2}} + \frac{1}{2\pi m\mu}\,\frac{x_1}{|x|^2} + O\big(|x|^{-3/2}\big),\\
\Gamma_{23}^{(\infty)}(x) &= -\Gamma_{32}^{(\infty)}(x) = -\frac{\sqrt{m}(|x| - x_1)e^{-m(|x|-x_1)}}{2\sqrt{2\pi}\mu|x|^{3/2}} + O\big(|x|^{-3/2}\big),\\
\Gamma_{33}^{(\infty)}(x) &= -\frac{\eta_1\sqrt{m}(|x| - x_1)e^{-m(|x|-x_1)}}{4\sqrt{2\pi}\mu^2|x|^{3/2}} + O\big(|x|^{-3/2}\big).
\end{aligned} \tag{28}$$

In particular, (28) implies

$$\big|\Gamma_{kj}^{(\infty)}(x)\big| \le c_1 \frac{e^{-m(|x|-x_1)}}{|x|^{1/2}} + c_2|x|^{-3/2}, \quad k, j = 1, 2, 3. \tag{29}$$

In a similar manner we can obtain the estimates for the derivatives of $\Gamma_{kj}^{(\infty)}$ as well:

$$\big|\partial^\alpha \Gamma_{kj}^{(\infty)}(x)\big| \le c_1 \frac{e^{-m(|x|-x_1)}}{|x|^{1/2}} + c_2|x|^{-2}, \quad |\alpha| \ge 1, \quad k, j = 1, 2, 3. \tag{30}$$

*Remark.* As follows from (25), (29), in the case of Ozeen's linearization the fundamental matrix has order $O(|x|^{-1/2})$ at infinity, but, as shown in [4], the fundamental matrix of the system obtained in the case of Stokes' linearization has order $O(\ln|x|)$ at infinity, i.e. it is unbounded. Therefore the properties of solutions of the external boundary value problems are different in the two cases.

We also note that no such difference between the fundamental solutions is observed for Stokes' and Ozeen's linearizations. In that case both fundamental solutions have order $O(|x|^{-1})$ at infinity [3], [5].

**3. Regular Solution Representation Formulas. The Uniqueness Theorems.** Let $D^+$ be a finite domain in $\mathbb{R}^2$ bounded by the piecewise-smooth contour $S$; $(u,p)$ and $(u',p')$ be regular pairs satisfying the conditions

$$\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}=\frac{\partial u_1'}{\partial x_1}+\frac{\partial u_2'}{\partial x_2}=0.$$

If, besides, $L(\partial_x)u-\eta\frac{\partial u}{\partial x_1}-G(\partial_x)p\in L_1(D^+)$, $L(\partial_x)u'-\eta\frac{\partial u'}{\partial x_1}-G(\partial_x)p'\in L_1(D^+)$, then the following identities of the Green formula type are fulfilled:

$$\int_{D^+}\Big[u\Big(L(\partial_x)u-\eta\frac{\partial u}{\partial x_1}-G(\partial_x)p\Big)+E(u,u)\Big]dx= \\ =\int_S\Big[u\Big(P(\partial_y,n)u-\frac{1}{2}n_1\eta u-Np\Big)\Big]^+d_yS, \tag{31}$$

$$\int_{D^+}\Big[u'\Big(L(\partial_x)u-\eta\frac{\partial u}{\partial x_1}-G(\partial_x)p\Big)- \\ -u\Big(L(\partial_x)u'+\eta\frac{\partial u'}{\partial x_1}-G(\partial_x)p\Big)\Big]dx= \\ =\int_S\Big[u'\Big(P'(\partial_y,n)u-\frac{1}{2}n_1\eta u-Np\Big)- \\ -u\Big(P(\partial_y,n)u'+\frac{1}{2}n_1\eta u'-Np'\Big)\Big]^+d_yS, \tag{32}$$

where

$$E(u,u')=(\mu+\alpha)\sum_{i,j=1}^{2}v_{ij}v_{ji}'+(\mu-\alpha)\sum_{i,j=1}^{2}v_{ji}v_{ij}'+\gamma\sum_{i=1}^{2}\omega_i\omega_i',$$

$$v_{ij}=\frac{\partial v_j}{\partial x_i}+(j-i)\omega,\ \ i,j=1,2;\quad \omega_i=\frac{\partial\omega}{\partial x_i},\ \ i=1,2.$$

$E(u,u')$ is an analogue of the energetic form. In particular,

$$E(u,u')=(\mu+\alpha)(v_{12}^2+v_{21}^2)+2(\mu-\alpha)v_{12}v_{21}+ \\ +2\mu(v_{11}^2+v_{22}^2)+\gamma(\omega_1^2+\omega_2^2). \tag{33}$$

One can easily verify that for the form (33) to be positive definite with respect to $v_{ij}$, $\omega_i$, it is necessary and sufficient that the conditions

$$\mu>0,\quad \alpha>0,\quad \gamma>0 \tag{34}$$

be fulfilled.

If we use the area potentials

$$U(F,x)=\frac{1}{2}\int_{D^\pm}\rho\Gamma(x-y)F(y)\,dy,$$
$$q(F,x)=\frac{1}{2}\int_{D^\pm}\rho Q(x-y)F(y)\,dy,$$

where $F=(F_1,F_2,F_3)\in\mathbb{C}^{0,h}(D^\pm)$, $0<h\le 1$, and is the finite vector in the case of $D^-$, then the boundary value problems for the systems (3) and (4) can be reduced to the corresponding problems for the homogeneous systems [3]

$$\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}=0,\quad L(\partial_x)u-G(\partial_x)p-\eta\frac{\partial u}{\partial x_1}=0 \tag{35}$$

and

$$\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}=0,\quad L(\partial_x)u-G(\partial_x)p+\eta\frac{\partial u}{\partial x_1}=0. \tag{36}$$

Therefore in the sequel we will consider the boundary value problems only for the homogeneous systems (35) and (36).

Like in the three-dimensional case [3], from (32) we can obtain the following formula for representation of solutions of the system (3) in the domain $D^+$:

$$\begin{aligned}u(x)=\frac{1}{2}\int_S\Gamma(x-y)\big[P(\partial_y,n)u(y)-\frac{1}{2}n_1\eta u(y)-N(y)p(y)\big]^+d_yS-\\ -\frac{1}{2}\int_S\big[P(\partial_y,n)\Gamma'(x-y)+\frac{1}{2}n_1\eta\Gamma'(x-y)-\\ -N(y)*Q(x-y)\big]'u^+(y)d_yS,\end{aligned} \tag{37}$$

$$\begin{aligned}p(x)=\frac{1}{2}\int_S Q(x-y)\big[P(\partial_y,n)u(y)-\frac{1}{2}n_1\eta u(y)-N(y)p(y)\big]^+d_yS-\\ -\frac{1}{2}\int_S\big[P(\partial_y,n)Q(x-y)+\frac{1}{2}n_1\eta Q(x-y)-\\ -\eta_1 Q(x-y)N(y)\big]u^+(y)d_yS.\end{aligned} \tag{38}$$

Here the prime above the matrix denotes the operation of transposition and

$$N(y)*Q(z)=\left\|\begin{matrix}n_1(y)Q_1(z) & n_1(y)Q_2(z)\\ n_2(y)Q_1(z) & n_2(y)Q_2(z)\end{matrix}\right\|.$$

Taking the properties (29) and (30) of the matrix of fundamental solutions into account and repeating the reasoning from [3], one can prove that the formulas (37), (38) are also valid for regular solutions in the domain $D^-$.

Now let us prove that the equality

$$\int_{D^-} E(u,u)\,dx = -\int_S u^-\big[P(\partial_y,n)u - \frac{1}{2}n_1\eta u - Np\big]^- d_yS \qquad (39)$$

is fulfilled for the regular solution $(u,p)$ of the homogeneous system (35) in $D^-$.

To this effect we have to use the formula (31) in the domain $D^-\cap B(0,R)$, where $B(0,R)$ is the circle of radius $R$ centred at $x=0$ and containing the domain $D^+$. Recalling that the pair $(u,p)$ is the solution of the system (35), we obtain

$$\int_{D^-\cap B(0,R)} E(u,u)\,dx = -\int_S u^-\big[P(\partial_y,n)u - \frac{1}{2}n_1\eta u - Np\big]^- d_yS + \mathcal{I}(R),$$

where

$$\mathcal{I}(R) = \int_{\partial B(0,R)} u\big[P(\partial_y,n)u - \frac{1}{2}n_1\eta u - Np\big]\,d_yS. \qquad (40)$$

Obviously, to prove (39) it is enough to prove

$$\lim_{R\to\infty}\mathcal{I}(R) = 0. \qquad (41)$$

From (16), (25), (29) and the formula of representation of regular solutions in the domain $D^-$ it follows that in the neighbourhood of infinity

$$\big|\partial^\alpha u(x)\big| = c_1\frac{e^{-m(|x|-x_1)}}{|x|^{1/2}} + O(|x|^{-1}), \quad |\alpha|\geq 0,$$
$$|p(x)| = O(|x|^{-1}).$$

Taking these estimates into account in (40) and passing to the limit, we obtain (41). The equality (39) is proved.

Let us turn to proving the uniqueness theorems. In particular, we will prove

**Theorem 3.** *Each solution of the homogeneous problem* $(\mathrm{I})_0^+$ *has the form*

$$u = 0, \qquad p = p_0, \qquad (42)$$

*where* $p_0$ *is an arbitrary constant. The homogeneous Problems* $(II)_0^+$*,* $(I)_0^-$ *and* $(II)_0^-$ *can have only the trivial solutions* $u=0$*,* $p=0$*.*

*Proof.* Let $(u,p)$ be a solution of anyone of the considered problems. Then by virtue of (31) and (39)

$$\int_{D^\pm} E(u,u)\,dx = 0.$$

Therefore $E(u,u)=0$, $x\in D^+$ $(x\in D^-)$. Hence on account of (34)

$$\frac{\partial v_j}{\partial x_i}+(j-i)\omega=0,\quad \frac{\partial \omega}{\partial x_i}=0,\ \ i,j=1,2.$$

The general solution of the resulting system has the form

$$v_1=ax_2+b_1,\quad v_2=-ax_1+b_2,\quad \omega=a. \tag{43}$$

Since for the external domain the regular solution must vanish at infinity, the external homogeneous boundary value problems have trivial solutions. In the case of Problem $(\mathrm{I})_0^+$, (43) also implies that $v_1=v_2=\omega=0$; substituting these value in the equation (35) we obtain

$$\frac{\partial p}{\partial x_1}=\frac{\partial p}{\partial x_2}=0,$$

whence it follows that $p=const$.

Next, let us consider Problem $(\mathrm{II})_0^+$. By virtue of (43) and the homogeneous boundary condition we have $a=0$, i.e. $v_i=b_i$. Then (35) yields $p=p_0$. Considering again the boundary condition, we obtain

$$\eta_1 n_1 b_k + 2p_0 n_k = 0,\quad k=1,2.$$

Since the boundary of $D^+$ is not the straight line, we conclude that $b_1=b_2=p_0=0$. $\square$

**4. Reduction of the Boundary Value Problems to Integral Equations.** Like in the three-dimensional case, we introduce the simple-layer potentials

$$V(\varphi)(x)=\int_S \Gamma(x-y)\varphi(y)\,d_yS,\quad a(\varphi)(x)=\int_S Q(x-y)\varphi(y)\,d_yS$$

and the double-layer potentials

$$\begin{aligned}W(\varphi)(x)=\int_S \big[&P(\partial_y,n)\Gamma'(x-y)+\frac{1}{2}n_1\eta\Gamma'(x-y)-\\&-N(y)*Q(x-y)\big]'\varphi(y)d_yS,\\ b(\varphi)(x)=\int_S \big[&P(\partial_y,n)Q(x-y)+\frac{1}{2}n_1\eta Q(x-y)-\\&-\eta_1 Q_1(x-y)N(y)\big]\varphi(y)d_yS.\end{aligned}$$

Denote by $\widetilde{V}(\varphi)$, $\widetilde{a}(\varphi)$, $\widetilde{W}(\varphi)$, $\widetilde{b}(\varphi)$ the potential obtained from $V(\varphi)$, $a(\varphi)$, $W(\varphi)$, $b(\varphi)$ after replacing $\eta_1$ and $\eta_2$ by $-\eta_1$ and $-\eta_2$, respectively (note that $\widetilde{a}(\varphi)$ coincides with $a(\varphi)$).

We will seek for the solution of Problem $(\mathrm{I})^{\pm}$ $[(\widetilde{\mathrm{I}})^{\pm}]$ in the form of double-layer potentials

$$u(x)=W(\varphi)(x),\quad p(x)=b(\varphi)(x)$$
$$\Big[\,u(x)=\widetilde{W}(\varphi)(x),\quad p(x)=\widetilde{b}(\varphi)(x)\,\Big]$$

and the solution of Problem $(\mathrm{II})^{\pm}$ $[(\widetilde{\mathrm{II}})^{\pm}]$ in the form of single-layer potentials

$$u(x)=V(\psi)(x),\quad p(x)=a(\psi)(x)$$
$$\Big[\,u(x)=\widetilde{V}(\varphi)(x),\quad p(x)=a(\varphi)(x)\,\Big].$$

Then, as was done in [3], we obtain, for the densities $\varphi$, $\psi$, the singular integral equations

$$\mp\varphi(z)+\int_S\big[P(\partial_y,n)\Gamma'(z-y)+\frac{1}{2}n_1\eta\Gamma'(z-y)-\\ -N(y)*Q(y-z)\big]'\varphi(y)d_yS=f(z), \qquad (\mathrm{I})^{\pm}$$

$$\mp\varphi(z)+\int_S\big[P(\partial_y,n)\Gamma(z-y)-\frac{1}{2}n_1\eta\Gamma(z-y)-\\ -N(y)*Q(y-z)\big]'\varphi(y)d_yS=f(z), \qquad (\widetilde{\mathrm{I}})^{\pm}$$

$$\pm\psi(z)+\int_S\big[P(\partial_y,n)\Gamma(z-y)-\frac{1}{2}n_1\eta\Gamma(z-y)-\\ -N(z)*Q(y-z)\big]\psi(y)d_yS=f(z), \qquad (\mathrm{II})^{\pm}$$

$$\pm\psi(z)+\int_S\big[P(\partial_y,n)\Gamma'(z-y)+\frac{1}{2}n_1\eta\Gamma'(z-y)-\\ -N(z)*Q(y-z)\big]\psi(y)d_yS=f(z). \qquad (\widetilde{\mathrm{II}})^{\pm}$$

The investigation of these equations leads to the following result [3, 6]:

**Theorem 4.** *If $S\in\mathcal{L}_{k+1}(h')$, $f\in\mathbb{C}^{k,h}(S)$, $0<h<h'\le 1$, $k=1,2,\dots$, then the Fredholm theorems hold for the pairs of equations* $(\mathrm{I})^{+}$ *and* $(\widetilde{\mathrm{II}})^{-}$, $(\mathrm{I})^{-}$ *and* $(\widetilde{\mathrm{II}})^{+}$, $(\mathrm{II})^{+}$ *and* $(\widetilde{\mathrm{I}})^{-}$, $(\mathrm{II})^{-}$ *and* $(\widetilde{\mathrm{I}})^{+}$ *in the space* $\mathbb{C}^{h,k}(S)$.

**5. Existence Theorems for the Boundary Value Problems.** In this paragraph we present the existence theorems for the boundary value problems. Their proofs are left out because they repeat the ones given in [3].

**Theorem 5.** *If* $S \in \mathcal{L}_2(h')$, $f \in \mathbb{C}^{1,h}(S)$, $0 < h < h' \leq 1$, *and* $f$ *satisfies the condition*

$$\int_S Nf\,dS = 0, \tag{44}$$

*then Problem* $(\mathrm{I})^+$ *has a regular solution. Moreover, if the condition*

$$\int_S p\,dS = 0$$

*is fulfilled, then this solution is unique.*

We observe that the condition (44) is not only sufficient, but also necessary for the existence of a regular solution of Problem $(\mathrm{I})^+$.

**Theorem 6.** *If* $S \in \mathcal{L}_2(h')$, $f \in \mathbb{C}^{1,h}(S)$, $0 < h < h' \leq 1$, *then Problem* $(\mathrm{I})^-$ *has a unique regular solution.*

**Theorem 7.** *If* $S \in \mathcal{L}_2(h')$, $f \in \mathbb{C}^{0,h}(S)$, $0 < h < h' \leq 1$, *then Problem* $(\mathrm{II})^+$ *has a unique regular solution.*

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