

ON CONVERGENCE SUBSYSTEMS OF ORTHONORMAL SYSTEMS

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ABSTRACT. It is proved that for any sequence $\{R_k\}_{k=1}^{\infty}$ of real numbers satisfying $R_k \geq k$ ($k \geq 1$) and $R_k = o(k \log_2 k)$, $k \rightarrow \infty$, there exists a orthonormal system $\{\varphi_n(x)\}_{n=1}^{\infty}$, $x \in (0; 1)$, such that none of its subsystems $\{\varphi_{n_k}(x)\}_{k=1}^{\infty}$ with $n_k \leq R_k$ ($k \geq 1$) is a convergence subsystem.

Let $\{\varphi_n(x)\}$ be an orthonormal system (ONS) on $(0; 1)$. It is called a convergence system if the series $\sum c_n \varphi_n(x)$ is convergent almost everywhere whenever the sequence $\{c_n\}$ of real numbers satisfies $\sum c_n^2 < \infty$.

It is well-known [1] that not every ONS $\{\varphi_n(x)\}$ is a convergence system. However [2], [3], each of them contains some convergence subsystem $\{\varphi_{n_k}(x)\}$. A question was formulated later [4] whether there exists a common estimate of growth rate of numbers n_k in the class of all ONS. B.S.Kashin [5] answered this question in the affirmative: one can determine a sequence of positive numbers $\{R_k\}$ such that from any ONS it is possible to choose a convergence subsystem $\{\varphi_{n_k}\}$ with $n_k \leq R_k$, $1 \leq k < \infty$. In the same paper [5] the problem of finding $\{R_k\}$ with a minimal admissible growth order is formulated and the hypothesis $R_k = k^{1+\varepsilon}$ ($\varepsilon > 0$) is conjectured. G.A.Karagulyan [6] proved that one can take $R_k = \lambda^k$, $\lambda > 1$. However, this upper estimate is rougher than the one expected in [5].

In this paper we shall give the proof of the theorem providing the lower estimate for $\{R_k\}$.

Theorem. *For any sequence $\{R_k\}_{k=1}^{\infty}$ of real numbers satisfying*

$$R_k \geq k \quad (k \geq 1) \quad \text{and} \quad R_k = o(k \log_2 k), \quad k \rightarrow \infty, \quad (1)$$

there exists an ONS $\{\varphi_n(x)\}_{n=1}^{\infty}$, $x \in (0; 1)$, such that none of its subsystems $\{\varphi_{n_k}(x)\}_{k=1}^{\infty}$ with $n_k \leq R_k$ ($k \geq 1$) is a convergence subsystem.

Several lemmas are needed to prove this theorem.

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Lemma 1 (H.Rademacher [7]). For any ONS $\{\psi_n(x)\}_{n=1}^N$, $x \in (0; 1)$, and any collection of real numbers $\{c_n\}_{n=1}^N$

$$\int_0^1 \left(\max_{1 \leq j \leq N} \left| \sum_{n=1}^j c_n \psi_n(x) \right| \right)^2 dx \leq c \log_2^2(N+1) \sum_{n=1}^N c_n^2, \quad 1 \leq N < \infty.^1$$

Lemma 2. For any $N \geq 1$ there exists an ONS

$$\psi(N) := \{\psi_n^N(x)\}_{n=1}^N, \quad x \in (0; 1),$$

satisfying, for any collection of natural numbers $1 \leq n_1 < n_2 < \dots < n_m \leq N$ ($1 \leq m \leq N$), the inequality

$$\int_0^1 \max_{1 \leq j \leq m} \left| \sum_{k=1}^j \psi_{n_k}^N(x) \right| dx \geq c \frac{\log_2 N}{\sqrt{N}} m. \quad (2)$$

Proof. We shall assure that the requirements of this lemma are satisfied by the ONS usually used in the proof of the Menshov-Rademacher theorem. The functions $\psi_n^N(x)$, $1 \leq n \leq N$, belonging to this ONS (see [8], p.295) have, in particular, the following properties:

$$(i) \quad \psi_n^N(x) := \begin{cases} \frac{c\sqrt{N}}{s-n}, & x \in \left(\frac{s-1}{2N}, \frac{2s-1}{4N} \right), \\ \frac{c\sqrt{N}}{n-s}, & x \in \left(\frac{2s-1}{4N}, \frac{s}{2N} \right), \\ 0, & x \in \left(\frac{n-1}{2N}, \frac{n}{2N} \right); \end{cases} \quad 1 \leq s \leq N, \quad s \neq n;$$

(ii) $\psi_n^N(x)$ is constant on each of the intervals

$$\left(\frac{s-1}{4N}, \frac{s}{4N} \right), \quad 2N+1 \leq s \leq 4N;$$

(iii)

$$\int_0^1 \psi_n^N(x) dx = 0;$$

(iv) $\psi_n^N(x)$ is extended from $(0; 1)$ onto $(-\infty; \infty)$ with period 1.

Denote $\delta_s := \left(\frac{s-1}{2N}, \frac{2s-1}{4N} \right)$, $1 \leq s \leq N$. When $x \in \delta_s$, because of (i) we have $\psi_p^N(x) \geq 0$ for $1 \leq p \leq s$ and $\psi_p^N(x) \leq 0$ for $s \leq p \leq N$ ($1 \leq s \leq N$).

¹Here and in what follows c denotes positive absolute constants which, in general, may differ from one equality (inequality) to another.

Therefore for fixed numbers $1 \leq n_1 < n_2 < \dots < n_m \leq N$ ($1 \leq m \leq N$) and for each $x \in \delta_s$ ($1 \leq s \leq N$) we obtain

$$\begin{aligned} \sum_{k=1}^m |\psi_{n_k}^N(x)| &= \sum_{k:n_k \leq s} + \sum_{k:n_k > s} = \left| \sum_{k:n_k \leq s} \psi_{n_k}^N(x) \right| + \\ &+ \left| \sum_{k:n_k > s} \psi_{n_k}^N(x) \right| \leq 3 \max_{1 \leq j \leq m} \left| \sum_{k=1}^j \psi_{n_k}^N(x) \right|. \end{aligned}$$

Hence

$$\begin{aligned} \int_{\delta_s} \max_{1 \leq j \leq m} \left| \sum_{k=1}^j \psi_{n_k}^N(x) \right| dx &\geq \frac{1}{3} \int_{\delta_s} \sum_{k=1}^m |\psi_{n_k}^N(x)| dx = \\ &= \frac{c}{\sqrt{N}} \sum_{\substack{k:1 \leq k \leq m, \\ n_k \neq s}} \frac{1}{|n_k - s|}, \quad 1 \leq s \leq N; \\ \int_0^1 \max_{1 \leq j \leq m} \left| \sum_{k=1}^j \psi_{n_k}^N(x) \right| dx &\geq \sum_{s=1}^N \int_{\delta_s} \max_{1 \leq j \leq m} \left| \sum_{k=1}^j \psi_{n_k}^N(x) \right| dx \geq \\ &\geq \frac{c}{\sqrt{N}} \sum_{s=1}^N \sum_{\substack{k:1 \leq k \leq m, \\ n_k \neq s}} \frac{1}{|n_k - s|} = \\ &= \frac{c}{\sqrt{N}} \sum_{k=1}^m \sum_{\substack{s:1 \leq s \leq N, \\ s \neq n_k}} \frac{1}{|n_k - s|} \geq c \frac{\log_2 N}{\sqrt{N}} m. \quad \blacksquare \end{aligned}$$

Remark 1. For any positive integer Q , $\{\psi_n^N(Qx)\}_{n=1}^N$ is an ONS on $(0; 1)$ also satisfying the inequality (2).

Remark 2. Let N_0, N_1, Q_0, Q_1, p be positive integers. If $Q_1 = 4pN_0Q_0$, then functions belonging to different collections $\{\psi_n^{N_0}(Q_0x)\}_{n=1}^{N_0}$ and $\{\psi_n^{N_1}(Q_1x)\}_{n=1}^{N_1}$ are mutually orthogonal and pairwise stochastically independent on $(0; 1)$.

Both conclusions follow readily from (i)–(iv).

Lemma 3. *Suppose a function $f \in L^1_{(0,1)}$ is not equivalent to zero and*

$$A := \left\{ x \in (0; 1) : |f(x)| > \frac{1}{2} \|f\|_{L^1_{(0,1)}} \right\}.$$

Then

$$\text{mes } A \geq \|f\|_{L^1_{(0,1)}}^2 / 4 \|f\|_{L^2_{(0,1)}}^2. \quad (3)$$

Proof. Indeed, we obtain (using Hölder's inequality)

$$\begin{aligned} \|f\|_{L^1(0;1)} &= \int_A |f(x)|dx + \int_{(0;1)\setminus A} |f(x)|dx \leq \\ &\leq (\text{mes } A)^{1/2} \cdot \|f\|_{L^2(0;1)} + \frac{1}{2} \|f\|_{L^1(0;1)}, \end{aligned}$$

which immediately implies (3). \square

Lemma 4. *Let N, Q, m ($1 \leq m \leq N$) be positive integers. Then for any collection of natural numbers $1 \leq n_1 < n_2 < \dots < n_m \leq N$*

$$\text{mes} \left\{ x \in (0; 1) : \max_{1 \leq j \leq m} \left| \sum_{k=1}^j \psi_{n_k}^N(Qx) \right| > \frac{J}{2} \right\} \geq c \frac{m}{N},$$

where $\psi_n^N(x) \in \psi(N)$, $1 \leq n \leq N$ (see Lemma 2), and

$$J := \int_0^1 \max_{1 \leq j \leq m} \left| \sum_{k=1}^j \psi_{n_k}^N(Qx) \right| dx.$$

Proof. By Lemmas 1 and 2 (see also Remark 1) we have

$$\begin{aligned} \int_0^1 \left(\max_{1 \leq j \leq m} \left| \sum_{k=1}^j \psi_{n_k}^N(Qx) \right| \right)^2 dx &\leq cm \log_2^2(m+1), \\ J^2 &\geq c \frac{m^2}{N} \log_2^2 N; \quad 1 \leq m \leq N < \infty. \end{aligned}$$

Thus, applying Lemma 3, we obtain

$$\text{mes} \left\{ x \in (0; 1) : \max_{1 \leq j \leq m} \left| \sum_{k=1}^j \psi_{n_k}^N(Qx) \right| > \frac{J}{2} \right\} \geq c \frac{m}{N} \cdot \frac{\log_2^2 N}{\log_2^2(m+1)} \geq c \frac{m}{N}. \quad \blacksquare$$

Proof of the theorem. If the conditions (1) are fulfilled, then $R_k = k \log_2 k / \varepsilon(k)$, $2 \leq k < \infty$, where $\varepsilon(k)$ tends to infinity. Moreover, without loss of generality, it can be assumed that

- a) $R_1 = 1$;
- b) $\varepsilon(k)$ is a nondecreasing sequence of positive integers;
- c) the sets $\Delta_m := \{k : \varepsilon(k) = m\}$ have the form $(\nu_{m-1}; \nu_m] \cap \mathbb{N}$ ($m = 1, 2, \dots$), where $\nu_0 = 0$, $\log_2 \log_2 \log_2 \nu_m = p_m$ ($m \geq 1$) and $\{p_m\}_{m=1}^\infty$ is some increasing sequence of positive integers.

In particular, for $k \in \Delta_m$ ($m \geq 2$) we have

$$\begin{aligned} \log_2 k > \log_2 \nu_{m-1} &= 2^{2^{p_{m-1}}} \geq 2^{2^{m-1}} > m = \varepsilon(k), \\ \varepsilon(k \log_2 k) &\leq \varepsilon(\nu_m \log_2 \nu_m) < \varepsilon(\nu_{m+1}) = m+1 \leq 2\varepsilon(k). \end{aligned} \quad (4)$$

Denote

$$T_1 := 0, \quad T_k := 2^{\lceil k \log_2 k \log_2 \log_2 k \rceil}, \quad 2 \leq k < \infty, \quad (5)$$

and $E_m := \{k : R_k \leq T_m\}$, $2 \leq m < \infty$ (here $[x]$ is the integer part of the number x).

Since the function $\varphi(x) = x\varepsilon(x)/\log_2 x$ increases on $(e; \infty)$, taking into account (4) and (5), we have, for $m > \nu_1$,

$$\begin{aligned} E_m &= \{1; 2\} \cup \{k \geq 3 : R_k \leq T_m\} = \{1; 2\} \cup \{k \geq 3 : \frac{k \log_2 k}{\varepsilon(k)} \leq T_m\} = \\ &= \{1; 2\} \cup \{k \geq 3 : \varphi\left(\frac{k \log_2 k}{\varepsilon(k)}\right) \leq \varphi(T_m)\} \supset \\ &\supset \{1; 2\} \cup \{k \geq 3 : 2k \leq \varphi(T_m)\} = \{k \geq 1 : k \leq \frac{1}{2}\varphi(T_m)\}. \end{aligned}$$

Therefore for a large m ($m \geq m_1 > \nu_1$)

$$|E_m| > \frac{1}{2}\varphi(T_m) - 1 \geq \frac{1}{3} \frac{T_m \varepsilon(T_m)}{\log_2 T_m}, \quad (6)$$

$$|E_{m+1}| \geq \frac{1}{3} \frac{T_{m+1} \varepsilon(T_{m+1})}{\log_2 T_{m+1}} \geq \frac{1}{3} \frac{T_{m+1}}{T_m \log_2 T_{m+1}} \cdot T_m \geq 2T_m; \quad (7)$$

Because of

$$\sum_{k \in \Delta_m} \frac{1}{\log_2 T_k} = \sum_{k \in \Delta_m} \frac{1}{\lceil k \log_2 k \log_2 \log_2 k \rceil} > \frac{1}{8}(p_m - p_{m-1}) \geq \frac{1}{8} \quad (m \geq 2)$$

we can select a subsequence $T_{q_k} \equiv \tilde{T}_k$ ($1 \leq k < \infty$; $\tilde{T}_1 = 0$, $\tilde{T}_2 = \nu_1$) such that

$$\frac{1}{2m^2} < \sum_{k:q_k \in \Delta_m} \frac{1}{\log_2 \tilde{T}_k} \equiv \sum_{k:q_k \in \Delta_m} \frac{1}{\log_2 T_{q_k}} \leq \frac{1}{m^2}, \quad m \geq 2, \quad (8)$$

and hence

$$\begin{aligned} &\sum_{k:q_k \in \Delta_m} \frac{\varepsilon(\tilde{T}_k)}{\log_2 \tilde{T}_k} = \sum_{k:q_k \in \Delta_m} \frac{\varepsilon(T_{q_k})}{\log_2 T_{q_k}} \geq \\ &\geq \sum_{k:q_k \in \Delta_m} \frac{\varepsilon(q_k)}{\log_2 T_{q_k}} = m \sum_{k:q_k \in \Delta_m} \frac{1}{\log_2 T_{q_k}} > \frac{1}{2m}, \quad m \geq 2. \end{aligned} \quad (9)$$

Let

$$\begin{aligned} N_m &:= \tilde{T}_m - \tilde{T}_{m-1}, \\ Q_2 &:= 1, \quad Q_{m+1} := 4N_m Q_m, \quad 2 \leq m < \infty. \end{aligned} \quad (10)$$

Consider the orthonormal collections

$$\{\psi_n^{N_m}(Q_m x)\}_{n=1}^{N_m}, \quad 2 \leq m < \infty,$$

and construct with their aid the desired ONS $\{\varphi_n(x)\}_{n=1}^\infty$ as

$$\varphi_n(x) := \psi_k^{N_m}(Q_m x), \quad (11)$$

where $n \in (\tilde{T}_{m-1}; \tilde{T}_m]$, $k = n - \tilde{T}_{m-1}$, $2 \leq m < \infty$, $x \in (0; 1)$ (the orthonormality follows from (10) and Remark 2).

Let $\{n_k\}_{k=1}^\infty$ be a sequence of positive integers with $k \leq n_k \leq R_k$, $1 \leq k < \infty$. We set

$$\begin{aligned} G_m &:= \{k : \tilde{T}_{m-1} < n_k \leq \tilde{T}_m\}, \\ M_m &:= |G_m|, \\ a_n &:= ((1 + M_m) \log_2 \tilde{T}_m)^{-1/2} \text{ for } n \in (\tilde{T}_{m-1}; \tilde{T}_m]; \quad m \geq 2. \end{aligned}$$

On account of (8)

$$\begin{aligned} \sum_{k=1}^\infty a_{n_k}^2 &= \sum_{m=2}^\infty \sum_{k: \tilde{T}_{m-1} < n_k \leq \tilde{T}_m} a_{n_k}^2 = \sum_{m=2}^\infty \frac{M_m}{(1 + M_m) \log_2 \tilde{T}_m} < \\ &< \sum_{k=2}^\infty \frac{1}{\log_2 T_{q_k}} = \sum_{m=2}^\infty \sum_{k: q_k \in \Delta_m} \frac{1}{\log_2 T_{q_k}} < \infty. \end{aligned}$$

It is thus sufficient to show that the series

$$\sum_{k=1}^\infty a_{n_k} \varphi_{n_k}(x) \quad (12)$$

diverges on some set of positive measure.

Note that

$$\begin{aligned} G_m &= \{k : n_k \leq \tilde{T}_m\} \setminus \{k : n_k \leq \tilde{T}_{m-1}\} \supset \\ &\supset \{k : R_k \leq \tilde{T}_m\} \setminus \{k : k \leq \tilde{T}_{m-1}\}, \quad m \geq 2. \end{aligned}$$

Consequently in view of (6) and (7)

$$\begin{aligned} M_m &\geq |\{k : R_k \leq \tilde{T}_m\}| - |\{k : k \leq \tilde{T}_{m-1}\}| = |\{k : R_k \leq T_{q_m}\}| - T_{q_{m-1}} > \\ &> |E_{q_m}| - \frac{1}{2}|E_{q_{m-1}+1}| \geq \frac{1}{2}|E_{q_m}| \geq \frac{1}{6} \frac{\tilde{T}_m \varepsilon(\tilde{T}_m)}{\log_2 \tilde{T}_m}, \quad m \geq m_1. \end{aligned} \quad (13)$$

Hence by (11) and Lemma 2

$$\begin{aligned} \tilde{J}_m &:= \int_0^1 \max_{\tilde{T}_{m-1} < j \leq \tilde{T}_m} \left| \sum_{k: \tilde{T}_{m-1} < n_k \leq j} a_{n_k} \varphi_{n_k}(x) \right| dx \geq \\ &\geq c \cdot \frac{1}{\sqrt{(1 + M_m) \log_2 \tilde{T}_m}} \cdot \frac{\log_2 N_m}{\sqrt{N_m}} \cdot M_m \geq \\ &\geq c \sqrt{\frac{M_m \log_2 \tilde{T}_m}{\tilde{T}_m}} \geq c \sqrt{\varepsilon(\tilde{T}_m)}, \quad m \geq m_1, \end{aligned}$$

and therefore

$$\lim_{m \rightarrow \infty} \tilde{J}_m = \infty. \tag{14}$$

If A_m denotes the set

$$\left\{ x \in (0; 1) : \max_{\tilde{T}_{m-1} < j \leq \tilde{T}_m} \left| \sum_{k: \tilde{T}_{m-1} < n_k \leq j} a_{n_k} \varphi_{n_k}(x) \right| > \frac{1}{2} \tilde{J}_m \right\}, \quad m \geq 2,$$

then by (11), Lemma 4, (13) and (9)

$$\begin{aligned} \text{mes } A_m &\geq \frac{M_m}{N_m} > \frac{M_m}{\tilde{T}_m} > \frac{1}{6} \frac{\varepsilon(\tilde{T}_m)}{\log_2 \tilde{T}_m}, \quad m \geq m_1; \\ \sum_{m=2}^{\infty} \text{mes } A_m &\geq \frac{1}{6} \sum_{k=m_1}^{\infty} \frac{\varepsilon(\tilde{T}_k)}{\log_2 \tilde{T}_k} = \frac{1}{6} \sum_{k: q_k \geq q_{m_1}} \frac{\varepsilon(T_{q_k})}{\log_2 T_{q_k}} > \\ &> \frac{1}{6} \sum_{k: q_k > \nu_{m_1}} \frac{\varepsilon(T_{q_k})}{\log_2 T_{q_k}} = \frac{1}{6} \sum_{m=1+m_1} \sum_{k: q_k \in \Delta_m} \frac{\varepsilon(T_{q_k})}{\log_2 T_{q_k}} = \infty. \end{aligned} \tag{15}$$

It is easy to verify (see (10),(11), Remark 2) that $\{A_m\}_{m=2}^{\infty}$ is a sequence of stochastically independent sets. Therefore by (15) and the Borel-Cantelli lemma

$$\text{mes} \left(\limsup_{m \rightarrow \infty} A_m \right) = 1.$$

Hence we conclude because of (14) and the definition of sets A_m that the series (12) diverges almost everywhere on $(0;1)$. ■

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