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STABILIZATION OF FUNCTIONS AND ITS APPLICATION

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Abstract. The concepts of polynomial stabilization, strong polynomial stabilization, and strong stabilization are introduced for a fundamental system of solutions of linear differential equations. Some criteria of such kinds of stabilizations and applications to the theory of existence and uniqueness of solutions of ordinary differential equations are given. An abstract scheme of the obtained results is presented for Banach spaces.

1. Polynomial and Strong Polynomial Stabilization

Let us introduce concepts of stabilization and strong stabilization of functions. We begin by considering stabilization as $t \to +\infty$ of a function to a polynomial

$$P(t) = \sum_{m=0}^{n-1} c_m t^m \tag{1}$$

of degree at most $n-1$, where n is a fixed natural number.

Definition 1. An $n-1$ times differentiable function $x(t)$ on the infinite half-interval $[t_0, +\infty)$, $t_0 \in \mathbb{R}$ ($\mathbb{R}$ is the real line), is said to stabilize as $t \to +\infty$ to the polynomial (1) if

$$\lim_{t\to+\infty} \big(x(t) - P(t)\big)^{(j)} = 0, \quad j = 0, 1, \dots, n-1. \tag{2}$$

Let us write in this case $x(t) \sim P(t)$.

If such a polynomial exists (for a given function), then it is unique.

We introduce the notation

$$(I_m x)(t) = \int_t^{+\infty} dt_1 \int_{t_1}^{+\infty} dt_2 \cdots \int_{t_{m-1}}^{+\infty} x(t_m)\, dt_m, \quad m \in \mathbb{N}.$$

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It is possible to obtain a good enough description of functions stabilizing to polynomials in the class of functions having n and not $n-1$ derivatives as assumed by Definition 1.

Theorem 1. *A function $x(t)$ having a locally integrable derivative of order n on the half-interval $[t_0, +\infty)$ stabilizes as $t \to +\infty$ to a polynomial of degree at most $n-1$ iff the integral*

$$(I_n x^{(n)})(t_0) = \int_{t_0}^{+\infty} dt_1 \int_{t_1}^{+\infty} dt_2 \cdots \int_{t_{n-1}}^{+\infty} x^{(n)}(t_n)\, dt_n \tag{3}$$

converges.

Theorem 2. *If the integral* (3) *converges, then the function $x(t)$ stabilizes as $t \to +\infty$ to the the given polynomial* (1) *iff*

$$x(t) = P(t) + (-1)^n (I_n x^{(n)})(t). \tag{4}$$

The property (2) of the polynomial $P(t)$ is analogous to that of the Taylor polynomial of a function for the finite point t_0 when $t \to t_0$. However, in contrast to the latter polynomial, the polynomial with the property (2) for a given function $x(t)$ exists only for one number n.

The conditions of Theorem 1 are fulfilled if

$$\int_{t_0}^{+\infty} t^{m-1} |x^{(n)}(t)|^p \, dt < +\infty, \quad 1 \le p < +\infty, \quad m \in \mathbb{N}, \quad m > pn.$$

This case was considered by S.L. Sobolev [1]. V.N. Sedov [2] and the author [3] obtained some generalizations when the integral $\int\limits_{t_0}^{+\infty} \varphi(t) |x^{(n)}(t)|^p \, dt$ is finite for a nonnegative function φ. The general case, i.e., Theorems 1 and 2, is treated in [4, 5].

The coefficients of the polynomial (1) to which the function $x(t)$ stabilizes as $t \to +\infty$ can be calculated (see [4]) by the formula

$$\begin{aligned} c_{n-m} = \frac{1}{(n-m)!} \Bigg[&\sum_{j=1}^{m} \frac{(-1)^{m-j}}{(m-j)!} x^{(n-j)}(t_0) t_0^{m-j} + \\ &+ (-1)^{m-1} \sum_{k=0}^{m-1} \frac{t_0^k}{k!} (I_{m-k} x^{(n)})(t_0) \Bigg], \quad m = 1, 2, \ldots, n. \end{aligned}$$

If the function $x(t)$ is $n-1$ times differentiable on $[t_0, +\infty)$, then there exists its one and only one representation

$$x(t) = \sum_{m=0}^{n-1} y_{x,m}(t) t^m \tag{5}$$

such that

$$x^{(k)}(t) = \sum_{m=k}^{n-1} \frac{m!}{(m-k)!} y_{x,m}(t) t^{m-k}, \quad k = 0, 1, \ldots, n-1, \ t \geq t_0, \quad (6)$$

i.e., coefficients $y_{x,m}(t)$ behave if they were constants by $n-1$ times differentiation of the expression occurring on the right-hand side of the equality (5). Such representations of functions will be called polynomial Lagrange representations. Representations of this kind emerge when we use Lagrange's method of variation of constants for solving linear nonhomogeneous ordinary differential equations.

Definition 2. An $n-1$ times differentiable function $x(t)$ on the half-interval $[t_0, +\infty)$ is said to strongly stabilize as $t \to +\infty$ to the polynomial (1) if

$$\lim_{t \to +\infty} y_{x,m}(t) = c_m, \quad m = 0, 1, \ldots, n-1, \quad (7)$$

where $y_{x,m}(t)$ are the coefficients of the polynomial Lagrange representation of $x(t)$.

In this case let us write $x(t) \approx P(t)$.

If the function $x(t)$ has n derivatives, then for derivatives of the coefficients of its polynomial Lagrange representation we have the formula

$$y'_{x,n-k}(t) = \frac{(-1)^{k-1}}{(n-k)!(k-1)!} x^{(n)}(t) t^{k-1}, \quad k = 1, 2, \ldots, n.$$

Theorem 3. *A function $x(t)$ having a locally integrable derivative of order n on the half-interval $[t_0, +\infty)$ strongly stabilizes as $t \to +\infty$ to the polynomial* (1) *iff the integral*

$$\int_{t_0}^{+\infty} t^{n-1} x^{(n)}(t)\, dt \quad (8)$$

converges.

We observe that if the integral (8) converges, then so does the integral (3), but some examples show that the converse statement is wrong [4]. Thus if some function strongly stabilizes as $t \to +\infty$ to a polynomial, then it stabilizes to a polynomial too, but not conversely. More exactly, the following theorem is valid.

Theorem 4. *If the integral* (8) *converges, then a function $x(t)$ stabilizes as $t \to +\infty$ to the polynomial* (1) *iff this function strongly stabilizes as $t \to +\infty$ to the same polynomial.*

We note the special role of polynomials

$$Q_r(t) = \sum_{j=0}^{r} \frac{(-1)^j}{j!} x^j, \quad r = 0, 1, \ldots,$$

for the polynomial Lagrange representation of functions.

Theorem 5. *The coefficients $y_{x,m}(t)$ of the polynomial Lagrange representation of an $n-1$ times differentiable function $x(t)$ on $[t_0, +\infty)$ can be calculated by the formula*

$$y_{x,m}(t) = \frac{1}{m!} Q_{n-m-1}\Big(t \frac{d^m}{dt^m}\Big) x(t), \quad m = 0, 1, \ldots, n-1.$$

The following criterion plays an important role for strong stabilization to polynomials.

Theorem 6. *If the integral* (8) *converges, then a function $x(t)$, having a locally integrable derivative of order n on $[t_0, +\infty)$, strongly stabilizes to the polynomial* (1) *iff the identity*

$$x(t) = P(t) + \sum_{m=0}^{n-1} \frac{(-1)^{n-m}}{m!(n-m-1)!} t^m \int_t^{+\infty} s^{n-m-1} x^{(n)}(s)\, ds \tag{9}$$

holds.

We introduce some Banach spaces for stabilized and strongly stabilized functions.

Let $\widetilde{X}_t$ be a set of all $n-1$ times continuously differentiable functions on the half-interval $[t, +\infty)$, $t \geq t_0$ which stabilize as $t \to +\infty$ to polynomilas of degree at most $n-1$, i.e., $\widetilde{X}_t = \{C^{n-1}[t, +\infty) : \exists P \sim x\}$.

We shall use the notation

$$P_x(t) = \sum_{m=0}^{n-1} c_{x,m} t^m \tag{10}$$

for the polynomial to which the given function $x(t)$ stabilizes as $t \to +\infty$ and assume that

$$\mathbf{c}_x = (c_{x,0}, c_{x,1}, \ldots, c_{x,n-1}). \tag{11}$$

Theorem 7. *The set $\widetilde{X}_t$ is a Banach space with the norm*

$$\|x\|_t = \|x - P_x\|_{C^{n-1}[t,+\infty)} + |\mathbf{c}_x|. \tag{12}$$

The polynomial stabilization is continuous with respect to the norm (12).

Let $\widetilde{\widetilde{X}}_t$ be a set of all functions $n-1$ times continuously differentiable on the half-interval $[t,+\infty)$, $t \geq t_0$, and strongly stabilized as $t \to +\infty$ to polynomials of degree at most $n-1$, i.e., $\widetilde{\widetilde{X}} = \{x \in C^{n-1}[t,+_\infty) : \exists P \approx x\}$, and let $\mathbf{y}_x = (y_{x,0}, y_{x,1}, \dots, y_{x,n-1})$, where $y_{x,m} = y_{x,m}(t)$ are the coefficients of the polynomial Lagrange representation of the given function $x(t)$. Thus, if $x(t) \approx P(t)$, then

$$\lim_{t\to+\infty} \mathbf{y}_x(t) = \mathbf{c}_x. \tag{13}$$

Theorem 8. *The set* $\widetilde{\widetilde{X}}_t$ *is a Banach space with the norm* $|||x|||_t = \|y_x\|_{C_n[t,+\infty)}$, *where* $C_n[t,+\infty)$ *is a Banach space of all* n*-dimensional vector functions with the uniform norm, continuous and bounded on the half-interval* $[t,+\infty)$.

2. Some Applications of Polynomial and Strong Polynomial Stabilizations to the Existence and Uniqueness of Solutions of Ordinary Differential Equations

Let us consider a differential equation

$$x^{(n)}(t) = f(t, x, x', \dots, x^{(n-1)}), \tag{14}$$

where $f : [t_0,+\infty) \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function.

First we shall study the problem of stabilization of solutions of equation (14) to the given polynomial (1):

$$x(t) \sim P(t). \tag{15}$$

Note that we do not obtain simpler problems by the change of the variables $t = 1/s$ or $x = x_1,\ x' = x_2, \dots, x^{(n-1)} = x_n$.

Theorem 9. *A solution* $x(t)$ *of equation* (14) *stabilizes as* $t \to +\infty$ *to the polynomial* (1) *iff it is a solution of the integral equation*

$$x(t) = P(t) + (-1)^n \big(I_n f(\cdot, x, x', \dots, x^{(n-1)})\big)(t). \tag{16}$$

It is useful to note that in the case of strong stabilization of solutions of equation (14) we naturally obtain another integral equation which is equivalent to the differential equation (14) (see Theorem 11 below).

Definition 3. Let g be a function such that $g : [t_0,+\infty) \times \mathbb{R}^n \to \mathbb{R}$ and let Y be a set of some functions $n-1$ times differentiable on the half-interval $[t_0,+\infty)$.

The integral $(I_n g(\cdot, x, x', \dots, x^{(n-1)}))(t_0)$ is called strongly uniformly convergent (on the n-dimensional half-interval $[t_0, +\infty)^n$) with respect to the set Y if for every $\varepsilon > 0$ there exists $t_\varepsilon \geq t_0$ such that for all functions $x(t) \in Y$ and for all $t > t_\varepsilon$ we have the inequalities $\left|\left(I_m g(\cdot, x, x', \dots, x^{(n-1)})\right)(t)\right| < \varepsilon$, $m = 1, 2, \dots, n$.

We introduce the notation

$$\widetilde{X}_T(0) = \{x \in \widetilde{X}_T \,:\, x \sim 0\}, \quad \widetilde{X}_T(P) = \widetilde{X}_T(0) + P,$$
$$\widetilde{Q}_T(P, r) = \{x \,:\, \widetilde{X}_T(P) \,:\, \|x - P\|_T \leq r\}, \quad T \geq t_0,$$
$$\mathbf{x} = (x, x', \dots, x^{(n-1)}), \text{ in particular, } \mathbf{P} = (P, P', \dots, P^{(n-1)}), \tag{17}$$
$$f(t, \mathbf{x}) = f(t, x, x', \dots, x^{(n-1)}).$$

Theorem 10. *If the polynomial* (1) *is given, if for every $r > 0$ the integral $\left(I_n f(\cdot, x)\right)(t_0)$ is strongly uniformly convergent with respect to the ball $\widetilde{Q}_{t_0}(P, r)$, if for every $r > 0$ there exists a function $\varphi_r : [t_0, +\infty) \to [0, +\infty)$ such that $(I_n \varphi_r)(t_0) < +\infty$ and for all $t \geq t_0$, $\boldsymbol{\xi} \in \mathbb{R}^n$, $\boldsymbol{\eta} \in \mathbb{R}^n$, $|\boldsymbol{\xi}| \leq r$, $|\boldsymbol{\eta}| \leq r$ the inequality*

$$|f(t, \mathbf{P} + \boldsymbol{\eta}) - f(t, \mathbf{P} + \boldsymbol{\xi})| \leq \varphi_r(t) |\boldsymbol{\eta} - \boldsymbol{\xi}| \tag{18}$$

holds, then there exists a $T \geq t_0$ such that on the half-interval $[T, +\infty)$ there exists one and only one solution of equation (14) *which stabilizes as $t \to +\infty$ to the given polynomial* (1).

An equation of the type $x' = f(x)/(1 + t^2)$, where for every $r > 0$ the function $f(x)$ is bounded on the set of all functions $x(t) \in \widetilde{X}_{t_0}$ belonging to the ball $\|x\|_{C[t_0, +\infty)} \leq r$, is an example of equations satisfying the conditions of Theorem 10.

Note that solutions of equation (14), for which the conditions of Theorem 10 are fulfilled, can have no absolutely integrable derivatives. The simplest example of such an equation is $x' = \sin t / t$.

Let us now consider the case of strong stabilization of solutions as $t \to +\infty$ to a polynomial.

Theorem 11. *A solution $x(t)$ of the equation* (14) *strongly stabilizes as $t \to +\infty$ to the polynomial* (1) *iff it is a solution of the integral equation*

$$x(t) = P(t) + \sum_{m=0}^{n-1} \frac{(-1)^{n-m}}{m!(n-m-1)!} t^m \int_t^{+\infty} s^{n-m-1} f(s, x, x', \dots, x^{(n-1)})\, ds.$$

Setting now $\boldsymbol{\xi} = (\xi_0, \xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^n$, $P(t, \boldsymbol{\xi}) = \sum_{m=0}^{n-1} \xi_m t^m$, $\widetilde{\widetilde{Q}}_T(r) = \{x \in \widetilde{\widetilde{X}} \,:\, |||x|||_T \leq r\}$ and using the notation (17), we obtain $\mathbf{P}(t, \xi) = (P(t, \xi), P'(t, \xi), \dots, P^{(n-1)}(t, \xi))$.

Theorem 12. *If the polynomial* (1) *is given, if for every $r > 0$ the integral*

$$\int_{t_0}^{+\infty} t^{n-1} f(t, \mathbf{x})\, dt$$

uniformly converges with respect to the ball $\widetilde{\widetilde{Q}}_{t_0}(r)$, if for every $r > 0$ there exists a function $\psi_r : [t_0, +\infty) \to [0, +\infty)$ such that

$$\int_{t_0}^{+\infty} t^{n-1} \psi_r(t)\, dt < +\infty \tag{19}$$

and for all $t \geq t_0$, $\boldsymbol{\xi} \in \mathbb{R}^n$, $\boldsymbol{\eta} \in \mathbb{R}^n$, $|\boldsymbol{\xi}| \leq r$, $|\boldsymbol{\eta}| \leq r$, we have the inequality

$$|f(t, \mathbf{P}(t, \boldsymbol{\eta})) - f(t, \mathbf{P}(t, \boldsymbol{\xi}))| \leq \psi_r(t) |\boldsymbol{\eta} - \boldsymbol{\xi}|, \tag{20}$$

then there exists a $T \geq t_0$ such that on the half-interval $[T, +\infty)$ there exists one and only one solution of (14) *which strongly stabilizes as $t \to +\infty$ to the given polynomial* (1).

It is obvious that, in contrast to simple stabilization, in the case of strong stabilization we need some other generalization of the Lipschitz condition (compare the relations (18) and (20)).

The theorem about continuous dependence of solutions of the equation (14) on the stabilization data holds in the space $\widetilde{\widetilde{X}}_{t_0}$.

Theorem 13. *Let the polynomials $P(t, \mathbf{a}_j)$, $\mathbf{a}_j \in \mathbb{R}^n$, $j = 1, 2$, of degree at most $n-1$ be given. If the solution $x_j(t)$ of equation* (14) *is defined on the half-interval $[t_0, +\infty)$, if it strongly stabilizes as $t \to +\infty$ to the polynomial $P(t, \mathbf{a}_j)$, $j = 1, 2$, and if for some $r > \max_{j=1,2} |||x_j|||_{t_0}$ there exists a function $\psi_r(t)$ satisfying the conditions* (19), (20), *then the inequality*

$$|||x_2 - x_1|||_{t_0} \leq |\mathbf{a}_2 - \mathbf{a}_1| n \exp n \int_{t_0}^{+\infty} t^{n-1} \psi_r(t)\, dt$$

is valid.

3. General Case of the Strong Stabilization Problem

Let the equation

$$Lx = f(t, x, x', \dots, x^{(n-1)}), \tag{21}$$

be given, where

$$L = \frac{d^n}{dt^n} + \sum_{k=0}^{n-1} p_m(t) \frac{d^m}{dt^m} \tag{22}$$

$p_m(t)$ are continuous functions on the interval (a,b), $m = 0,1,\dots,n-1$, $-\infty \le a < b \le +\infty$.

In the case of polynomial stabilization we have $L = \frac{d^n}{dt^n}$, $a = t_0 \in \mathbb{R}$, and $b = +\infty$. For an arbitrarily chosen linear operator L it is possible to write equation (14) in the form (21), and conversely. Of course the right-hand sides of the equations will be different. So equation (21) is not an equation of a new type but only a new notation.

Assume that

$$v_1, v_2, \dots, v_n \tag{23}$$

in some fundamental system of solutions of the equation

$$Lx = 0. \tag{24}$$

Let $x(t)$ be an $n-1$ times differentiable function on the interval (a,b) and

$$x(t) = \sum_{j=1}^{n} y_{x,j}(t) v_j(t), \quad a < t < b, \tag{25}$$

be its Lagrange representation with respect to the system (23), i.e. a representation such that

$$x^{(m)}(t) = \sum_{j=1}^{n} y_{x,j}(t) v_j^{(m)}(t), \quad m = 0,1,\dots,n-1, \;\; a < t < b. \tag{26}$$

For every function $n-1$ times differentiable on the interval (a,b) there exists one and only one Lagrange representation (25), since the determinant of the system of linear equations (26) with respect to the variables $y_{x,j}$, $j = 1,2,\dots,n$, is the Wronskian of the system (23).

Let now $v(t) \in \ker L$; therefore

$$v(t) = \sum_{j=1}^{n} c_j v_j(t), \tag{27}$$

where c_j are some constants, $j = 1,2,\dots,n$; also let k, l be some nonnegative integers, $1 \le k + l \le n$.

Definition 4. An $n-1$ times differentiable function on the interval (a,b) is said to strongly (k,l)-stabilize to the function (27) if

$$\lim_{t\to a} y_{x,j}(t) = c_j, \;\; j = 1,2,\dots,k,$$
$$\lim_{t\to b} y_{x,j}(t) = c_j, \;\; j = k+1, k+2, \dots, k+l.$$

In this case let us write $x(t) \underset{(k,l)}{\approx} v(t)$.

The general problem is to find solutions of equation (21) which strongly (k,l)-stabilize to a given function $v(t) \in \ker L$. Let us briefly discuss this problem. It includes the classical boundary problems on finite segments, the Cauchy problem, and some new problems.

Indeed, if all coefficients $p_m(t)$, $m = 0, 1, \dots, n-1$, of the operator (22) are continuous on the half-interval $(a, b]$, $b \in \mathbb{R}$, then all functions (23) with all their derivatives up to order n inclusive are continuous at the point $t = b$. Therefore the system of identities (26) implies as $t \to b$ that

$$x^{(m)}(b) = \sum_{j=1}^{n} c_j v_j^{(m)}(b), \quad m = 0, 1, \dots, n-1.$$

From this we conclude that giving the function $v(t) \in \ker L$, i.e., giving the coefficients $c_1, c_2, \dots, c_n$, is equivalent to giving the Cauchy data $x(b), x'(b), \dots, x^{(n-1)}(b)$. Therefore in this case the problem of strong $(0,n)$-stabilization of solutions of equation (21) is equivalent to the Cauchy problem.

In the case where $k \geq 1$, $l \geq 1$ and $-\infty < a < b < +\infty$, it is possible to see (when the coefficients $p_m(t)$, $m = 0, 1, \dots, n-1$, are continuous at $t = a$ and $t = b$) that the problem of strong (k,l)-stabilization of solutions to a given $v \in \ker L$ is equivalent to a classical boundary value problem of the type in which $x(a)$, $x'(a)$, $\dots$ $x^{(k-1)}(a)$, $x(b)$, $x'(b)$, $\dots$, $x^{(l-1)}(b)$ are given.

An example of the new problem is given in [6]. There the Euler equation

$$Lx + f = 0 \tag{28}$$

with L as a quadratic integral functional depending on a function and its derivatives up to order n inclusive is considered. Under some restrictions imposed on the coefficients of the integrand of the given functional the existence and uniqueness of a generalized solution on the half-interval $[t_0, +\infty)$ are proved when the following stabilization data are given:

1) the solution stabilizes as $t \to +\infty$ to some polynomial (1);

2) the values $x^{(i_1)}(t_0), \dots, x^{(i_k)}(t_0)$ and the coefficients $c_{j_1}, c_{j_2}, \dots, c_{j_l}$ of the polynomial (1) are given.

Some conditions are established for the indices

$$\{i_\mu\}_{\mu=1}^{\mu=k}, \quad \{j_\nu\}_{\nu=1}^{\nu=l} \tag{29}$$

when one and only one generalized solution exists.

Let the indices (29) be increasing sequences of integers belonging to the set $\{0, 1, \dots, n-1\}$:

$$0 \le i_1 < i_2 < \dots < i_k \le n-1, \quad 1 \le k \le n,$$
$$0 \le j_1 < j_2 < \dots < j_l \le n-1, \quad 1 \le l \le n.$$

If $k+l = n$, we introduce the notation $\{\bar{i}_\nu\}_{\nu=1}^{\nu=l}$ for the complement of the set of indices $\{i_\mu\}$ to the set $\{0, 1, \dots, n-1\}$. Assume that $\bar{i}_1 < \bar{i}_2 < \dots < \bar{i}_l$.

If $k + l = n$, then the conditions $j_1 \le \bar{i}_1$, $j_2 \le \bar{i}_2$, ... , $j_l \le \bar{i}_l$ are called the Pólya ones. When one of the sets $\{i_\mu\}_{\mu=1}^{\mu=k}$ or $\{j_\nu\}_{\nu=1}^{\nu=l}$ is empty, the system (29) is also said to satisfy the Pólya conditions.

These conditions were introduced by Pólya. He proved (see [7]) that there exists one and only one polynomial (1) with the given values $P^{(i_\mu)}(0)$, $P^{(j_\nu)}$, $\mu = 1, 2 \dots, k$, $\nu = 1, 2, \dots, l$, $k+l = n$, iff the system of indices (29) satisfies the Pólya conditions. It is evidently a purely algebraic problem.

In the general case, i.e., when $k + l \le 2n$, the system (29) is called complete if it contains some subsystem satisfying the Pólya conditions.

For the complete system (29) we evidently have $n \le k + l \le 2n$.

If the system (29) is complete, then there exists one and only one generalized solution $x(t)$ of the Euler equation (28) with the given values $x^{(i_\mu)}(t_0)$, c_{j_ν}, $\mu = 1, 2, \dots, k$, $\nu = 1, 2, \dots, l$ where c_{j_ν} are some coefficients of the polynomial (1) to which the solution $x(t)$ stabilizes as $t \to +\infty$. It is interesting to note that in contrast to the Pólya case it is a purely analytic problem. If the system of indices (29) is not complete, then there exist examples for which the problem under consideration has more than one solution.

One can prove that the function n times continuously differentiable on the interval (a, b) strongly $(k, n-k)$-stabilizes ($k = 0, 1, \dots, n-1$) to the function $v(t) \in \ker L$ iff the identity

$$x(t) = v(t) + \int_a^b G_k(t, s) Lx(s) ds \tag{30}$$

holds.

Here $G_k(t, s)$ is the generalized Green's function. This function strongly $(k, n-k)$-stabilizes to zero (at the ends of the interval (a, b)), but it and its appropriate derivatives do not in general tend to zero as $t \to a$ and $t \to b$, as it should be if $G_k(t, s)$ were the ordinary Green's function.

Let us assume now that the solution of equation (21) strongly $(0, n)$-stabilizes to a function $v(t) \in \ker L$. Then the standard change of variables $x = x_1$, $x' = x_2$, ... , $x^{(n-1)} = x_n$z is recommended. We obtain a system of the type

$$Lx = f(t, \mathbf{x}), \quad \mathbf{x} = (x_1, x_2, \dots, x_n),$$
$$L = \frac{d}{dt} + A, \tag{31}$$

where A is a continuous matrix of order $n \times n$ on the interval (a, b), and $f : (a, b) \times \mathbb{R}^n \to \mathbb{R}^n$.

Let

$$\mathbf{v}_i = (v_{i1}, v_{i2}, \dots, v_{in}), \quad i = 1, 2, \dots, n, \tag{32}$$

be a fundamental system of the homogeneous equation $L\mathbf{x} = 0$,

$$V = (v_{ij}), \quad i, j = 1, 2, \dots, n, \tag{33}$$

be a fundamental matrix of the system (32), $\mathbf{x}(t)$ be a differentiable function on the interval (a, b),

$$\mathbf{x}(t) = \sum_{i=1}^{n} y_{x,i}(t)\mathbf{v}_i(t) \tag{34}$$

(by analogy with scalar functions this representation of the vector function $\mathbf{x}(t)$ will be called the Lagrange representation); also let

$$\mathbf{y}_{\mathbf{x}}(t) = (y_{\mathbf{x},1}(t), y_{\mathbf{x},2}(t), \dots, y_{\mathbf{x},n}(t)). \tag{35}$$

The vector function $\mathbf{x}(t)$ is called strongly stabilized as $t \to b$ to the function

$$\mathbf{v}(t) = \sum_{i=1}^{n} c_i v_i(t) \in \ker L \tag{36}$$

if

$$\lim_{t \to b} \mathbf{y}_{\mathbf{x}}(t) = \mathbf{c}, \quad \mathbf{c} = (c_1, c_2, \dots, c_n). \tag{37}$$

The function $\mathbf{x}(t)$ strongly stabilizes to the function (35) iff the identity

$$\mathbf{x}(t) = V\mathbf{c} - V \int_t^b V^{-1}(s)L\mathbf{x}(s)ds$$

holds.

The problem of strongly $(0, n)$-stabilized solutions of equation (21) is reduced to the problem of strongly stabilized solutions of equation (31). The formula (34) implies $\mathbf{x} = V\mathbf{y}$ and for the vector function $\mathbf{y}$ we obtain the equation (see [8]) $\mathbf{y}' = V^{-1}f(t, V\mathbf{y})$; thus the condition (37) has to be fulfilled.

As in the case of strong stabilization to polynomials, one can obtain the existence and uniqueness theorem for solutions of the system (31) in a neighborhood of the point $t = b$ only if these solutions strongly stabilize as $t \to b$ to some function $\mathbf{v} \in ker\, L$ (see [9]).

4. Abstract Scheme

The main idea of this paper is to establish (under some restrictions) that for every given solution $v(t)$ of the linear homogeneous equation $Lx = 0$ there exists one and only one solution $x(t)$ of the nonhomogeneous equation $Lx = f(t, x)$ if only this solution $x(t)$ stabilizes to the solution $v(t)$.

The first question in the case of abstract spaces is connected with defining the concept of stabilization, since in the function case this concept is based on the concept of the limit of functions. It is recommended to use for this purpose the generalization of the representations of functions (4), (9), and (30).

Let X and Y be linear spaces, $L : X \to Y$, $F : X \to Y$, $S : Y \to X$, $Y = L(X)$, where L and S are linear operators, and F is in general a nonlinear operator; also let $LS = Id$, where Id is the identity operator of the space Y onto itself. Then the following decomposition in the direct sum holds: $X = \ker L \oplus S(Y)$.

Definition 5. An element $x \in X$ is called S-stabilized to an element $v \in ker\, L$ if $x = v + SLx$.

In this case we can write $x \underset{S}{\sim} v$.

Under the above assumptions, for every element $v \in ker\, L$ the equations $Lx = Fx$ and $x = v + SFx$ are equivalent on the set of all elements of the space X which are S-stabilized to the given element $v \in ker\, L$.

If X, Y are Banach spaces and SF is a contracting operator, then for any $v \in ker\, L$ there exists one and only one solution $x \underset{S}{\sim} v$ in the space X.

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