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A SELF-ADJOINT "SIMULTANEOUS CROSSING OF THE AXIS"

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Abstract. By constructing the corresponding Green's function in a trapezoidal domain, we establish the existence of self-adjoint realizations of $A = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}$ incorporating boundary conditions of the form $u(s,0) = u(s,T) = 0$. Such operators correspond to the historically important concept of a "simultaneous crossing of the axis" for vibrating strings.

1. Introduction. Given Poisson's equation

$$\begin{gathered} \Delta u \equiv u_{xx} + u_{yy} = f(x,y) \quad \text{in} \quad R, \\ u = 0 \quad \text{on} \quad \partial R \end{gathered} \tag{1.1}$$

in a rectangle

$$R = \{(x,y) : 0 \le x \le h,\ 0 \le y \le k\},$$

separation of variables leads to the Fourier series solution

$$u(x,y) = \sum a_{mn} \sin \frac{m\pi x}{h} \sin \frac{n\pi y}{k}$$

with

$$a_{mn} = \frac{-4}{hk\pi^2(\frac{m^2}{h^2} + \frac{n^2}{k^2})} \iint_R f(x,y) \sin \frac{m\pi x}{h} \sin \frac{n\pi y}{k} dx\, dy. \tag{1.2}$$

The validity of this formal solution is related to the fact that Δ has a self-adjoint realization in $L^2(R)$ corresponding to the boundary condition $u = 0$ on ∂R.

If one naively attempts to use the same approach to

$$\begin{gathered} Au \equiv u_{xx} - u_{yy} = f(x,y) \quad \text{in} \quad R, \\ u = 0 \quad \text{on} \quad \partial R, \end{gathered} \tag{1.3}$$

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separation of variables again provides the same form of series solution. However, now

$$a_{mn} = \frac{-4}{hk\pi^2(\frac{m^2}{h^2} - \frac{n^2}{k^2})} \iint_R f(x,y) \sin\frac{m\pi x}{h} \sin\frac{n\pi y}{k} dx\, dy, \tag{1.4}$$

and the fact that the Dirichlet problem is not well posed for (1.3) is reflected by the sign change in going from (1.2) to (1.4). While the series solution for $u(x,y)$ makes formal sense for irrational values of h/k, its instability precludes the existence of a Green's function $G(x,y;\xi,\eta)$ and a corresponding representation of this "solution" in the form

$$u(x,y) = \iint_R G(x,y;\xi,\eta) f(\xi,\eta) d\xi\, d\eta.$$

One therefore would not expect to find a self-adjoint realization of A in $L^2(R)$ corresponding to $u = 0$ on ∂R.

The fact remains, however, that problems such as (1.3) have a certain appeal. Replacing x by a spatial variable s and y by a temporal variable t, it is natural, in the theory of vibrating strings, to consider

$$\begin{gathered} Au \equiv u_{tt} - u_{ss} = f(s,t), \\ u(s,0) = u(s,T) = 0. \end{gathered} \tag{1.5}$$

Here the boundary conditions can be interpreted as calling for a "simultaneous crossing of the axis" at $t = 0$ and $t = T$. As described by Cannon and Dostrovski [1], the physical concept played an important role in early attempts by both Brook Taylor and Johann Bernoulli to model vibrating strings.

Also, if one interprets the classical Sturmian theory for $(py')' + qy = 0$ in terms of the motion of a mass $p(t)$ subject to a linear restoring force $-q(t)y$, then it becomes very attractive to consider (1.5) as part of an effort to generalize Sturmian theory to hyperbolic PDEs (see, for example, [4]). While one would not expect to find a self-adjoint realization of A corresponding to (1.3), there do exist both historical and mathematical reasons for seeking self-adjoint realizations which incorporate (1.5).

The purpose of this paper is to show that, by considering (1.5) on a trapezoid

$$R = \{(s,t) : t \le s \le L - t;\ 0 \le t \le T\} \tag{1.6}$$

with $L \ge 2T > 0$, it becomes possible to establish self-adjoint realizations of (1.5) in terms of additional boundary conditions on the characteristics

$$s - t = 0 \quad \text{and} \quad s + t = L \quad (0 \le t \le T).$$

In case $L = 2T$, these results are related to ones obtained by Kalmenov [2], albeit by very different techniques.

2. Fundamental Singularities. If one seeks a representation of solutions of $y'' = f(x)$ in the form

$$y = \int_a^{x^-} \Gamma(x,\xi)f(\xi)d\xi + \int_{x^+}^b \Gamma(x,\xi)f(\xi)d\xi,$$

two applications of Leibniz's rule readily lead to the conditions

$$\Gamma_{xx} = 0 \quad \text{for} \quad x \neq \xi, \quad \Gamma(x,x^-) - \Gamma(x,x^+) = 0,$$

and

$$\Gamma_x(x,x^-) - \Gamma_x(x,x^+) = 1$$

as characterizing a fundamental singularity for $\frac{d^2}{dx^2}$. In [3] this familiar idea is extended to obtain a characterization of a fundamental singularity for $\frac{\partial^2}{\partial x \partial y}$. Representing the solution of $u_{xy} = f(x,y)$ in the form

$$u(x,y) = \int_c^{y^-} \Big[\int_a^{x^-} \Gamma(x,y;\xi,\eta)f(\xi,\eta)d\xi + \int_{x^+}^b \Gamma(x,y;\xi,\eta)f(\xi,\eta)d\xi \Big] d\eta +$$
$$+ \int_{y^+}^d \Big[\int_a^{x^-} \Gamma(x,y;\xi,\eta)f(\xi,\eta)d\xi + \int_{x^+}^b \Gamma(x,y;\xi,\eta)f(\xi,\eta)d\xi \Big] d\eta,$$

repeated applications of Leibniz's rule lead to the following characterization of a fundamental singularity for $\frac{\partial^2}{\partial x \partial y}$:

(i) $\Gamma_{xy} = 0 \quad \text{for} \quad x \neq \xi \quad \text{and} \quad y \neq \eta$,

(ii) $\Gamma_x(x,y;\xi,y^-) = \Gamma_x(x,y;\eta,y^+)$,
$\Gamma_y(x,y;x^-,\eta) = \Gamma_y(x,y;x^+,\eta)$,

(iii) $\Gamma(x,y;x^-,y^-) - \Gamma(x,y;x^-,y^+) - \Gamma_x(x,y;x^+,y^-) + {} + \Gamma(x,y;x^+,y^+) = 1$.

Transforming such singularities into the (s,t)-plane by $t = y + x$, $s = y - x$ (and taking note of the fact that $|J\begin{pmatrix} x, & y \\ s, & t \end{pmatrix}| = \frac{1}{2}$), one obtains, as a special case, the following

Lemma 2.1. *If $G(s,t;\sigma,\tau)$ satisfies*

$$G(s,t;\sigma,\tau) = \begin{cases} \frac{1}{4} & \text{for } |t-\tau| > |s-\sigma| \\ 0 & \text{for } |s-\sigma| > |t-\tau| \end{cases} \tag{2.1}$$

or

$$G(s,t;\sigma,\tau) = \begin{cases} -\frac{1}{4} & \textit{for } |s-\sigma| > |t-\tau| \\ 0 & \textit{for } |t-\tau| > |s-\sigma|, \end{cases} \tag{2.2}$$

then G is a fundamental singularity for $A = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}$. Furthermore, G is symmetric in the sense that $G(s,t;\sigma,\tau) = G(\sigma,\tau,s,t)$.

Proof. Transforming back into the (ξ,η)-plane, the above values for G yield

$$\Gamma(x,y;x^-,y^-) - \Gamma(x,y;x^-,y^+) - \Gamma(x,y;x^+,y^-) + \Gamma(x,y;x^+,y^+) = 1. \qquad \square$$

Lemma 2.2. *If γ is a real constant and $G(s,t;\sigma,\tau)$ satisfies*

$$G(s,t;\sigma,\tau) = \begin{cases} 0 & \textit{for } |t-\tau| > |s-\sigma| \\ \gamma & \textit{for } s-\sigma > |t-\tau| \\ -\gamma & \textit{for } |\sigma-s| > |t-\tau|, \end{cases}$$

then G is "nonsingular" in the sense that

$$A \iint_R G(s,t;\sigma,\tau) f(\sigma,\tau) d\sigma\, d\tau = 0$$

for R a neighborhood of (s,t) in the (σ,τ)-plane.

Proof. Transforming back into the (ξ,η)-plane, the above values for G yield

$$\Gamma(x,y;x^-,y^-) - \Gamma(x,y;x^-,y^+) - \Gamma(x,y;x^+,y^-) + \Gamma(x,y;x^+,y^+) = 0$$

in place of (iii).

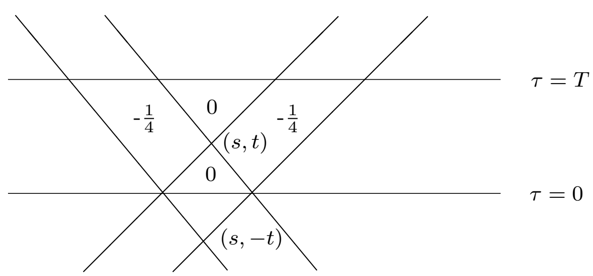

Figure 1

3. Construction of Green's Functions. To construct symmetric Green's functions corresponding to (1.5), we shall apply the method of images to symmetric fundamental singularities of the form (2.1). In order to satisfy $u(s,0)=0$, we consider a pair of fundamental singularities to form

$$H(s,t;\sigma,\tau)=G(s,t;\sigma,\tau)-G(s,-t;\sigma,\tau).$$

Restricting $H(s,t;\sigma,\tau)$ to the strip $0\le t\le T$, we have

$$H(s,t;\sigma,\tau)=\begin{cases}0 & \text{for } |t-\tau|>|s-\sigma|\\ -\frac{1}{4} & \text{for } |\tau+t|>|\sigma-s|>|t-\tau|\end{cases}$$

(see Figure 1). Since H is composed of symmetric singularities, it is again symmetric.

Noting that $H(s,T;\sigma,\tau)\ne 0$, we now reflect about $t=T$ to consider $H(s,t;\sigma,\tau)-H(s,2T-t;\sigma,\tau)$ as depicted in Figure 2.

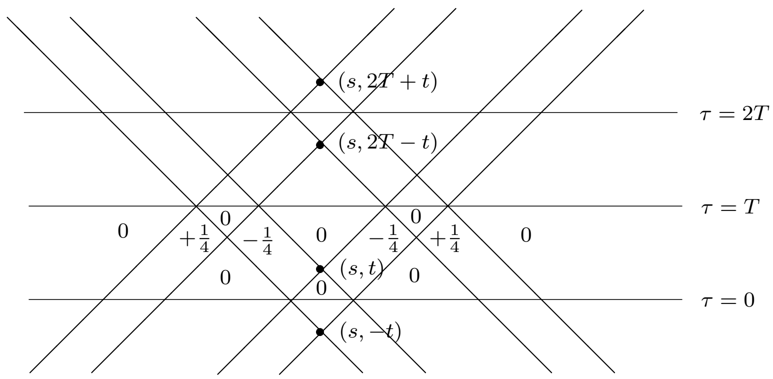

Figure 2

As confirmed by Lemmas 1.1 and 1.2, this again yields a symmetric fundamental singularity for A, one which corresponds to $u(s,0)=0$. While this construction does not, in general, correspond to $u(s,T)=0$, continued reflections about $\tau=0$ and $\tau=T$ do eventually achieve this condition for bounded domains. This fact is evident from the fundamental singularity depicted in Figure 3, which vanishes except in rectangular regions defined by characteristics emanating from (s,t) and reflected by $\tau=0$ and $\tau=T$.

Since all these rectangular regions approach the empty set as $t\to 0$ or $t\to T$, we see that

$$u(s,t)=A^{-1}f\equiv\iint_R H(s,t;\sigma,\tau)f(\sigma,\tau)d\sigma\,d\tau$$

does satisfy $u(s,0)=u(s,T)=0$. Since $H(s,t;\sigma,\tau)=H(\sigma,\tau;s,t)$, A^{-1} is a completely continuous self-adjoint operator in $L^2(R)$ whose range manifests "a simultaneous crossing of the axis".

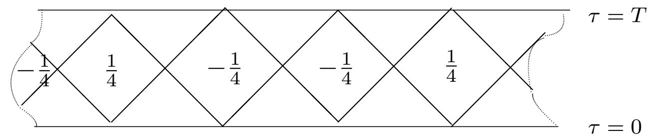

Figure 3

4. The Spacelike Boundary. Given a domain R contained in the rectangle $0 \le s \le S$, $0 \le t \le T$, the construction of §3 yields a self-adjoint realization of $A = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}$ whose domain satisfies $u(s,0) = u(s,T) = 0$. What this construction fails in general to do is to characterize the domain of A in terms of boundary conditions for which

$$\iint_R (vAu - uAv) ds\, dt = 0.$$

It is here that the geometry of the problem enters in an essential way.

We consider a trapezoidal region

$$R(k,\theta) = \{(s,t) : 0 \le t \le s \le 2kT + \theta - t;\ 0 \le t \le T\},$$

where k is a positive integer and $0 \le \theta < 2T$. In this case it will be possible to determine boundary conditions on the characteristics

$$t - s = 0 \quad \text{and} \quad t + s = 2kT + \theta$$

which, together with $u(s,0) = u(s,T) = 0$, characterize self-adjoint realizations of A.

In the case $\theta = 0$, the broken characteristic connecting (T,T) with $((2k-1)T, T)$ divides R into $2k-1$ congruent triangles (see Figure 4 for $k = 2$).

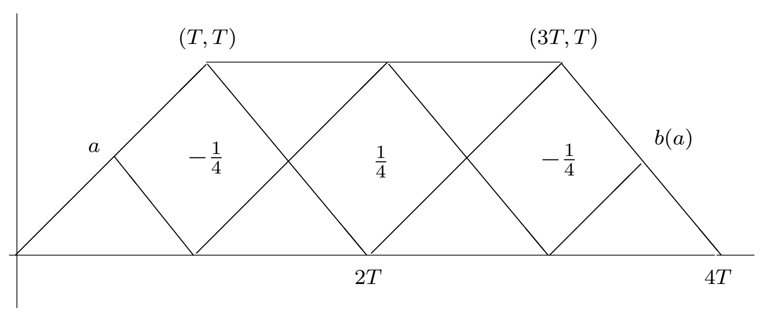

Figure 4

Corresponding to $a = (s,s)$ on the characteristic $t - s = 0$ there is a point $b(a) = ((2k-1)T + s, T - s)$ on the characteristic $t + s = 2kT$. The pairs of parallel lines

$$t - s = \nu T \quad \text{and} \quad t - s = \nu T + 2s,$$
$$t + s = \nu T + 2s \quad \text{and} \quad t + s = 2(\nu + 1)T$$

define a sequence of $2k-1$ rectangles in R in which the fundamental singularity $H(s,t;\sigma,\tau)$ of §3 assumes the values $\frac{(-1)^\nu}{4}$ for $\nu = 1, \dots, 2k-1$. However, this is the same function $H(s,t;\sigma,\tau)$ obtained by locating the fundamental singularity at $b(a)$. These observations establish the following.

Theorem 4.1. *Given a trapezoidal domain*

$$R(k,0) = \{(s,t) : 0 \le t \le s \le 2kT - t;\ 0 \le t \le T\},$$

the boundary conditions

$$\begin{aligned} u(s,0) &= 0 \quad \textit{for} \quad 0 \le s \le 2kT, \\ u(s,T) &= 0 \quad \textit{for} \quad T \le s \le (2k-1)T, \\ u(s,s) &= u((2k-1)T + s, T - s) \quad \textit{for} \quad 0 \le s \le T, \end{aligned}$$

correspond to a self-adjoint realization of $A = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}$ *in* $L^2(R)$.

Remark. For $k = 1$, the trapezoid becomes a characteristic triangle and we obtain the self-adjoint operator studied by Kalmenov [2].

There remains the problem of a general trapezoid with $0 < \theta < 2T$ for which self-adjoint boundary conditions will involve four points:

$$a = (s,s) \quad \text{for} \quad 0 < s < T,$$
$$b(a) = \begin{cases} ((2k-1)T + \frac{\theta}{2} + s,\ T + \frac{\theta}{2} - s) & \text{for} \quad s > \frac{\theta}{2} \\ ((2kT + \frac{\theta}{2} + s,\ \frac{\theta}{2} - s) & \text{for} \quad s < \frac{\theta}{2} \end{cases}$$
$$c = \big(\frac{\theta}{2}, \frac{\theta}{2}\big), d = \big(2kT + \frac{\theta}{2}, \frac{\theta}{2}\big).$$

Here a and $b(a)$ can be connected by broken characteristics reflected by the lines $t = 0$ and $t = T$. There is a similar relationship between c and $((2k-1)T + \theta, T)$ and between d and (T,T).

Our principal result is the following

Theorem 4.2. *Given a trapezoidal domain*

$$R(k,\theta) = \{(s,t) : 0 \le t \le s \le 2kT - t + \theta;\ 0 \le t \le T\},$$

the boundary conditions

$$u(s,0)=0 \quad \textit{for} \quad 0\le s\le 2kT+\theta,$$
$$u(s,T)=0 \quad \textit{for} \quad T\le s\le (2k-1)T+\theta,$$
$$u(a)+\frac{1}{2}u(c)=u(b)+\frac{1}{2}u(d),$$

correspond to a self-adjoint realization of $A=\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial s^2}$ *in* $L^2(R)$.

Proof. The proof consists of applying the construction of §3 to $(s,t)=a,b,c,$ and d and noting the rectangles in which this construction assigns the values $\pm\frac{1}{4}$. In the figures below (for $k=2$) we denote the value $\pm\frac{1}{4}$ resulting from point a by $\pm a$, etc.

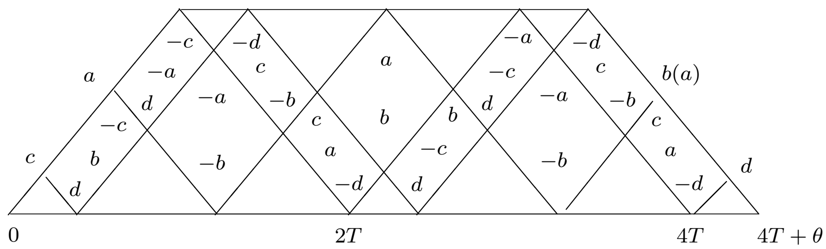

Figure 5

A rather tedious calculation shows that this construction *always* decomposes R into rectangles in which H assumes one of the following values:

$$\pm(a+b),$$
$$\pm(a+c-d),$$
$$\pm(b-c+d).$$

Since all of these expressions are made to vanish by choosing

$$a=\frac{1}{4},\quad b=-\frac{1}{4},\quad c=-\frac{1}{8},\quad d=\frac{1}{8}$$

it follows that the composite singularity corresponding to

$$u(a)-u(b)-\frac{1}{2}u(c)+\frac{1}{2}u(d)$$

vanishes identically in $R\times R$. Therefore all functions in the range of

$$A^{-1}f=\iint_R H(s,t;\sigma,\tau)f(\sigma,\tau)\,d\sigma\,d\tau$$

will satisfy

$$u(a)-u(b)+\frac{u(d)-u(c)}{2}=0.$$

Remarks.

1. Other self-adjoint boundary conditions can be obtained by choosing $a = \frac{1}{4}$, $b = -\frac{1}{4}$, c arbitrary, and $d = c + \frac{1}{4}$.

2. By way of physical interpretation of these results, it seems that a simultaneous crossing of the axis is an unreasonable requirement for a driven string tied down at both end points. If, however, one is willing to shorten the string by moving in from both ends at the speed of propagation, then it is possible to manipulate these ends so as to achieve a simultaneous crossing of the axis for arbitrary $T > 0$.

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