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ON COHOMOTOPY-TYPE FUNCTORS

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Abstract. This article deals with Chogoshvili cohomotopy functors which are defined by extending a cohomology functor given on some special auxiliary subcategories of the category of topological spaces. The question of choosing these subcategories is discussed. In particular, it is shown that in the singular case to define absolute groups it is sufficient that auxiliary subcategories should have as objects only spheres S^n, Moore spaces $P^n(t) = S^{n-1} \cup_t e^n$, and one-point unions of these spaces.

In [2, 3] for any cohomology theory $H = \{H^n\}$ given on some category K of pairs of topological spaces the sequence $\Pi = \{\Pi^n\}, \quad n = 0, 1, 2, \dots,$ of contravariant functors Π^n is constructed from K into the category of abelian groups with the coboundary operator $\delta^{\#}$ which commutes with the induced homomorphisms $\varphi^{\#}$, $\varphi \in K$. Functors Π^n possess the properties of semi-exactness and homotopy and are connected with H by the natural transformations $d : H^n \to \Pi^n$ which are the natural equivalences on a certain subcategory K_n of K. Constructing such functors is reduced to the problem of extending the functor given on an auxiliary subcategory K_n to the whole category K. The problem is solved by means of the theory of inverse systems of groups with sets of homomorphisms of Hurewiz, Dugundji, and Dowker [4].

Functors Π^n are dual to the homotopy functors associated in the sense of Bauer [1] with a given homological structure. It should be noted that functors Π^n have some of the basic properties of the Borsuk cohomotopy, but they differ from the latter.

In [5-7] functors Π^n were investigated under the assumption that K is the category of pairs of topological spaces with a base point and base point preserving maps, and H is the singular integral theory of cohomology. We will adhere to the same assumption throughout this paper (base points are not indicated here). To define functors Π^n we need auxiliary subcategories K_n. We consider the problem of choosing these subcategories.

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For convenience we recall the definition of a limit of the inverse system of groups with sets of homomorphisms (see [4]). Let ω be a partially ordered set, and $\{G_\sigma\}$ a system of abelian groups indexed by the elements of ω. Furthermore, let, for each pair $\rho<\sigma$, sets $H_{\sigma\rho}\subset Hom\,(G_\sigma,G_\rho)$ be given such that if $\rho<\sigma<\tau$ and $\varphi_1\in H_{\sigma\rho}$, $\varphi_2\in H_{\tau\sigma}$, then the composition $\varphi_1\varphi_2\in H_{\tau\rho}$. Then, by definition, $\lim\limits_{\leftarrow} G_\sigma$ is a subgroup of the group ΠG_σ and its elements are elements $g=\{g_\sigma\}\in\Pi G_\sigma$ such that for each pair $\rho<\sigma$ and $\varphi\in H_{\sigma\rho}$ we have $\varphi(g_\sigma)=g_\rho$.

It should be noted that this theory of [4] is essentialy Kan's extension theory in its early stage, but quite sufficient for our purpose.

The results of this paper were announced earlier in [6,7].

1. Preliminaries

In this section we will give the definitions of subcategories K_n and functors Π^n and discuss some of their properties.

Let e^m be the unit m-cell of the m-dimensional euclidean space $\mathbb{R}^m$. By e^0 we denote some fixed point (base point). Let $\widetilde{K}$ be the small full subcategory of K whose objects are all finite CW-complexes X for which $X^0=e^0$ and X^k is the adjunction space obtained by adjoining a finite number of e^k to X^{k-1}, $k>0$. We denote by $\widetilde{\widetilde{K}}$ the small full subcategory of K whose objects are all CW-pairs (X,X') for which X and X' are the objects of $\widetilde{K}$.

Now we shall define auxiliary subcategories K_n, $n>3$, by the following two conditions (cf. [2, 5]):

1) K_n is an arbitrary small full subcategory of K; each object of K_n is a pair (X,X') of linearly and simply connected spaces satisfing the conditions that $\pi_2(X,X')=1$, the homology modules $H_*(X)$ and $H_*(X')$ are of finite type, $H^i(X,X')=0,\ i<n$, and $H^i(X)=H^i(X')=0,\ 0<i<n-1$;

2) K_n contains all possible objects of $\widetilde{\widetilde{K}}$.

We denote by F_n^r an auxiliary subcategory of objects only of $\widetilde{\widetilde{K}}$.

If $n=3$, we assume that K_3 is an arbitrary (containing all possible objects of $\widetilde{K}$) small full subcategory of K whose all objects are linearly, and simply connected spaces X for which $H_*(X)$ is a module of finite type and $H^2(X)=0$.

Let (R,R') be an object of K. Consider a set of indices $\omega(R,R';n)$ of all pairs $\alpha=(X,X';f)$, where (X,X') is an object of K_n and $f:(X,X')\to(R,R')$ is a continuous map of K. Let $\omega(R,R';n)$ be ordered as follows: $\alpha<\beta$, where $\beta=(Y,Y';g)$ if there is a map $\varphi:(X,X')\to(Y,Y')$ of K_n such that

$$g\varphi=f. \tag{1}$$

Assume that to every $\alpha \in \omega(R, R'; n)$ there corresponds the n-dimensional cohomology group $H_\alpha = H^n(X, X')$ and to every ordered pair $\alpha < \beta$ there corresponds the set of homomorphisms $\{\varphi^*\}$, where $\varphi^* : H^n(Y, Y') \to H^n(X, X')$ are the induced homomorphisms in the H theory.

We have obtained the inverse system of the group H_α with sets of homomorphisms. Cohomotopic groups of Chogoshvili are determined by the formula $\Pi^n(R, R'; K_n) = \lim_{\leftarrow} H_\alpha$.

We denote by $\Pi^n(R; K_n)$ the absolute group $\Pi^n(R, *; K_n)$, where $*$ is a base point, and by p_α the α-coordinate of an element $p \in \Pi^n(R, R'; K_n)$. Note that for $n = 3$ we have determined the absolute groups only.

Let

$$\alpha = (X, X'; f), \quad \beta = (X, X'; g),$$
$$\alpha, \beta \in \omega(R, R'; n), \quad p \in \Pi^n(R, R'; K_n)$$

and let the maps f and g be homotopic, i.e., $f \sim g$. Let I be the unit segment.

Lemma 1.1. *If the subcategory K_n contains, alongside with (X, X'), the pair $(X \times I, X' \times I)$, then $p_\alpha = p_\beta$.*

Proof. See [5], p. 83. □

Let

$$\alpha = (X, X'; f), \quad \beta = (Y, Y'; g),$$
$$\alpha, \beta \in \omega(R, R'; n), \quad p \in \Pi^n(R, R'; K_n).$$

Moreover, let us have a map $\varphi : (X, X') \to (Y, Y')$ from K_n such that the maps $g\varphi$ and f are homotopic, i.e.,

$$g\varphi \sim f. \tag{2}$$

Lemma 1.2. *If the subcategory K_n contains, alongside with (X, X'), the pair $(X \times I, X' \times I)$, then $\varphi^*(p_\beta) = p_\alpha$.*

Proof. Consider the index $(X, X'; g\varphi) = \alpha_1 < \beta$ and apply Lemma 1.1 to α_1 and α. □

Let $\alpha = (X, X'; f)$, where f is null-homotopic, and $p \in \Pi^n(R, R'; K_n)$.

Corollary 1.3. *If the subcategory K_n contains, alongside with (X, X'), the pair $(X \times I, X' \times I)$, then $p_\alpha = 0$.*

Proof. Apply Lemma 1.2 to the homotopy commutative diagram

$$\begin{tikzcd} & (R,R') & \\ (X,X') \arrow[ur, "f"] \arrow[rr, "o"] & & (X,X') \arrow[ul, "o"'] \end{tikzcd} ,$$

where o denotes the constant map. □

2. Main Results

Let K_n' be a subcategory of K_n'', where K_n' and K_n'' are two auxiliary subcategories, and let $(R,R') \in K$, $p \in \Pi^n(R,R';K_n'')$, $\alpha \in {}'\omega(R,R';n) \subset {}''\omega(R,R';n)$. Then, as one can easily verify, the formula $[\lambda(p)]_\alpha = p_\alpha$ defines the restriction homomorphism $\lambda : \Pi^n(R,R';K_n'') \to \Pi^n(R,R';K_n')$.

In Section 3 we will prove

Theorem 2.1. *The homomorphism λ defines the natural equivalence of the functors $\Pi^n(-,-;K_n'')$ and $\Pi^n(-,-;K_n')$, $n > 3$. In particular, all functors $\Pi^n(-,-;K_n)$ are naturally equivalent to the functor $\Pi^n(-,-;F_n^r)$.*

Remark 2.2. Theorem 2.1 shows that in choosing a subcategory K_n we can restrict ourselves only to the finite CW-pairs. On the other hand, from Theorem 2.1 it follows that for the convenience of construction and proof we can regard an arbitrary admissible pair as an object of K_n.

Remark 2.3. One can easily show that Theorem 2.1 holds for the absolute groups when $n > 2$. Moreover, in defining the absolute groups, to choose the subcategory K_n we can restrict ourselves to the absolute pairs $(X,*)$, i.e., $X' = *$ (see the definition of K_n and [5]).

In the remainder of this section we will consider the absolute groups only. Therefore to define the functors Π^n, $n > 2$, we can use auxiliary subcategories F_n^a consisting of finite CW-complexes. More exactly, F_n^a is a full subcategory $\widetilde{K}$ whose objects are all spaces X for which $\pi_1(X) = 1$ and $H^2(X) = \cdots = H^{n-1}(X) = 0$.

We intend here to study the problem dealing with the possibility of further reducing subcategories K_n provided that groups $\Pi^n(R;K_n)$ and the results from [5-7] remain unchanged. To this effect, relying on Lemma 1.2, in the definition of $\Pi^n(R;K_n)$ we replace condition (1) by condition (2) (see Section 1). We will stick to this definition in the sequel.

Let S^n denote the n-dimensional unit sphere of the euclidean space $\mathbb{R}$ and e^n the unit disk. Denote by $P^n(t)$, $t > 1$, $n > 2$, the Moore space $S^{n-1} \cup_t e^n$. Also assume that $P^n(1) = S^n$.

Consider now the full subcategory F_n^b of F_n^a whose objects are all finite one-point unions of spaces $P^n(t)$, $t \geq 1$. The subcategory F_n^b will be regarded as an auxiliary subcategory.

The following theorem will be proved in Section 4.

Theorem 2.4. *The restriction homomorphism defines the natural equivalence of the functors* $\Pi^n(-;F_n^a)$ *and* $\Pi^n(-;F_n^b)$, $n > 2$.

We introduce the following notations:
1) $P_j^n(t) = P^n(t)$, where j is a positive integer, $n > 2,\ t \geq 1$;
2) $X_k^n = \vee_{t=1}^k \big(\vee_{j=1}^k P_j^n(t) \big)$;
3) $Q^n = \varinjlim X_k$ (by inclusion maps $X_k^n \to X_{k+1}^n$).

Let Q_n be the full subcategory of K consisting of one object Q^n, $n > 2$. The subcategory will also be regarded as an auxiliary subcategory. Note that the module $H_*(Q^n)$ is not obviously of the finite type. Therefore none of the above-defined auxiliary subcategories contains Q^n.

The following theorem will be proved in Section 5.

Theorem 2.5. *The functors* $\Pi^n(-;F_n^b)$ *and* $\Pi^n(-;Q_n)$ *are naturally equivalent,* $n > 2$.

3. Proof of Theorem 2.1

Let us prove that λ is natural. Assume $f : (S,S') \to (R,R')$ to be an arbitrary map of K. Consider the diagram

$$\begin{tikzcd} & (R,R') & \\ (M,M') \arrow[ur, "f"] \arrow[rr, "\varphi"] & & (N,N') \arrow[ul, "f\varphi^{-1}"'] \end{tikzcd}$$

Let

$$\alpha = (X,X';g) \in {'\omega}(S,S',n) \subset {''\omega}(S,S';n),$$
$$\beta = f(\alpha) = (X,X';fg) \in {'\omega}(R,R';n) \subset {''\omega}(R,R';n)$$

and let $p \in \Pi^n(R,R';K_n'')$. We have

$$\big[\lambda(f^\#(p))\big]_\alpha = \big[f^\#(p)\big]_\alpha = p_\beta,$$
$$\big[f^\#(\lambda(p))\big]_\alpha = \big[\lambda(p)\big]_\beta = p_\beta,$$

which proves that λ is natural.

Let (X,X') be an arbitrary object of some auxiliary subcategory K_n. Using the standard technique of the homotopy theory, we can construct a CW-pair (A,B) from the subcategory K_n and a map $\varphi : (A,B) \to (X,X')$ such that homomorphisms φ^* induced by φ in the H theory will be isomorphisms up to any pregiven dimension. Let now $p \in \Pi^n(R,R';K_n)$ and $\alpha =$

$(X, X'; f) \in \omega(R, R'; n)$. Consider the index $\beta = (A, B; f\varphi) \in \omega(R, R'; n)$. Then $\beta < \alpha$ and therefore $p_\beta = \varphi^*(p_\alpha)$. Hence $p_\alpha = \varphi^{*-1}(p_\beta)$. From the above reasoning and the definition of auxiliary subcategories it now follows that if $p \in \Pi^n(R, R'; K''_n)$ and $\lambda(p) = 0$, then $p = 0$. Thus λ is a monomorphism.

Let L_n, $n > 3$, be a full subcategory of the category K whose objects are all pairs (X, X') for which X and X' are linearly and simply connected spaces, $\pi_2(X, X') = 1$, the homology modules $H_*(X)$ and $H_*(X')$ are of the finite type, $H^i(X, X') = 0$, $i < n$, and $H^i(X) = H^i(X') = 0$, $0 < i < n-1$.

Consider some full subcategories of L_n:

1) $L_n^{(1)}$ are CW-pairs;

2) $L_n^{(2)}$ are CW-pairs with a finite number of cells in all demensions;

3) $L_n^{(3)}$ are finite CW-pairs;

4) $L_n^{(4)} = \widetilde{\widetilde{K}} \cap \text{Ł}_n^{(3)} = F_n^r$.

We will gradually extend the thread defining element $p \in \Pi^n(R, R'; F_n^r)$ from the category $L_n^{(4)}$ onto $L_n^{(3)}$, then onto $L_n^{(2)}$, $L_n^{(1)}$ and, finally, onto L_n. Such an extension already implies that λ is epimorphic.

Let (M, M') be an arbitrary object of $L_n^{(3)}$; also let $f : (M, M') \to (R, R')$ be an arbitrary map. Consider the diagram

$$\begin{tikzcd}
\Pi^n(R, R'; K''_n) \arrow[r, "\lambda"] \arrow[d, "f^\#"] & \Pi^n(R, R'; K'_n) \arrow[d, "f^\#"] \\
\Pi^n(S, S'; K''_n) \arrow[r, "\lambda"] & \Pi^n(S, S'; K'_n)
\end{tikzcd}$$

where $(N, N') \in L_n^{(4)}$ and φ is a homeomorphism. Let $\alpha = (M, M'; f)$ and $\beta = (N, N'; f\varphi^{-1})$. It is assumed that $p_\alpha = \varphi^*(p_\beta)$. We will show that p_α does not depend on the choice of the homeomorphism φ. Consider the diagram

$$\begin{tikzcd}
 & (R, R') & \\
 & (M, M') \arrow[u, "f"'] \arrow[dl, "\varphi_1"] \arrow[dr, "\varphi_2"'] & \\
(N, N') \arrow[uur, "f\varphi_1^{-1}"] \arrow[rr, "g"] & & (\bar{N}, \bar{N}') \arrow[uul, "f\varphi_2^{-1}"']
\end{tikzcd}$$

where φ_1 and φ_2 are two different homeomorphisms and $g = \varphi_2\varphi_1^{-1}$. The indices β_1 and β_2 will be defined similarly to β. We have $(f\varphi_2^{-1})g = f\varphi_2^{-1}\varphi_2\varphi_1^{-1} = f\varphi_1^{-1}$. Therefore $\beta_1 < \beta_2$. Then $\varphi_2^*(p_{\beta_2}) = (g\varphi_1)^*(p_{\beta_2}) = \varphi_1^*(g^*(p_{\beta_2})) = \varphi_1^*(p_{\beta_1})$.

Consider the commutative diagram

$$\begin{tikzcd}
& (R,R') & \\
(M,M') \arrow[ur, "f"] \arrow[rr, "g"] \arrow[d, "\varphi"] & & (M_0,M_0') \arrow[ul, "f_0"'] \arrow[d, "\varphi_0"] \\
(N,N') \arrow[rr, "g_0"] & & (N_0,N_0')
\end{tikzcd}$$

where φ and φ_0 are homeomorphisms, $g_0 = \varphi_0 g \varphi^{-1}$.

Let

$$\alpha = (M, M'; f), \quad \alpha_0 = (M_0, M_0'; f_0),$$
$$\beta = (N, N'; f\varphi^{-1}), \quad \beta_0 = (N_0, N_0'; f_0\varphi_0^{-1}).$$

We have

$$(f_0\varphi_0^{-1})g_0 = f_0\varphi_0^{-1}\varphi_0 g\varphi^{-1} = f_0 g\varphi^{-1} = f\varphi^{-1}.$$

Therefore $\beta < \beta_0$. Then

$$g^*(p_{\alpha_0}) = g^*(\varphi_0^*(p_{\beta_0})) = (\varphi_0 g)^*(p_{\beta_0}) = (g_0\varphi)^*(p_{\beta_0}) =$$
$$= \varphi^*(g_0^*(p_{\beta_0})) = \varphi^*(p_\beta) = p_\alpha.$$

We have thus extended the thread of the element p onto the category $L_n^{(3)}$.

Let, now, $(M, M') \in L_n^{(2)}$ and $f : (M, M') \to (R, R')$ be an arbitrary map. By $i : X_k \to X$ we denote here the standard embedding, where X_k is the k-skeleton of the CW-complex X. Let $N_1 > N > n + 1$ be arbitrary integers.

Consider the commutative diagram

$$\begin{tikzcd}
& (R,R') & \\
(M_N,M_N') \arrow[ur, "f_N"] \arrow[r, "i"'] & (M_{N_1},M'_{N_1}) \arrow[u, "f_{N_1}"] \arrow[r, "i_1"'] & (M,M') \arrow[ul, "f"']
\end{tikzcd}$$

where $f_{N_1} = fi_1$ and $f_N = f_{N_1}i$. Also consider the indices

$$\alpha = (M_N, M_N'; f_N), \quad \beta = (M_{N_1}, M_{N_1}'; f_{N_1}), \quad \gamma = (M, M'; f).$$

Assume $p_\gamma = i_1^{*-1}(p_\beta)$. We have

$$(i_1 i)^{*-1}(p_\alpha) = i_1^{*-1}(i^{*-1}(p_\alpha)) = i_1^{*-1}(p_\beta),$$

where the last equality evidently follows from the fact that $\alpha < \beta$. Therefore p_γ does not depend on the choice of the number N_1.

Now consider the diagram

$$\begin{tikzcd}
 & (R,R') & \\
(M,M') \arrow[ur, "f"] \arrow[rr, "\widetilde{\varphi}"] \arrow[d, "i"] & & (T,T') \arrow[ul, "f_1"'] \arrow[d, "i_1"] \\
(M_N,M'_N) \arrow[rr, "\widetilde{\varphi}_1"] & & (T_N,T'_N)
\end{tikzcd}$$

where $\varphi : (M,M') \to (T,T')$ is a map such that $f_1\varphi = f$, $\widetilde{\varphi}$ is a cellular approximation of φ and $\widetilde{\varphi_1} = \widetilde{\varphi}|(M_N,M'_N)$. We have $f \sim f_1\widetilde{\varphi}$ and $(f_1 i_1)\widetilde{\varphi_1} = (f_1\widetilde{\varphi})i \sim fi$. Also consider the indices $\alpha = (M_N,M'_N;fi)$, $\beta = (T_N,T'_N;f_1 i_1)$. Applying Lemma 1.2, we have $\varphi^*(i_1^{*-1}(p_\beta)) = (\widetilde{\varphi^* i_1^{*-1}})(p_\beta) = i^{*-1}(\widetilde{\varphi_1}^*(p_\beta)) = i^{*-1}(p_\alpha)$.

We have thus extended the thread of the element p onto the category $L_n^{(2)}$.

Let now $(\bar{M},\bar{M}')$ be an arbitrary object of $L_n^{(1)}$ and let $\bar{g} : (\bar{M},\bar{M}') \to (R,R')$ be an arbitrary map. Using the standard technique of the homotopy theory, we can, under our assumptions, construct a map $\varphi : (M,M') \to (\bar{M},\bar{M}')$ such that $(M,M') \in L_n^{(2)}$ and φ is a homotopy equivalence. Let $g = \bar{g}\varphi$. Assume $p_\alpha = \varphi^{*-1}(p_\beta)$, where $\alpha = (\bar{M},\bar{M}';\bar{g})$, $\beta = (M,M';g)$. We will show that p_α does not depend on the choice of φ. Consider the diagram

$$\begin{tikzcd}
 & (R,R') & \\
(M,M') \arrow[ur, "g"] \arrow[rr, "\varphi"] & & (\bar{M},\bar{M}') \arrow[ul, "\bar{g}"'] \\
 & (\widetilde{M},\widetilde{M}') \arrow[ul, "\varphi_1"] \arrow[ur, "\widetilde{\varphi}"'] &
\end{tikzcd}$$

where $(\widetilde{M},\widetilde{M}') \in L_n^{(2)}$, $\widetilde{\varphi}$ and φ_1 are homotopy equivalences, and $\varphi \sim \widetilde{\varphi}\varphi_1$. Let $\widetilde{\beta} = (\widetilde{M},\widetilde{M}';\bar{g}\widetilde{\varphi})$. Then

$$g = \bar{g}\varphi \sim \bar{g}\widetilde{\varphi}\varphi_1 = (\bar{g}\widetilde{\varphi})\varphi_1.$$

Applying Lemma 1.2, we have

$$\widetilde{\varphi}^{*-1}(p_{\widetilde{\beta}}) = \widetilde{\varphi}^{*-1}(\varphi_1^{*-1}(p_\beta)) = \varphi^{*-1}(p_\beta).$$

Consider now an arbitrary map

$$\varphi_0 : (\bar{M},\bar{M}') \to (\bar{T},\bar{T}')$$

from the category $L_n^{(1)}$ and the diagram

$$\begin{tikzcd}
 & (R,R') & \\
(\bar{M},\bar{M}') \arrow[ur, "\bar{g}"] \arrow[rr, "\varphi_0"] \arrow[d, "\varphi"] & & (\bar{T},\bar{T}') \arrow[ul, "\bar{g}_1"'] \arrow[d, shift left, "\widetilde{\varphi}_1"] \\
(M,M') \arrow[rr, "\widetilde{\varphi}_0"] & & (T,T') \arrow[u, shift left, "\varphi_1"]
\end{tikzcd}$$

where $\bar{g} = \bar{g}_1\varphi_0$, φ and φ_1 are homotopy equivalences, $\widetilde{\varphi}_1$ is the homotopy inverse of φ_1, $\widetilde{\varphi_0} = \widetilde{\varphi_1}\varphi_0\varphi$ and $(M,M'),(T,T') \in L_n^{(2)}$. We have $(\bar{g}_1\varphi_1)\widetilde{\varphi_0} = \bar{g}_1\varphi_1\widetilde{\varphi_1}\varphi_0\varphi \sim \bar{g}_1\varphi_0\varphi = \bar{g}\varphi$. Also, consider the indices $\beta = (M,M';\bar{g}\varphi)$, $\beta_1 = (T,T';\bar{g}_1\varphi_1)$. Then, applying Lemma 1.2, we have $\varphi_0^*(\varphi_1^{*-1}(p_{\beta_1})) = (\varphi_0^*\varphi_1^{*-1})(p_{\beta_1}) = (\varphi^{*-1}\widetilde{\varphi_0}^*)(p_{\beta_1}) = \varphi^{*-1}(p_\beta)$.

Let, finally, (X,X') be an arbitrary object of L_n, and let

$$w_X : (S(X),S(X')) \to (X,X')$$

be the natural projection of the singular complex $S(X)$ onto X. Let $f : (X,X') \to (R,R')$ be an arbitrary map. Also, consider the indices $\alpha = (X,X';f)$, $\beta = (S(X),S(X');fw_X)$. Assume $p_\alpha = w_X^{*-1}(p_\beta)$ and consider the commutative diagram

$$\begin{tikzcd}
 & (R,R') & \\
(X,X') \arrow[ur, "f"] \arrow[rr, "\varphi"] \arrow[d, "\omega_X"] & & (Y,Y') \arrow[ul, "g"'] \arrow[d, "\omega_Y"] \\
(S(X),S(X')) \arrow[rr, "\bar{\varphi}"] & & (S(Y),S(Y'))
\end{tikzcd}$$

where $g\varphi = f$ and $\bar{\varphi}$ is the cellular map induced by φ. We have $(gw_Y)\overline{\varphi} = g\varphi w_X = fw_X$. Also, consider the indices $\beta_X = (S(X),S(X');f\omega_X)$, $\beta_Y = (S(Y),S(Y');g\omega_Y)$. Then

$$\varphi^*\big(\omega_Y^{*-1}(p_{\beta_Y})\big) = \omega_X^{*-1}\big(\bar{\varphi}^*(p_{\beta_Y})\big) = \omega_X^{*-1}(p_{\beta_X}).$$

This completes the proof of Theorem 2.1. ■

4. Proof of Theorem 2.4

Consider some full subcategories of F_n^a (see Section 2):

0) $K_n^{(0)} = F_n^a$;

1) $K_n^{(1)}$-objects are CW-complexes with one vertex and without cells of dimensions $1,2,\ldots,n-2$;

2) $K_n^{(2)}$-objects are CW-complexes with one vertex and cells of dimensions $n-1$, n and $n+1$ only;

3) $K_n^{(3)}$-objects are CW-complexes with one vertex and cells of dimensions $n-1$ and n only;

4) $K_n^{(4)} = F_n^b$.

Let $R = (R, *)$ be an arbitrary space from K. All subcategories $K_n^{(i)}$, $0 \le i \le 4$, will be regarded as auxiliary ones.

Let

$$\lambda_i : \Pi^n(R; K_n^{(i)}) \to \Pi^n(R; K_n^{(i+1)}), \quad 0 \le i \le 3,$$

be the natural restriction homomorphisms.

Let L'_n and L''_n be two small full subcategories of the category K consisting of the spaces $(X, *)$, and $L'_n \subset L''_n$. It is assumed that the following condition is satisfied: for each $X \in L''_n$ there is $Y \in L'_n$ such that Y has the same homotopy type as X. Consider L'_n and L''_n as auxiliary subcategories. Let $\bar\lambda : \Pi^n(R; L''_n) \to \Pi^n(R; L'_n)$ be the natural restriction homomorphism.

Lemma 4.1. *$\bar\lambda$ is a natural isomorphism.*

Proof. We prove first that the homomorphism $\bar\lambda$ is a monomorphism. Let $p \in \Pi^n(R; L''_n)$, $\bar\lambda(p) = 0$ and $\alpha = (X; f) \in {}''\omega(R; n)$ be an arbitrary index. Let $Y \in L'_n$ and the map $\varphi : Y \to X$ be a homotopy equivalence. Consider the index $\beta = (Y; f\varphi) \in {}'\omega(R; n) \subset {}''\omega(R; n)$. We have $\beta < \alpha$. Then $\varphi^*(p_\alpha) = p_\beta = [\bar\lambda(p)]_\beta = 0$. Therefore $p_\alpha = 0$ and $p = 0$.

Let us now prove that the homomorphism $\bar\lambda$ is an epimorphism. Let $q \in \Pi^n(R; L'_n)$ and $\alpha = (X; f) \in {}''\omega(R; n)$ be an arbitrary index. Let $Y \in L'_n$ and the map $\varphi : Y \to X$ be a homotopy equivalence. Consider the index $\beta = (Y; f\varphi) \in {}'\omega(R; n)$ and assume $p_\alpha = \varphi^{*-1}(q_\beta)$. We will show that p_α does not depend on the choice of φ. Consider the diagram

$$\begin{tikzcd}
& R & \\
Y \arrow[ur, "f\varphi"] \arrow[r, "\varphi"] \arrow[d, equal, "id"] & X \arrow[u, "f"] \arrow[r, shift right, "\widetilde{\varphi}_1"'] & Y_1 \arrow[ul, "f\varphi_1"'] \arrow[l, shift right, "\varphi_1"'] \arrow[d, equal, "id"] \\
Y \arrow[rr, "\widetilde{\varphi}_1\varphi"] & & Y_1
\end{tikzcd}$$

where $Y, Y_1 \in L'_n$, φ and φ_1 are homotopy equivalences, and $\widetilde{\varphi}_1$ is the homotopy inverse of φ_1. Consider the index $\gamma = (Y_1; f\varphi_1) \in {}'\omega(R; n)$.

Then $(f\varphi_1)(\widetilde{\varphi}_1\varphi) \sim f\varphi$ and therefore $\beta < \gamma$. In this case

$$\begin{aligned}\varphi_1^*\big(\varphi^{*-1}(q_\beta)\big) &= \varphi_1^*\big(\varphi^{*-1}\big((\widetilde{\varphi}_1\varphi)^*(q_\gamma)\big)\big) = \\ &= (\varphi_1^*\varphi^{*-1}\varphi^*\widetilde{\varphi}_1^*)(q_\gamma) = (\varphi_1^*\widetilde{\varphi}_1^*)(q_\gamma) = q_\gamma.\end{aligned}$$

Hence $\varphi^{*-1}(q_\beta) = \varphi_1^{*-1}(q_\gamma)$.

We will show that the set $\{p_\alpha\}$ defines an element of the group $\Pi^n(R; L''_n)$. Consider the diagram

$$\begin{tikzcd}
& R & \\
X_0 \arrow[ur, "f_0"] \arrow[rr, "\varphi"] & & X_1 \arrow[ul, "f_1"'] \arrow[d, shift left, "\widetilde{\varphi}_1"] \\
Y_0 \arrow[u, "\varphi_0"] \arrow[rr, "\widetilde{\varphi}_1\varphi\varphi_0"] & & Y_1 \arrow[u, shift left, "\varphi_1"]
\end{tikzcd}$$

where $f_1\varphi = f_0$, φ_0 and φ_1 are homotopy equivalences, $\widetilde{\varphi}_1$ is the homotopy inverse of φ_1, $X_0, X_1 \in L''_n$, and $Y_0, Y_1 \in L'_n$. Then

$$(f_1\varphi_1)(\widetilde{\varphi}_1\varphi\varphi_0) \sim f_1\varphi\varphi_0 = f_0\varphi_0.$$

Also, consider the indices

$$\alpha = (X_0; f_0), \quad \beta = (X_1; f_1), \quad \alpha, \beta \in {}''\omega(R; n),$$
$$\alpha_1 = (Y_0; f_0\varphi_0), \quad \beta_1 = (Y_1; f_1\varphi_1), \quad \alpha_1, \beta_1 \in {}'\omega(R; n) \subset {}''\omega(R; n).$$

Then $\alpha < \beta$, $\alpha_1 < \beta_1$ and we have

$$\varphi^*(p_\beta) = \varphi^*\big(\varphi_1^{*-1}(q_{\beta_1})\big) = \varphi_0^{*-1}\big((\widetilde{\varphi}_1\varphi\varphi_0)^*(q_{\beta_1})\big) = \varphi_0^{*-1}(q_{\alpha_1}) = p_\alpha.$$

Finally, let us prove that $\bar{\lambda}(p) = q$. Assume that $X \in L'_n \subset L''_n$ and

$$\alpha = (X; f) \in {}'\omega(R; n) \subset {}''\omega(R; n).$$

Define p_α by taking $\varphi = id : X \to X$. Then

$$[\bar{\lambda}(p)]_\alpha = p_\alpha = id^{*-1}(q_\alpha) = q_\alpha.$$

This completes the proof of Lemma 4.1. $\square$

As a consequence of the foregoing lemma we have

Proposition 4.2. λ_0 *is a natural isomorphism.*

Proof. Every CW-complex from $K_n^{(0)}$ is homotopically equivalent to some CW-complex from $K_n^{(1)}$. $\square$

Proposition 4.3. λ_1 *is a natural isomorphism.*

Proof. We will prove in the first place that λ_1 is a monomorphism. Let $p \in \Pi^n(R; K_n^{(1)})$ and $\lambda_1(p) = 0$. Consider the index $\alpha = (X; f) \in {}^{(1)}\omega(R; n)$, $X \in K_n^{(1)}$, and the diagram

$$\begin{tikzcd} & R & \\ X^{n+1} \arrow[ur, "f i_X"] \arrow[rr, "i_X"] & & X \arrow[ul, "f"'] \end{tikzcd} \tag{3}$$

where X^{n+1} is the $(n+1)$-skeleton of X and i_X is the standard embedding. Let

$$\beta = (X^{n+1}; f\, i_X) \in {}^{(2)}\omega(R; n) \subset {}^{(1)}\omega(R; n).$$

Then $\beta < \alpha$, and we have $i_X^*(p_\alpha) = p_\beta = [\lambda_1(p)]_\beta = 0$. Therefore $p = 0$.

Assume now that $p \in \Pi^n(R; K_n^{(2)})$ and $\alpha = (X; f) \in {}^{(1)}\omega(R; n)$.

Consider the diagram (3), the index β, and assume that $q_\alpha = i_X^{*-1}(p_\beta)$.

We will show that the set $\{q_\alpha\}$ defines an element of $\Pi^n(R; K_n^{(1)})$. Consider the diagram

$$\begin{tikzcd} & R & \\ X \arrow[ur, "f"] \arrow[rr, shift left, "\widetilde{\varphi}"] \arrow[rr, shift right, "\varphi"'] & & Y \arrow[ul, "g"'] \\ X^{n+1} \arrow[u, "i_X"] \arrow[rr, "\varphi_1"'] & & Y^{n+1} \arrow[u, "i_Y"'] \end{tikzcd}$$

where $g\varphi \sim f$, $\widetilde{\varphi}$ is a cellular approximation of φ and $\varphi_1 = \widetilde{\varphi}|X^{n+1}$. Then $(g i_Y)\varphi_1 = g\widetilde{\varphi} i_X \sim f i_X$. Also, consider the indices

$$\alpha_1 = (Y; g) \in {}^{(1)}\omega(R; n),$$
$$\beta_1 = (Y^{n+1}; g i_Y) \in {}^{(2)}\omega(R; n).$$

We have $\beta < \beta_1$ and

$$\varphi^*(q_{\alpha_1}) = \widetilde{\varphi}^*(q_{\alpha_1}) = \widetilde{\varphi}^*\big(i_Y^{*-1}(q_{\beta_1})\big) = i_X^{*-1}\big(\varphi_1^*(q_{\beta_1})\big) = i_X^{*-1}(q_\beta) = q_\alpha.$$

Finally, let us prove that $\lambda_1(q) = p$.

Consider the index $\alpha = (X; f) \in {}^{(2)}\omega(R; n)$, where $X = X^{n+1} \in K_n^{(2)} \subset K_n^{(1)}$, and the diagram

$$\begin{tikzcd} & R & \\ X^{n+1} \arrow[ur, "f"] \arrow[rr, "i_X"] & & X \arrow[ul, "f"'] \end{tikzcd}$$

where $i_X = id$. Then $[\lambda_1(q)]_\alpha = i_X^{*-1}(p_\alpha) = p_\alpha$.

This completes the proof of Proposition 4.3. $\square$

Proposition 4.4. λ_2 *is a natural isomorphism.*

Proof. Let $q \in \Pi^n(R; K_n^{(2)})$ and $\lambda_2(q) = 0$. Consider the index $\alpha = (P_{n+1}; f) \in {}^{(2)}\omega(R; n)$, where $P_{n+1} \in K_n^{(2)}$. Let $P_{n+1}^{(n)}$ be the n-skeleton of P_{n+1} and $i : P_{n+1}^{(n)} \to P_{n+1}$ be the standard embedding. We have the commutative diagram

$$\begin{tikzcd} & R & \\ P_{n+1}^{(n)} \arrow[ur, "fi"] \arrow[rr, "i"] & & P_{n+1} \arrow[ul, "f"'] \end{tikzcd} \tag{4}$$

Also consider the index

$$\beta = (P_{n+1}^{(n)}; fi) \in {}^{(3)}\omega(R; n) \subset {}^{(2)}\omega(R; n).$$

We have $\beta < \alpha$. Let

$$\to H^n(P_{n+1}, P_{n+1}^{(n)}) \to H^n(P_{n+1}) \xrightarrow{i^*} H^n(P_{n+1}^{(n)}) \to$$

be a part of the cohomological exact sequence for the pair $(P_{n+1}, P_{n+1}^{(n)})$. Since $H^n(P_{n+1}, P_{n+1}^{(n)}) = 0$, i^* is a monomorphism. Then $i^*(q_\alpha) = q_\beta = [\lambda_2(q)]_\beta = 0$. Therefore $q_\alpha = 0$ and $q = 0$. Hence λ_2 is a monomorphism.

Let now $q \in \Pi^n(R; K_n^{(3)})$. Consider again the commutative diagram (4) and the corresponding indices α and β. We will prove below that $q_\beta \in Im\, i^*$. Therefore $p_\alpha = i^{*-1}(q_\beta)$ is the correct definition. Let us show that the set $\{p_\alpha\}$ defines an element $p \in \Pi^n(R; K_n^{(2)})$. Consider the diagram

$$\begin{tikzcd} & R & \\ P_{n+1} \arrow[ur, "f"] \arrow[rr, shift left, "\varphi"] \arrow[rr, shift right, "\widetilde{\varphi}"'] & & \bar{P}_{n+1} \arrow[ul, "f_1"'] \\ P_{n+1}^{(n)} \arrow[u, "i"] \arrow[rr, "\varphi_1"'] & & \bar{P}_{n+1}^{(n)} \arrow[u, "i_1"'] \end{tikzcd}$$

where $f_1\varphi \sim f$, $\widetilde{\varphi}$ is a cellular approximation of φ and $\varphi_1 = \widetilde{\varphi}|p_{n+1}^{(n)}$. Consider the indices

$$\alpha_1 = (\bar{p}_{n+1}; f_1) \in {}^{(2)}\omega(R;n), \ \ \beta_1 = (\bar{p}^{(n)}_{n+1}; f_1 i_1) \in {}^{(3)}\omega(R;n).$$

Since $(f_1 i_1)\varphi_1 = f_1 \widetilde{\varphi} i \sim fi$, we obtain $\beta < \beta_1$ and therefore $\varphi_1^*(q_{\beta_1}) = q_\beta$. We have $i^{*-1}(\varphi_1^*(q_{\beta_1})) = \widetilde{\varphi}^*(i_1^{*-1}(q_{\beta_1}))$ and

$$\varphi^*(p_{\alpha_1}) = \widetilde{\varphi}^*(p_{\alpha_1}) = \widetilde{\varphi}^*(i_1^{*-1}(q_{\beta_1})) = i^{*-1}(\varphi_1^*(q_{\beta_1})) = i^{*-1}(q_\beta) = p_\alpha.$$

Finally, we will prove that $\lambda_2(p) = q$. Let

$$\alpha = (P_{n+1}; f) \in {}^{(3)}\omega(R;n) \subset {}^{(2)}\omega(R;n)$$

be an arbitrary index, $P_{n+1} \in K_n^{(3)}$. Thus $P^{(n)}_{n+1} = P_{n+1}$. Then the map $i : P^{(n)}_{n+1} \to P_{n+1}$ is the identity map: $i = id$. Therefore for $\beta = (P^{(n)}_{n+1}; fi)$ we have $\beta = \alpha$. In this case

$$[\lambda_2(p)]_\alpha = p_\alpha = i^{*-1}(q_\beta) = id^{*-1}(q_\alpha) = q_\alpha.$$

It remains to prove that $q_\beta \in Im\, i^*$. Consider the characteristic map of the CW-complex P_{n+1}

$$\Phi : \big(C(\vee S^n), \vee S^n\big) \to \big(P_{n+1}, P^{(n)}_{n+1}\big),$$

where $C(\vee S^n)$ denotes the cone over $\vee S^n$ and $\vee$ denotes the finite one-point union of spaces. Let $\varphi = \Phi|(\vee S^n)$. Consider the commutative diagram

$$\begin{tikzcd} & R & \\ \vee S^n \arrow[ur, "fi\varphi"] \arrow[r, "\varphi"] & P^{(n)}_{n+1} \arrow[u, "fi"] \arrow[r, "i"] & P_{n+1} \arrow[ul, "f"'] \end{tikzcd}$$

Since $i\varphi \sim 0$, we obtain $fi\varphi \sim 0$. Let $\gamma = (\vee S^n; fi\varphi) \in {}^{(3)}\omega(R;n)$. We have $\gamma < \beta$ and $q_\gamma = 0$ (see the proof of Corollary 1.3). In this case $\varphi^*(q_\beta) = q_\gamma = 0$.

Now consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^n(P_{n+1}) \arrow[r, "i^*"] & H^n(P^{(n)}_{n+1}) \arrow[r, "\delta"] \arrow[d, "\varphi^*"] & H^{n+1}(P_{n+1}, P^{(n)}_{n+1}) \arrow[d, "\Phi^*"] & \\ & 0 \arrow[r] & H^n(\vee S^n) \arrow[r, "\delta"] & H^{n+1}(C(\vee S^n), \vee S^n) \arrow[r] & 0 \end{tikzcd}$$

where Φ^* is an isomorphism. Then, since $\varphi^*(q_\beta) = 0$, we have $\delta(q_\beta) = 0$. Therefore $q_\beta \in Im\, i^*$. This completes the proof of Proposition 4.4. $\square$

Lemma 4.5. *For each CW-complex P from the category $K_n^{(3)}$ there is a CW-complex $\bar{P}$ from the category $K_n^{(4)}$ such that $\bar{P}$ has the homotopy type of P.*

Proof. By the condition $H_i(P) = 0$, $i \neq 0, n-1, n$; $H_{n-1}(P) \approx \pi_{n-1}(P)$ are finite abelian groups and $H_n(P)$ is a finitely generated free abelian group. Thus the group $\pi_{n-1}(P)$ can be represented in the form

$$\pi_{n-1}(P) \approx Z_{r_1} \oplus Z_{r_2} \oplus \cdots \oplus Z_{r_t},$$

where Z_{r_i}, $i = 1, 2, \dots, t$, are cyclic groups of order r_i. Consider the corresponding system $\xi_i : S^{n-1} \to P$, $1 \leq i \leq t$, of generators in the group $\pi_{n-1}(P)$ and define, by means of ξ_i, the map $f : \vee_{i=1}^{t} P^n(r_i) \to P$. Then f induces isomorphisms in homotopy and homology in dimensions $\leq n-1$. Now consider a system $h_1, h_2, \dots, h_s$ of generators in the group $H_n(P)$. The Hurewicz homomorphism $\pi_n(P) \to H_n(P)$ for the space P is an epimorphism. In this case we can consider maps $\varphi_k : S^n \to P$, $k = 1, 2, \dots, s$, such that $\varphi_{k*}(1) = h_k$, where $1 \in H_n(S^n)$. Assume

$$\bar{P} = \big(\vee_{i=1}^{t} P^n(r_i)\big) \bigvee \big(\vee_{k=1}^{s} P_k^n(1)\big),$$

where $P_k^n(1) = P^n(1) = S^n$, and define by means of the maps f and φ_k the map $\varphi = f \vee (\vee \varphi_k) : \bar{P} \to P$. Then φ induces isomorphisms of all homology groups. Therefore, under our assumptions, the map φ is a homotopy equivalence. This proves Lemma 4.5. $\square$

Lemmas 4.1 and 4.5 imply

Proposition 4.6. *λ_3 is a natural isomorphism.*

Propositions 4.2 - 4.4 and 4.6 imply Theorem 2.4.

5. Proof of Theorem 2.5

Let $h \in H^n(Q^n)$ and $i_{j,t} : P^n(t) \to Q^n$ be standard embeddings. Assume

$$\varepsilon(h) = \{i_{j,t}^*(h)\} \in \prod_{j,t} H^n\big(P_j^n(t)\big).$$

Obviously, we have

Lemma 5.1. *The correspondence*

$$\varepsilon : H^n(Q^n) \to \prod_{j,t} H^n\big(P_j^n(t)\big)$$

is an isomorphism.

In the sequel, for convenience, the subcategories F_n^b and Q_n will be denoted by $K_n^{(4)}$ and $K_n^{(5)}$. Let $R=(R,*)$ be an arbitrary space from K. Let $q\in\Pi^n(R;K_n^{(4)})$ and

$$\alpha=(Q^n;f)\in {}^{(5)}\omega(R;n)$$

be an arbitrary index. Assume $f_{j,t}=fi_{j,t}$ and consider the indices

$$\alpha_{j,t}=(P^n(t);f_{j,t})\in {}^{(4)}\omega(R;n).$$

Let

$$p_\alpha=[\lambda_4(q)]_\alpha=\varepsilon^{-1}\big(\{q_{\alpha_{j,t}}\}\big).$$

We will show that the set $\{p_\alpha\}$ defines an element of the group $\Pi^n(R;K_n^{(5)})$ and the natural isomorphism

$$\lambda_4:\Pi^n(R;K_n^{(4)})\to\Pi^n(R;K_n^{(5)}).$$

By $\vee$ we will denote the symbol of finite one-point union of spaces. Consider the commutative diagram

$$\begin{tikzcd}
 & R & \\
P^n(t) \arrow[ur, "f_{k,l}"] \arrow[r, "\widetilde{i}_{k,l}"] & \vee_{j,t}P_j^n(t) \arrow[u, "\widetilde{f}"'] \arrow[r, "\vee i_{j,t}"] & Q^n \arrow[ul, "f"']
\end{tikzcd}$$

where $\widetilde{i}_{k,l}$ and $\vee i_{j,t}$ are standard embeddings, and the indices

$$\alpha=(Q^n;f)\in {}^{(5)}\omega(R;n),$$
$$\beta=(\vee_{j,t}P_j^n(t);\widetilde{f})\in {}^{(4)}\omega(R;n),$$
$$\alpha_{k,l}=(P^n(t);f_{k,l})\in {}^{(4)}\omega(R;n).$$

We have $(\vee i_{j,t})\widetilde{i}_{k,l}=i_{k,l}$. Therefore $\alpha_{k,l}<\beta$. Then

$$\widetilde{i}^*_{k,l}\big((\vee i_{j,t})^*(p_\alpha)\big)=i^*_{k,l}(p_\alpha)=q_{\alpha_{k,l}}.$$

Since this is true for arbitrary k and l, we have

$$(\vee i_{j,t})^*(p_\alpha)=q_\beta. \tag{5}$$

Now consider the diagram

$$\begin{tikzcd} & R & \\ P^n(t) \arrow[ur, "g"] \arrow[rr, "\varphi"] \arrow[dr, "\varphi_1"'] & & Q^n \arrow[ul, "f"'] \\ & \vee_{j,t} P_j^n(t) \arrow[uu, "F"] \arrow[ur, "\vee i_{j,t}"'] & \end{tikzcd}$$

where $f\varphi \sim g$, $F = f(\vee i_{j,t})$, and $(\vee i_{j,t})\varphi_1 = \varphi$ (since $P^n(t)$ is a compact space and φ is a continuous map, it follows that there exists a map φ_1). Then

$$F\varphi_1 = f\big(\vee_{j,t}\, i_{j,t}\big)\varphi_1 = f\varphi \sim g.$$

Consider the indices

$$\beta = (P^n(t), g) \in {}^{(4)}\omega(R; n),$$
$$\beta_1 = \big(\vee_{j,t}\, P_j^n(t); F\big) \in {}^{(4)}\omega(R; n).$$

Then $\beta < \beta_1$, and by (5) we have

$$\varphi^*(p_\alpha) = \varphi_1^*\big((\vee_{j,t})^*(p_\alpha)\big) = \varphi_1^*(q_{\beta_1}) = q_\beta.$$

Thus

$$\varphi^*(p_\alpha) = q_\beta. \tag{6}$$

Finally, consider the diagram

$$\begin{tikzcd} & R & \\ P^n(t) \arrow[ur, "\varphi_1"] \arrow[r, "i_{j,t}"] & Q^n \arrow[u, "f"] \arrow[r, "\varphi"] & Q^n \arrow[ul, "f_1"'] \end{tikzcd}$$

where $f_1\varphi \sim f$ and $fi_{j,t} = \varphi_1$. Consider the indices

$$\alpha_1 = (Q^n; f_1) \in {}^{(5)}\omega(R; n),$$
$$\alpha_{j,t} = (P^n(t); \varphi_1) \in {}^{(4)}\omega(R; n).$$

Thus $\alpha < \alpha_1$. Therefore by (6) we have

$$i_{j,t}^*\big(\varphi^*(p_{\alpha_1})\big) = (\varphi i_{j,t})^*(p_{\alpha_1}) = q_{\alpha_{j,t}}.$$

Since this equality is true for arbitrary j and t, we obtain $\varphi^*(p_{\alpha_1}) = p_\alpha$. Therefore the set $\{p_\alpha\}$ defines an element $p \in \Pi^n(R; K_n^{(5)})$. The map λ_4 is now defined by setting $\lambda_4(q) = p$, where $q \in \Pi^n(R; K_n^{(4)})$.

Let $q_1, q_2 \in \Pi^n(R; K_n^{(4)})$. Then we have

$$i_{j,t}^*\big([\lambda_4(q_1+q_2)]_\alpha\big) = (q_1+q_2)_{\alpha_{j,t}} = (q_1)_{\alpha_{j,t}} + (q_2)_{\alpha_{j,t}} = i_{j,t}^*\big([\lambda_4(q_1)]_\alpha\big) + \\ + i_{j,t}^*\big([\lambda_4(q_2)]_\alpha\big) = i_{j,t}^*\big([\lambda_4(q_1)]_\alpha + [\lambda_4(q_2)]_\alpha\big) = i_{j,t}^*\big([\lambda_4(q_1) + \lambda_4(q_2)]_\alpha\big).$$

Since this equality holds for arbitrary j and t, we have

$$\lambda_4(q_1+q_2) = \lambda_4(q_1) + \lambda_4(q_2).$$

Let now $\varphi : S \to R$ be an arbitrary map. Consider the diagram

$$\begin{tikzcd} \Pi^n(R; K_n^{(4)}) \arrow[r, "\lambda_4"] \arrow[d, "\varphi^{\#}"] & \Pi^n(R; K_n^{(5)}) \arrow[d, "\varphi^{\#}"] \\ \Pi^n(S; K_n^{(4)}) \arrow[r, "\lambda_4"] & \Pi^n(S; K_n^{(5)}) \end{tikzcd} ,$$

the element $q \in \Pi^n(R; K_n^{(4)})$, indices $\alpha, \alpha_{j,t}$, and

$$\beta = \varphi(\alpha) = (Q^n; g) \in {}^{(5)}\omega(R; n),$$
$$\beta_{j,t} = (P^n(t); g_{j,t}) \in {}^{(4)}\omega(R; n),$$

where $g = \varphi f$, $g_{j,t} = g i_{j,t}$. Then $\beta_{j,t} = \varphi(\alpha_{j,t})$, and we have

$$\big[\varphi^{\#}(\lambda_4(q))\big]_\alpha = [\lambda_4(q)]_\beta = \varepsilon^{-1}\big(\{q_{\beta_{j,t}}\}\big).$$

On the other hand, we have

$$\big[\lambda_4(\varphi^{\#}(q))\big]_\alpha = \varepsilon^{-1}\big(\{[\varphi^{\#}(q)]_{\alpha_{j,t}}\}\big) = \varepsilon^{-1}\big(\{q_{\varphi(\alpha_{j,t})}\}\big) = \varepsilon^{-1}\big(\{q_{\beta_{j,t}}\}\big).$$

Thus λ_4 is a natural homomorphism.

We will prove that λ_4 is a monomorphism. Let $q \in \Pi^n(R; K_n^{(4)})$ and $\lambda_4(q) = 0$. Consider an arbitrary index

$$\beta = \big(\vee_{j,t}\, P_j^n(t); g\big) \in {}^{(4)}\omega(R; n)$$

and define the map $f : Q^n \to R$ by taking

$$f\big((\vee i_{j,t})(x)\big) = g(x), \quad x \in \vee_{j,t} P_j^n(t),$$
$$f\big(Q^n - (\vee i_{j,t})(\vee_{j,t} P_j^n(t))\big) = *.$$

Consider the index

$$\alpha = (Q^n; f) \in {}^{(5)}\omega(R; n)$$

and the commutative diagram

$$
\begin{tikzcd}
 & R & \\
\vee_{j,t} P_j^n(t) \arrow[ur, "g"] \arrow[rr, "\vee i_{j,t}"] & & Q^n \arrow[ul, "f"']
\end{tikzcd}
$$

then by (5) we have

$$q_\beta = \big(\vee_{j,t}\, i_{j,t}\big)^*\big([\lambda_4(q)]_\alpha\big) = \big(\vee_{j,t}\, i_{j,t}\big)^*(0) = 0.$$

Therefore $q = 0$ and λ_4 is a monomorphism.

Further, we will prove that λ_4 is an epimorphism. Let $p \in \Pi^n(R; K_n^{(5)})$ and

$$\alpha = (X; f) \in {}^{(4)}\omega(R; n)$$

be an arbitrary index, $X \in K_n^{(4)}$. Therefore the space X can be represented in the form $X = \vee_{j,t} P_j^n(t)$, where j is an index indicating a certain arrangement of the identical subspaces of X. Then we have the natural embedding $i : X \to Q^n$. Define the map $l : Q^n \to X$ by taking

$$
\begin{aligned}
&l(i(x)) = x, \quad x \in X;\\
&l(Q^n - i(X)) = *.
\end{aligned}
$$

Let $g = fl$. We have $gi = f$. Let

$$\beta = (Q^n; g) \in {}^{(5)}\omega(R; n).$$

Assume that

$$q_\alpha = i^*(p_\beta). \tag{7}$$

Consider a different arrangement of subspaces of X. Let the map $\widetilde{i}$ and the index $\widetilde{\beta} = (Q^n; \widetilde{g})$ be defined in the same way as i and β, respectively. Consider the commutative diagram

$$
\begin{tikzcd}
 & R & \\
Q^n \arrow[ur, "g"] \arrow[rr, "k"] & & Q^n \arrow[ul, "\widetilde{g}"'] \\
 & X \arrow[ul, "i"] \arrow[ur, "\widetilde{i}"'] &
\end{tikzcd}
$$

where the map k can be defined by a certain permutation of subspaces of Q^n. Then $\beta < \widetilde{\beta}$ and we have

$$\widetilde{i}^*(p_{\widetilde{\beta}}) = i^*(k^*(p_{\widetilde{\beta}})) = i^*(p_\beta).$$

Thus definition (7) is correct.

Consider the diagram

$$
\begin{tikzcd}
& R & \\
X_1 \arrow[ur, "f_1"] \arrow[rr, "\varphi"'] \arrow[d, shift left, "i_1"] & & X_2 \arrow[ul, "f_2"'] \arrow[d, shift left, "i_2"] \\
Q^n \arrow[u, shift left, "l_1"] \arrow[uur, "g_1"] \arrow[rr, "\widetilde{\varphi}"] & & Q^n \arrow[u, shift left, "l_2"] \arrow[uul, "g_2"']
\end{tikzcd}
$$

where $X_1, X_2 \in K_n^{(4)}$, $f_2\varphi \sim f_1$, $\widetilde{\varphi} = i_2\varphi l_1$, $f_1 = g_1 i_1$, $f_2 = g_2 i_2$. Then

$$
\begin{gathered}
\widetilde{\varphi} i_1 = i_2\varphi l_1 i_1 = i_2\varphi, \\
g_2\widetilde{\varphi} = f_2 l_2 i_2 \varphi l_1 = f_2\varphi l_1 \sim f_1 l_1 = g_1.
\end{gathered}
$$

Consider the indices

$$
\begin{aligned}
\alpha_t &= (X_t; f_t) \in {}^{(4)}\omega(R; n), \quad t = 1, 2; \\
\beta_t &= (Q^n; g_t) \in {}^{(5)}\omega(R; n), \quad t = 1, 2.
\end{aligned}
$$

Then $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$ and we have

$$
\varphi^*(q_{\alpha_2}) = \varphi^*\big(i_2^*(p_{\beta_2})\big) = i_1^*\big(\widetilde{\varphi}^*(p_{\beta_2})\big) = i_1^*(p_{\beta_1}) = q_{\alpha_1}.
$$

Therefore the set $\{q_\alpha\}$ defines an element $q \in \Pi^n(R; K_n^{(4)})$.

Finally, let us prove that $\lambda_4(q) = p$. Consider an arbitrary index

$$
\alpha = (Q^n; f) \in {}^{(5)}\omega(R; n)
$$

and the index

$$
\alpha_{j,t} = (P^n(t); f i_{j,t}) \in {}^{(4)}\omega(R; n).
$$

Then $i_{j,t} = i$ and we have

$$
i_{j,t}^*\big([\lambda_4(q)]_\alpha\big) = q_{\alpha_{j,t}} = i_{j,t}^*(p_\alpha).
$$

Since this equality is true for arbitrary j and t, we have $[\lambda_4(q)]_\alpha = p_\alpha$. Therefore $\lambda_4(q) = p$. This completes the proof of Theorem 2.5. ∎

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