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ON THE STABILITY OF SOLUTIONS OF LINEAR BOUNDARY VALUE PROBLEMS FOR A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. Linear boundary value problems for a system of ordinary differential equations are considered. The stability of the solution with respect to small perturbations of coefficients and boundary values is investigated.

Let $\mathcal{P}_0 : [a,b] \to \mathbb{R}^{n\times n}$ and $q_0 : [a.b] \to \mathbb{R}^n$ be integrable matrix- and vector-functions, respectively, $c_0 \in \mathbb{R}^n$, and let $l_0 : C([a,b];\mathbb{R}^n) \to \mathbb{R}^n$ be a linear continuous operator such that the boundary value problem

$$\frac{dx}{dt} = \mathcal{P}_0(t)x + q_0(t), \tag{1}$$

$$l_0(x) = c_0 \tag{2}$$

has the unique solution x_0. Consider the sequences of integrable matrix- and vector-functions, $\mathcal{P}_k : [a,b] \to \mathbb{R}^{n\times n}$ $(k = 1,2,\dots)$, and $q_k : [a,b] \to \mathbb{R}^n$ $(k = 1,2,\dots)$, respectively, the sequence of constant vectors $c_k \in \mathbb{R}^n$ $(k = 1,2,\dots)$ and the sequence of linear continuous operators $l_k : C([a,b];\mathbb{R}^n) \to \mathbb{R}^n$ $(k = 1,2,\dots)$. In [1,2], sufficient conditions are given for the problem

$$\frac{dx}{dt} = \mathcal{P}_k(t)x + q_k(t), \tag{3}$$

$$l_k(x) = c_k \tag{4}$$

to have a unique solution x_k for any sufficiently large k and

$$\lim_{k\to+\infty} x_k(t) = x_0(t) \quad \text{uniformly on } [a,b]. \tag{5}$$

In the present paper, necessary and sufficient conditions are established for the sequence of boundary value problems of the form (3),(4) to have the above-mentioned property.

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Throughout the paper the following notations and definitions will be used:

$\mathbb{R} =]-\infty, +\infty[$;

$\mathbb{R}^n$ is a space of real column n-vectors $x = (x_i)_{i=1}^n$ with the norm

$$\|x\| = \sum_{i=1}^n |x_i|;$$

$\mathbb{R}^{n\times n}$ is a space of real $n \times n$ matrices $X = (x_{ij})_{i,j=1}^n$ with the norm

$$\|X\| = \sum_{i,j=1}^n |x_{ij}|;$$

if $X = (x_{ij})_{i,j=1}^n \in \mathbb{R}^{n\times n}$, then $\operatorname{diag} X$ is a diagonal matrix with diagonal components $x_{11}, \dots, x_{nn}$; X^{-1} is an inverse matrix to X; E is an identity $n \times n$ matrix;

$C([a,b];\mathbb{R}^n)$ is a space of continuous vector-functions $x : [a,b] \to \mathbb{R}^n$ with the norm

$$\|x\|_c = \max\{\|x(t)\| : a \le t \le b\};$$

$\widetilde{C}([a,b];\mathbb{R}^n)$ and $\widetilde{C}([a,b];\mathbb{R}^{n\times n})$ are the sets of absolutely continuous vector- and matrix- functions, respectively;

$L([a,b];\mathbb{R}^n)$ and $L([a,b];\mathbb{R}^{n\times n})$ are the sets of vector- and matrix-functions $x : [a,b] \to \mathbb{R}^n$ and $X : [a,b] \to \mathbb{R}^{n\times n}$, respectively, whose components are Lebesgue-integrable;

$\|l\|$ is the norm of the linear continuous operator $l : C([a,b];\mathbb{R}^n) \to \mathbb{R}^n$.

The vector-function $x : [a,b] \to \mathbb{R}^n$ is said to be a solution of the problem (1),(2) if it belongs to $\widetilde{C}([a,b];\mathbb{R}^n)$ and satisfies the condition (2) and the system (1) a.e. on $[a,b]$.

Definition 1. We shall say that the sequence $(\mathcal{P}_k, q_k, l_k)$ $(k = 1, 2, \dots)$ belongs to $S(\mathcal{P}_0, q_0, l_0)$ if for every $c_0 \in \mathbb{R}^n$ and $c_k \in \mathbb{R}^n$ $(k = 1, 2, \dots)$ satisfying the condition

$$\lim_{k\to+\infty} c_k = c_0 \tag{6}$$

the problem (3),(4) has the unique solution x_k for any sufficiently large k and (5) holds.

Along with (1),(2) and (3),(4) we shall consider the corresponding homogeneous problems

$$\frac{dx}{dt} = \mathcal{P}_0(t)x, \tag{1_0}$$

$$l_0(x) = 0 \tag{2_0}$$

and

$$\frac{dx}{dt} = \mathcal{P}_k(t)x, \tag{3_0}$$

$$l_k(x) = 0. \tag{4_0}$$

Theorem 1. *Let*

$$\lim_{k\to+\infty} l_k(y) = l_0(y) \quad \textit{for } y \in \widetilde{C}([a,b];\mathbb{R}^n) \tag{7}$$

and

$$\lim_{k\to+\infty} \sup \|l_k\| < +\infty. \tag{8}$$

Then

$$\big((\mathcal{P}_k, q_k, l_k)\big)_{k=1}^{+\infty} \in S(\mathcal{P}_0, q_0, l_0) \tag{9}$$

if and only if there exist sequences of matrix- and vector-functions, $\Phi_k \in \widetilde{C}([a,b];\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$ *and* $\varphi_k \in \widetilde{C}([a,b];\mathbb{R}^n)$ $(k = 1,2,\dots)$*, respectively, such that*

$$\lim_{k\to+\infty} \sup \int_a^b \|\mathcal{P}_k^*(\tau)\|\, d\tau < +\infty \tag{10}$$

and

$$\lim_{k\to+\infty} \Phi_k(t) = 0, \tag{11}$$

$$\lim_{k\to+\infty} \varphi_k(t) = 0, \tag{12}$$

$$\lim_{k\to+\infty} \int_a^t \mathcal{P}_k^*(\tau)\, d\tau = \int_a^t \mathcal{P}_0(\tau)\, d\tau, \tag{13}$$

$$\lim_{k\to+\infty} \int_a^t q_k^*(\tau)\, d\tau = \int_a^t q_0(\tau)\, d\tau \tag{14}$$

uniformly on $[a,b]$*, where*

$$\mathcal{P}_k^*(t) \equiv [E - \Phi_k(t)] \cdot \mathcal{P}_k(t) - \Phi_k'(t), \tag{15}$$

$$q_k^*(t) \equiv [E - \Phi_k(t)][\mathcal{P}_k(t)\varphi_k(t) + q_k(t) - \varphi_k'(t)]. \tag{16}$$

Theorem 1′. *Let* (7)*and* (8) *be satisfied. The* (9) *holds if and only if there exist sequences of matrix- and vector-functions,* $\Phi_k \in \widetilde{C}([a,b];\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$ *and* $\varphi_k \in \widetilde{C}([a,b];\mathbb{R}^n)$ $(k = 1,2,\dots)$*, respectively, such that*

$$\lim_{k\to+\infty} \sup \int_a^b \|\mathcal{P}_k^*(\tau) - \operatorname{diag} \mathcal{P}_k^*(\tau)\|\, d\tau < +\infty \tag{17}$$

and the conditions (11)–(13) *and*

$$\lim_{k\to+\infty}\int_a^t \exp\Big(-\int_a^\tau \operatorname{diag}\mathcal{P}_k^*(s)ds\Big)\cdot q_k^*(\tau)\,d\tau = \\ = \int_a^t \exp\Big(-\int_a^\tau \operatorname{diag}\mathcal{P}_0(s)ds\Big)\cdot q_0(\tau)\,d\tau \tag{18}$$

are fulfilled uniformly on $[a,b]$, *where*

$$\mathcal{P}_k^*(t) \equiv [\mathcal{P}_k(t) - \Phi_k(t)\mathcal{P}_k(t) - \Phi_k'(t)]\cdot[E-\Phi_k(t)]^{-1} \tag{19}$$

and $q_k^*(t)$ *is the vector-function defined by* (16).

Before proving this theorems, we shall give a theorem from [1] and some of its generalizations.

Theorem 2_0. *Let the conditions* (6)–(8),

$$\lim_{k\to+\infty}\sup\int_a^b \|\mathcal{P}_k(\tau)\|\,d\tau < +\infty \tag{20}$$

hold and let the following conditions

$$\lim_{k\to+\infty}\int_a^t \mathcal{P}_k(\tau)\,d\tau = \int_a^t \mathcal{P}_0(\tau)\,d\tau, \tag{21}$$

$$\lim_{k\to+\infty}\int_a^t q_k(\tau)\,d\tau = \int_a^t q_0(\tau)\,d\tau \tag{22}$$

hold uniformly on $[a,b]$. *Then* (9) *is satisfied*[1].

Theorem 2. *Let there exist sequences of matrix- and vector-functions,* $\Phi_k \in \widetilde{C}([a,b];\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ *and* $\varphi_k \in \widetilde{C}([a,b];\mathbb{R}^n)$ $(k=1,2,\dots)$, *respectively, such that the conditions* (10),

$$\lim_{k\to+\infty}[c_k - l_k(\varphi_k)] = c_0 \tag{23}$$

hold and let the conditions (11),(13),(14) *be fulfilled uniformly on* $[a,b]$, *where* $\mathcal{P}_k^*(t)$ *and* $q_k^*(t)$ *are the matrix- and vector-functions defined by* (15) *and* (16), *respectively. Let, moreover, conditions* (7),(8) *hold. Then for any sufficiently large* k *the problem* (3),(4) *has the unique solution* x_k *and*

$$\lim_{k\to+\infty}\|x_k-\varphi_k-x_0\|_c = 0.$$

[1]See [1], Theorem 1.2.

Proof. The transformation $z = x - \varphi_k$ reduces the problem (3),(4) to

$$\frac{dz}{dt} = \mathcal{P}_k(t)z + r_k(t), \tag{24}$$

$$l_k(z) = c_{k1}, \tag{25}$$

where $r_k(t) \equiv \mathcal{P}_k(t)\varphi_k(t) + q_k(t) - \varphi_k'(t)$, $c_{k1} = c_k - l_k(\varphi_k)$ $(k = 1, 2, \dots)$.

Let us show that for any sufficiently large k the homogeneous problem (3_0),(4_0) has only trivial solution.

Suppose this proposition is invalid. It can be assumed without loss of generality that for every natural k the problem (3_0),(4_0) has the solution x_k for which

$$\|x_k\|_c = 1. \tag{26}$$

Moreover, it is evident that the vector-function x_k is the solution of the system

$$\frac{dx}{dt} = \mathcal{P}_k^*(t)x + [\Phi_k(t) \cdot x_k(t)]'. \tag{27}$$

According to (11) and (26)

$$\lim_{k \to +\infty} [\Phi_k(t) \cdot x_k(t)] = 0$$

uniformly on $[a, b]$. Therefore the conditions of Theorem 2_0 are fulfilled for the sequence of problems (27),(4_0). Hence

$$\lim_{k \to +\infty} \|x_k\|_c = 0,$$

which contradicts (26). This proves that the problem (3_0),(4_0) has only trivial solution.

From this fact it follows that for any sufficiently large k the problem (24),(25) has the unique solution z_k.

It can easily be shown that the vector-function z_k satisfies the system

$$\frac{dz}{dt} = \mathcal{P}_k^*(t)z + r_k^*(t), \tag{28}$$

where $r_k^*(t) = [\Phi_k(t) \cdot z_k(t)]' + q_k^*(t)$.

Show that

$$\lim_{k \to +\infty} \sup \|z_k\|_c < +\infty. \tag{29}$$

Let this proposal be invalid. Assume without loss of generality that

$$\lim_{k \to +\infty} \|z_k\|_c = +\infty. \tag{30}$$

Put
$$u_k(t) = \|z_k\|_c^{-1} \cdot z_k(t) \quad \text{for} \ \ t \in [a,b] \ \ (k=1,2,\dots).$$
Then in view of (25) and (28), for every natural k the vector-function $u_k(t)$ will be the solution of the boundary value problem
$$\frac{du}{dt} = \mathcal{P}_k^*(t)u + s_k(t),$$
$$l_k(u) = \|z_k\|_c^{-1} \cdot c_{k1},$$
where $s_k(t) = \|z_k\|_c^{-1} \cdot r_k^*(t)$. Equations (11),(14),(23), and (30) imply
$$\lim_{k\to+\infty} [\|z_k\|_c^{-1} \cdot c_{k1}] = 0$$
and
$$\lim_{k\to+\infty} \int_a^t s_k(\tau)\,d\tau = 0$$
uniformly on $[a,b]$. Hence, according to (10) and (13), the conditions of Theorem 2_0 are fulfilled for the sequence of the last boundary value problems. Therefore
$$\lim_{k\to+\infty} \|u_k\|_c = 0.$$
This equality contradicts the conditions $\|u_k\|_c = 1$ $(k=1,2,\dots)$. The inequality (29) is proved.

In view of (11),(14), and (29)
$$\lim_{k\to+\infty} \int_a^t r_k^*(\tau)\,d\tau = \int_a^t q_0(\tau)\,d\tau$$
uniformly on $[a,b]$.

Applying Theorem 2_0 to the sequence of the problems (28),(25), we again show that
$$\lim_{k\to+\infty} \|z_k - x_0\|_c = 0. \qquad \square$$

Corollary 1. *Let* (6)–(8),
$$\lim_{k\to+\infty} \sup \int_a^b \|\mathcal{P}_k(\tau) - \Phi_k(\tau)\mathcal{P}_k(\tau) - \Phi_k'(\tau)\|\,d\tau < +\infty$$
hold and let the conditions (11),(21),(22),
$$\lim_{k\to+\infty} \int_a^t \Phi_k(\tau)\mathcal{P}_k(\tau)\,d\tau = \int_a^t \mathcal{P}^*(\tau)\,d\tau$$
and
$$\lim_{k\to+\infty} \int_a^t \Phi_k(\tau)q_k(\tau)\,d\tau = \int_a^t q^*(\tau)\,d\tau$$

be fulfilled uniformly on $[a,b]$, *where* $\Phi_k \in \widetilde{C}([a,b];\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $\mathcal{P}^* \in L([a,b];\mathbb{R}^{n\times n})$, $q^* \in L([a,b];\mathbb{R}^n)$. *Let, moreover, the system*

$$\frac{dx}{dt} = \mathcal{P}_0^*(t)x + q_0^*(t),$$

where $\mathcal{P}_0^*(t) \equiv \mathcal{P}_0(t) - \mathcal{P}^*(t)$, $q_0^*(t) \equiv q_0(t) - q^*(t)$, *have a unique solution satisfying the condition* (2). *Then*

$$\big((\mathcal{P}_k, q_k, l_k)\big)_{k=1}^{+\infty} \in S(\mathcal{P}_0^*, q_0^*, l_0).$$

Proof. It suffices to assume in Theorem 2 that $\varphi_k(t) \equiv 0$ and to notice that

$$\lim_{k\to+\infty} \int_a^t [E - \Phi_k(\tau)] \cdot \mathcal{P}_k(\tau)\, d\tau = \int_a^t \mathcal{P}_0^*(\tau)\, d\tau$$

and

$$\lim_{k\to+\infty} \int_a^t [E - \Phi_k(\tau)] \cdot q_k(\tau)\, d\tau = \int_a^t q_0^*(\tau)\, d\tau$$

uniformly on $[a,b]$. □

Corollary 2. *Let* (6)–(8) *hold, and let there exist a natural number* m *and matrix-functions* $\mathcal{P}_{oj} \in L([a,b],\mathbb{R}^{n\times n})$ $(j=1,\dots,m)$ *such that*

$$\lim_{k\to+\infty} [\mathcal{P}_{km}(t) - \mathcal{P}_k(t)] = 0,$$
$$\lim_{k\to+\infty} \int_a^t [E + \mathcal{P}_{km}(\tau) - \mathcal{P}_k(\tau)] \cdot \mathcal{P}_k(\tau)\, d\tau = \int_a^t \mathcal{P}_0(\tau)\, d\tau,$$
$$\lim_{k\to+\infty} \int_a^t [E + \mathcal{P}_{km}(\tau) - \mathcal{P}_k(\tau)] \cdot q_k(\tau)\, d\tau = \int_a^t q_0(\tau)\, d\tau$$

uniformly on $[a,b]$, *where*

$$\mathcal{P}_{k1}(t) \equiv \mathcal{P}_k(t), \quad \mathcal{P}_{k\,j+1}(t) \equiv \mathcal{P}_{kj}(t) - \int_a^t [\mathcal{P}_{kj}(\tau) - \mathcal{P}_{0j}(\tau)]\, d\tau \quad (j=1,\dots,m).$$

Let, moreover,

$$\lim_{k\to+\infty} \sup \int_a^b \|[E + \mathcal{P}_{km}(\tau) - \mathcal{P}_k(\tau)] \cdot \mathcal{P}_k(\tau) + [\mathcal{P}_{km}(\tau) - \mathcal{P}_k(t)]'\| d\tau < +\infty.$$

Then (9) *holds.*

Theorem 2′. *Let there exist sequences of matrix- and vector-functions, $\Phi_k \in \widetilde{C}([a,b];\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ and $\varphi_k \in \widetilde{C}([a,b];\mathbb{R}^n)$ $(k=1,2,\dots)$, respectively, such that the conditions* (17),(23) *hold, and let the conditions* (11),(13), *and* (18) *be fulfilled uniformly on $[a,b]$. Here $\mathcal{P}_k^*(t)$ and $q_k^*(t)$ are the matrix- and vector-functions defined by* (19) *and* (16), *respectively. Then the conclusion of Theorem* 2 *is true.*

Proof. In view of (14), we may assume without loss of generality that for every natural k the matrix $E-\Phi_k(t)$ is invertible for $t\in[a,b]$.

For every $k\in\{0,1,\dots\}$ and $t\in[a,b]$ assume

$$\mathcal{P}_0^*(t)=\mathcal{P}_0(t),\ \ q_0^*(t)=q_0(t),\ \ \Phi_0(t)=0,\ \ \varphi_0(t)=0,$$
$$c_{k1}=c_k-l_k(\varphi_k),\ \ Q_k(t)=H_k(t)\cdot[\mathcal{P}_k^*(t)-\operatorname{diag}\mathcal{P}_k^*(t)]\cdot H_k^{-1}(t),$$
$$r_k(t)=H_k(t)\cdot q_k^*(t),$$

where

$$H_k(t)=\exp\Big(-\int_a^t \operatorname{diag}\mathcal{P}_k^*(\tau)\,d\tau\Big).$$

Moreover, assume

$$l_k^*(z)=l_k(x)\quad\text{for}\ \ z\in C([a,b];\mathbb{R}^n),$$

where $x(t)=[E-\Phi_k(t)]^{-1}\cdot H_k^{-1}(t)\cdot z(t)$.

From (13) it follows that $l_k^*: C([a,b];\mathbb{R}^n)\to\mathbb{R}^n$ $(k=0,1,\dots)$ is a sequence of linear continuous operators for which conditions (7) and (8) are satisfied.

For every $k\in\{0,1,\dots\}$ the transformation

$$z(t)=H_k(t)\cdot[E-\Phi_k(t)]\cdot[x(t)-\varphi_k(t)]\ \ \text{for}\ \ t\in[a,b] \tag{31}$$

reduces the problem (3),(4) to

$$\frac{dz}{dt}=Q_k(t)z+r_k(t), \tag{32}$$
$$l_k^*(z)=c_{k1} \tag{33}$$

and the problem (1),(2) to

$$\frac{dz}{dt}=Q_0(t)z+r_0(t), \tag{34}$$
$$l_0^*(z)=c_0. \tag{35}$$

In view of (13) and (17) from Lemma 1.1 ([1], p.9) it follows that

$$\lim_{k\to+\infty}\int_a^t Q_k(\tau)\,d\tau=\int_a^t Q_0(\tau)\,d\tau$$

uniformly on $[a,b]$. According to Theorem 2_0 from the above and from (7),(8),(17),(18),(23) it follows that the problem (32),(33) has the unique solution z_k for any sufficiently large k, and

$$\lim_{k\to+\infty} \|z_k - z_0\|_c = 0,$$

where z_0 is the unique solution of the problem (34),(35). Therefore (11),(13) and (31) show that the statement of the theorem is true. □

Corollary 3. *Let the conditions* (6)–(8),

$$\lim_{k\to+\infty} \sup \int_a^b \|\mathcal{P}_k(\tau) - \operatorname{diag} \mathcal{P}_k(\tau)\| \, d\tau < +\infty$$

hold and let (21) *and*

$$\lim_{k\to+\infty} \int_a^t \exp\Big(-\int_a^\tau \operatorname{diag} \mathcal{P}_k(s)ds\Big) \cdot q_k(\tau)\, d\tau =$$
$$= \int_a^t \exp\Big(-\int_a^\tau \operatorname{diag} \mathcal{P}_0(s)ds\Big) \cdot q_0(\tau)\, d\tau$$

be fulfilled uniformly on $[a,b]$. *Then* (9) *holds.*

Remark. As compared with Theorem 2_0 and the results of [2], it is not assumed in Theorems 2 and $2'$ that the equalities (21) and (22) hold uniformly on $[a,b]$. Below we will give an example of a sequence of boundary value problems for linear systems for which (9) holds but (21) is not fulfilled uniformly on $[a,b]$.

Example. Let $a = 0$, $b = 2\pi$, $n = 2$, and for every natural k and $t \in [0,2\pi]$, let

$$\mathcal{P}_k(t) = \begin{pmatrix} 0 & p_{k1}(t) \\ 0 & p_{k2}(t) \end{pmatrix}, \quad \mathcal{P}_0(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\varphi_k(t) = q_k(t) = q_0(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

$$p_{k1}(t) = \begin{cases} (\sqrt{k} + \sqrt[4]{k}) \sin kt & \text{for } t \in I_k, \\ \sqrt{k} \sin kt & \text{for } t \in [0,2\pi] \backslash I_k; \end{cases}$$

$$p_{k2}(t) = \begin{cases} -\alpha_k'(t) \cdot [1 - \alpha_k(t)]^{-1} & \text{for } t \in I_k, \\ 0 & \text{for } t \in [0,2\pi] \backslash I_k; \end{cases}$$

$$\beta_k(t) = \int_0^t [1 - \alpha_k(\tau)] \cdot p_{k1}(\tau)\, d\tau;$$

$$\alpha_k(t) = \begin{cases} 4\pi^{-1}(\sqrt[4]{k} + 1)^{-1} \sin kt & \text{for } t \in I_k, \\ 0 & \text{for } t \in [0,2\pi] \backslash I_k, \end{cases}$$

where $I_k = \cup_{m=0}^{k-1}]2mk^{-1}\pi, (2m+1)k^{-1}\pi[$. Let, moreover, for every $k \in \{0, 1, \dots\}$, $Y_k(t)$ be the fundamental matrix of the system (3_0) satisfying

$$Y_k(a) = E.$$

It can easily be shown that for every natural k we have

$$Y_0(t) = E, \quad Y_k(t) = \begin{pmatrix} 1 & \beta_k(t) \\ 0 & 1 - \alpha_k(t) \end{pmatrix} \quad \text{for } t \in [0, 2\pi]$$

and

$$\lim_{k \to +\infty} Y_k(t) = Y_0(t)$$

uniformly on $[0, 2\pi]$, since

$$\lim_{k \to +\infty} \|\alpha_k\|_c = \lim_{k \to +\infty} \|\beta_k\|_c = 0.$$

Note that

$$\lim_{k \to +\infty} \int_0^{2\pi} p_{k1}(t)\, dt = 2 \lim_{k \to +\infty} \sqrt[4]{k} = +\infty.$$

Therefore neither the conditions of Theorem 2_0 nor the results of [2] are fulfilled.

On the other hand, if we assume that

$$\Phi_k(t) = E - Y_k^{-1}(t) \quad \text{for } \ t \in [0, 2\pi] \ \ (k = 1, 2, \dots),$$

then the conditions of Theorems 2 and 2′ will be fulfilled, and if we put

$$\Phi_k(t) = \begin{pmatrix} \alpha_k(t) & \beta_k(t) \\ 0 & 0 \end{pmatrix} \quad \text{for } \ t \in [0, 2\pi] \ \ (k = 1, 2, \dots),$$

then in this case only the conditions of Theorem 2 will be fulfilled, since

$$\lim_{k \to +\infty} \sup \int_0^{2\pi} |p_{k2}(t)|\, dt = +\infty.$$

Proof of Theorem 1. The sufficiency follows from Theorem 2, since in view of (6),(8), and (12), condition (23) holds.

Let us show the necessity. Let $c_k \in \mathbb{R}^n$ $(k = 0, 1, \dots)$ be an arbitrary sequence satisfying (6) and let $e_j = (\delta_{ij})_{i=1}^n$ $(j = 1, \dots, n)$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

In view of (9), we may assume without loss of generality that for every natural k the problem (3),(4) has the unique solution x_k.

For any $k \in \{0, 1, \dots\}$ and $j \in \{1, \dots, m\}$ assume

$$y_{kj}(t) = x_k(t) - x_{kj}(t) \quad (t \in [a, b]),$$

where x_{oj} and x_{kj} $(k = 1, 2, \dots)$ are the unique solutions of (1) and (3) satisfying

$$l_0(x) = c_0 - e_j \quad \text{and} \quad l_k(x) = c_k - e_j,$$

respectively. Moreover, for every $k \in \{0, 1, \dots\}$ denote by $Y_k(t)$ the matrix-function whose columns are $y_{k1}(t), \dots, y_{kn}(t)$.

It can easily be shown that y_{oj} and y_{kj} satisfy (1_0) and (3_0), respectively, and

$$l_k(y_{kj}) = e_j \quad (j = 1, \dots, n;\ k = 0, 1, \dots). \tag{36}$$

If for some k and $\alpha_j \in \mathbb{R}$ $(j = 1, \dots, n)$

$$\sum_{j=1}^{n} \alpha_j y_{kj}(t) = 0 \quad (t \in [a, b]),$$

then, using (36), we have

$$\sum_{j=1}^{n} \alpha_j e_j = 0,$$

and therefore

$$\alpha_1 = \dots = \alpha_n = 0,$$

i.e., Y_0 and Y_k are the fundamental matrices of the systems (1_0) and (3_0), respectively. Hence, (5) implies

$$\lim_{k \to +\infty} Y_k^{-1}(t) = Y_0^{-1}(t) \quad \text{uniformly on } [a, b]. \tag{37}$$

Let, for every natural k and $t \in [a, b]$,

$$\Phi_k(t) = E - Y_0(t) Y_k^{-1}(t), \tag{38}$$

$$\varphi_k(t) = x_k(t) - x_0(t). \tag{39}$$

Let us show (10)–(14). Equations (11) and (12) are evident. Moreover, using the equality

$$[Y_k^{-1}(t)]' = -Y_k^{-1}(t) \mathcal{P}_k(t) \quad \text{for } t \in [a, b]\ (k = 1, 2, \dots),$$

it can be easily shown that

$$\mathcal{P}_k^*(t) = \mathcal{P}_0(t) Y_0(t) Y_k^{-1}(t) \quad \text{for } t \in [a, b]\ (k = 1, 2, \dots)$$

and

$$\int_a^t q_k^*(\tau)\, d\tau = Y_0(t) Y_k^{-1}(t) x_0(t) - Y_0(a) Y_k^{-1}(a) x_0(a) -$$

$$- \int_a^t \mathcal{P}_0(\tau) Y_0(\tau) Y_k^{-1}(\tau) x_0(\tau)\, d\tau \quad \text{for } t \in [a, b]\ (k = 1, 2, \dots).$$

Therefore, according to (37) the conditions (10),(13), and (14) are fulfilled uniformly on $[a, b]$. This completes the proof. $\square$

The proof of Theorem 1$'$ is analogous. We note only that Φ_k and φ_k are defined as above.

The behavior at $k \to +\infty$ of the solution of the Cauchy problem ($l_k(x) = x(t_0)$, $t_0 \in [a, b]$) and of the Cauchy-Nicoletti problem ($l_k(x) = (x_i(t_i))_{i=1}^n$, $t_i \in [a, b]$) is considered in [3-5]. Moreover, in [6] the necessary conditions for the stability of the Cauchy problem are investigated.

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