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ON SOME ENTIRE MODULAR FORMS OF WEIGHTS 5 AND 6 FOR THE CONGRUENCE GROUP $\Gamma_0(4N)$

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Abstract. Two entire modular forms of weight 5 and two of weight 6 for the congruence subgroup $\Gamma_0(4N)$ are constructed, which will be useful for revealing the arithmetical sense of additional terms in formulas for the number of representations of positive integers by quadratic forms in 10 and 12 variables.

In this paper N, a, k, n, r, s, t denote positive integers; u are odd positive integers; H, c, g, h, j, m, α, β, γ, δ, ξ, η are integers; A, B, C, D are complex numbers, and z, τ ($\operatorname{Im}\tau > 0$) are complex variables. Further, $(\frac{h}{u})$ is the generalized Jacobi symbol, $\binom{n}{t}$ a binomial coefficient, $\varphi(k)$ Euler's function, and $e(z) = \exp 2\pi iz$, $\eta(\gamma) = 1$ if $\gamma \geq 0$ and $\eta(\gamma) = -1$ if $\gamma < 0$.

Let

$$\Gamma = \Big\{\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big|\alpha\delta-\beta\gamma=1\Big\},$$
$$\Gamma_0(4N) = \Big\{\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\in\Gamma\Big|\gamma\equiv 0\,(\operatorname{mod}\,4N)\Big\}.$$

Definition. We shall say that a function F defined on $\mathcal{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$ is an entire modular form of weight r and character $\chi(\delta)$ for the congruence subgroup $\Gamma_0(4N)$ if

1) F is regular on $\mathcal{H}$,
2) for all substitutions from $\Gamma_0(4N)$ and all $\tau \in \mathcal{H}$

$$F\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) = \chi(\delta)(\gamma\tau+\delta)^r F(\tau),$$

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3) in the neighborhood of the point $\tau = i\infty$

$$F(\tau) = \sum_{m=0}^{\infty} A_m e(m\tau),$$

4) for all substitutions from Γ, in the neighborhood of each rational point $\tau = -\frac{\delta}{\gamma}$ $(\gamma \neq 0,\ (\gamma, \delta) = 1)$

$$(\gamma\tau + \delta)^r F(\tau) = \sum_{m=0}^{\infty} A'_m e\Big(\frac{n}{4N}\,\frac{\alpha\tau + \beta}{\gamma\tau + \delta}\Big).$$

For $\eta \neq 0,\ \xi,\ g,\ h,\ N$ with $\xi g + \eta h + \xi\eta N \equiv 0 \pmod 2$, put

$$S_{gh}\left(\begin{matrix}\xi\\ \eta\end{matrix}\ ;c,N\right) = \sum_{\substack{m \bmod N|\eta| \\ m \equiv c \pmod N}} (-1)^{h(m-c)/N} e\Big(\frac{\xi}{2N\eta}\big(m + \frac{g}{2}\big)^2\Big).$$

It is known ([2] p. 323, formula (2.4)–(2.6)) that

$$\begin{gathered} S_{g+2j,h}\left(\begin{matrix}\xi\\ \eta\end{matrix}\ ;c,N\right) = S_{gh}\left(\begin{matrix}\xi\\ \eta\end{matrix}\ ;c+j,N\right), \\ S_{g,h+2j}\left(\begin{matrix}\xi\\ \eta\end{matrix}\ ;c,N\right) = S_{gh}\left(\begin{matrix}\xi\\ \eta\end{matrix}\ ;c,N\right), \\ S_{gh}\left(\begin{matrix}\xi\\ \eta\end{matrix}\ ;c+N_j,N\right) = (-1)^{hj} S_{gh}\left(\begin{matrix}\xi\\ \eta\end{matrix}\ ;c,N\right). \end{gathered} \tag{1}$$

Let

$$\vartheta_{gh}(z|\tau; c, N) = \\ = \sum_{m \equiv c \pmod N} (-1)^{h(m-c)/N} e\big(\frac{1}{2N}\big(m + \frac{g}{2}\big)^2 \tau\big) e\big(\big(m + \frac{g}{2}\big)z\big), \tag{2}$$

hence

$$\frac{\partial^n}{\partial z^n} \vartheta_{gh}(z|\tau; c, N) = (\pi i)^n \sum_{m \equiv c \pmod N} (-1)^{h(m-c)/N} (2m+g)^n \times \\ \times e\Big(\frac{1}{2N}\big(m + \frac{g}{2}\big)^2 \tau\Big) e\big(\big(m + \frac{g}{2}\big)z\big). \tag{3}$$

Put

$$\begin{gathered} \vartheta_{gh}^{(n)}(\tau; c, N) = \frac{\partial^n}{\partial z^n} \vartheta_{gh}(z|\tau; c, N)\Big|_{z=0}, \\ \vartheta_{gh}^{(0)}(\tau; c, N) = \vartheta_{gh}(\tau; c, N) = \vartheta_{gh}(0|\tau; c, N). \end{gathered} \tag{4}$$

It is known ([2], p.318, formula (1.3); p. 321, formula (1.12); p.327, formulas (3.9), (3.5), (3.3), (3.7); p.324, formula (2.16); p.327, formulas (3.10), (3.11)) that

$$\vartheta_{g,h+2j}(z|\tau;c,N)=\vartheta_{gh}(z|\tau;c,N); \tag{5}$$

$$\begin{aligned}\vartheta_{gh}\big(\frac{z}{\tau}\big|-\frac{1}{\tau};c,N\big)=\big(-\frac{i\tau}{N}\big)^{1/2}e\big(\frac{Nz^2}{2\tau}\big)\times\\ \times\sum_{H \bmod N} e\Big(-\frac{1}{N}\big(c+\frac{g}{2}\big)\big(H+\frac{h}{2}\big)\Big)\vartheta_{hg}(z|\tau;H,N);\end{aligned} \tag{6}$$

$$\begin{aligned}\vartheta_{gh}\Big(\frac{z}{\gamma\tau+\delta}\Big|\frac{\alpha\tau+\beta}{\gamma\tau+\delta};c,N\Big)=\Big(\frac{-i(\gamma\tau+\delta)\operatorname{sgn}\gamma}{N|\gamma|}\Big)^{\frac{1}{2}}e\Big(\frac{N\gamma z^2}{2(\gamma\tau+\delta)}\Big)\times\\ \times\sum_{H \bmod N}\varphi_{g'gh}(c,H;N)\vartheta_{g'h'}(z|\tau;H,N)\quad(\gamma\neq 0),\end{aligned} \tag{7}$$

where

$$g'=\alpha g+\gamma h+\alpha\gamma N,\quad h'=\beta g+\delta h+\beta\delta N, \tag{8}$$

$$\begin{aligned}\varphi_{g'gh}(c,H;N)=e\Big(-\frac{\beta\delta}{2N}\big(H+\frac{g'}{2}\big)^2\Big)e\Big(\frac{\beta}{N}\big(c+\frac{g}{2}\big)\big(H+\frac{g'}{2}\big)\Big)\times\\ \times S_{g-\delta g',h+\beta g'}\Big(\begin{matrix}\alpha\\ \gamma\end{matrix};c-\delta H,N\Big);\end{aligned} \tag{9}$$

$$\begin{aligned}\vartheta_{gh}(z|\tau+\beta;c,N)=e\Big(\frac{\beta}{2N}\big(c+\frac{g}{2}\big)^2\Big)\vartheta_{g,h+\beta g+\beta N}(z|\tau;c,N),\\ \vartheta_{gh}(-z|\tau-\beta;c,N)=\\ =e\Big(-\frac{\beta}{2N}\big(c+\frac{g}{2}\big)^2\Big)\vartheta_{-g,-h+\beta g-\beta N}(z|\tau;-c,N).\end{aligned} \tag{10}$$

From (5) and (10), according to the notations (4), it follows that

$$\begin{aligned}\vartheta_{g,h+2j}(\tau;c,N)=\vartheta_{gh}(\tau;c,N),\\ \vartheta^{(n)}_{g,h+2j}(\tau;c,N)=\vartheta^{(n)}_{gh}(\tau;c,N);\end{aligned} \tag{11}$$

$$\begin{aligned}\vartheta^{(n)}_{gh}(\tau+\beta;c,N)=e\Big(\frac{\beta}{2N}\big(c+\frac{g}{2}\big)^2\Big)\vartheta^{(n)}_{g,h+\beta g+\beta N}(\tau;c,N),\\ \vartheta^{(n)}_{gh}(\tau-\beta;c,N)=\\ =(-1)^n e\Big(-\frac{\beta}{2N}\big(c+\frac{g}{2}\big)^2\Big)\vartheta^{(n)}_{-g,-h+\beta g-\beta N}(\tau;-c,N).\end{aligned} \tag{12}$$

From (2) and (3), according to the notations (4), in particular, it follows that

$$\vartheta_{gh}(\tau;0,N)=\sum_{m=-\infty}^{\infty}(-1)^{hm}e\Big(\frac{1}{2N}\big(Nm+\frac{g}{2}\big)^2\tau\Big),$$

$$\vartheta_{gh}^{(n)}(\tau;0,N)= \tag{13}$$

$$=(\pi i)^n\sum_{m=-\infty}^{\infty}(-1)^{hm}(2Nm+g)^n e\Big(\frac{1}{2N}\big(Nm+\frac{g}{2}\big)^2\tau\Big).$$

1.

Lemma 1. *For* $n\geq 0$

$$\vartheta_{gh}^{(n)}\big(-\frac{1}{\tau};c,N\big)=(Ni)^n\big(-\frac{i\tau}{N}\big)^{(2n+1)/2}\times$$

$$\times\sum_{H\,\mathrm{mod}\,N}e\Big(-\frac{1}{N}\big(c+\frac{g}{2}\big)\big(H+\frac{h}{2}\big)\Big)\times$$

$$\times\Big\{\vartheta_{hg}^{(n)}(\tau;H,N)+\sum_{t=1}^{n}\binom{n}{t}\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}\cdot\vartheta_{hg}^{(n-t)}(\tau;H,N)\Big\},$$

where

$$A_{tk}\Big|_{z=0}=\begin{cases}(2k)!\big(\frac{N\pi i}{\tau}\big)^k & \textit{if}\ \ t=2k\\ 0 \quad \textit{if}\ \ t\neq 2k\end{cases}$$

$$(t=1,2,\dots,n;\ k=1,2,\dots,t). \tag{1.1}$$

Proof. From (6), by Leibniz's formula, we obtain

$$\frac{\partial^n}{\partial z^n}\vartheta_{gh}\big(\frac{z}{\tau}\big|-\frac{1}{\tau};c,N\big)=\tau^n\big(\frac{-i\tau}{N}\big)^{1/2}\frac{\partial^n}{\partial z^n}\Big\{e\big(\frac{Nz^2}{2\tau}\big)\times$$

$$\times\sum_{H\,\mathrm{mod}\,N}e\Big(-\frac{1}{N}\big(c+\frac{g}{2}\big)\big(H+\frac{h}{2}\big)\Big)\vartheta_{hg}(z|\tau;H,N)\Big\}=$$

$$=(Ni)^n\big(\frac{-i\tau}{N}\big)^{(2n+1)/2}\times$$

$$\times\sum_{H\,\mathrm{mod}\,N}e\Big(-\frac{1}{N}\big(c+\frac{g}{2}\big)\big(H+\frac{h}{2}\big)\Big)\Big\{e\big(\frac{Nz^2}{2\tau}\big)\frac{\partial^n}{\partial z^n}\vartheta_{hg}(z|\tau;H,N)\Big\}+$$

$$+\sum_{t=1}^{n}\binom{n}{t}\frac{\partial^n}{\partial z^n}e\big(\frac{Nz^2}{2\tau}\big)\frac{\partial^{n-t}}{\partial z^{n-t}}\vartheta_{hg}(z|\tau;H,N)\Big\}. \tag{1.2}$$

According to formulae (a) and (b) of [1], p. 37[1]

$$\frac{\partial^t}{\partial z^t} e\Big(\frac{Nz^2}{2\tau}\Big) = A_{t1} e\Big(\frac{Nz^2}{2\tau}\Big) + \frac{A_{t2}}{2!} e\Big(\frac{Nz^2}{2\tau}\Big) + \cdots + \frac{A_{tt}}{t!}\, e\Big(\frac{Nz^2}{2\tau}\Big)$$
$$(t = 1, 2, \dots, n),$$

where

$$\begin{aligned} A_{tk} = {} & \frac{\partial^t}{\partial z^t}\Big(\frac{N\pi i z^2}{\tau}\Big)^k - k\Big(\frac{N\pi i z^2}{\tau}\Big)\frac{\partial^t}{\partial z^t}\Big(\frac{N\pi i z^2}{\tau}\Big)^{k-1} + \\ & + \frac{k(k-1)}{2!}\Big(\frac{N\pi i z^2}{\tau}\Big)^2 \frac{\partial^t}{\partial z^t}\Big(\frac{N\pi i z^2}{\tau}\Big)^{k-2} + \\ & + \cdots + (-1)^{k-1} k \Big(\frac{N\pi i z^2}{\tau}\Big)^{k-1} \frac{\partial^t}{\partial z^t}\Big(\frac{N\pi i z^2}{\tau}\Big) \quad (k = 1, 2, \dots, t), \end{aligned}$$

hence

$$\frac{\partial^t}{\partial z^t} e\Big(\frac{Nz^2}{2\tau}\Big)\Big|_{z=0} = \sum_{k=1}^{t} \frac{A_{tk}}{k!}\Big|_{z=0} \quad (t = 1, 2, \dots, n) \tag{1.3}$$

and

$$A_{tk}\big|_{z=0} = \frac{\partial^t}{\partial z^t}\Big(\frac{N\pi i z^2}{\tau}\Big)^k\Big|_{z=0} \quad (k = 1, 2, \dots, t). \tag{1.4}$$

Thus, in view of notations (4), the lemma follows from (1.2)–(1.4). □

Lemma 2. *If $\gamma \neq 0$, then for $n \geq 0$*

$$\begin{aligned} \vartheta_{gh}^{(n)}\Big(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; c, N\Big) = {} & (N|\gamma| i \operatorname{sgn} \gamma)^n \Big(-i(\gamma\tau + \delta)\frac{\operatorname{sgn}\gamma}{N|\gamma|}\Big)^{(2n+1)/2} \times \\ & \times \sum_{H \bmod N} \varphi_{g'gh}(c, H; N)\Big\{\vartheta_{g'h'}^{(n)}(\tau; H, N) + \\ & + \sum_{t=1}^{n} \binom{n}{t} \sum_{k=1}^{t} \frac{A_{tk}}{k!}\Big|_{z=0} \cdot \vartheta_{g'h'}^{(n-t)}(\tau; H, N)\Big\}, \end{aligned}$$

where $g'\, h'$ and $\varphi_{g'gh}(c, H; N)$ are defined by the formulas (8) *and* (9),

$$A_{tk}\Big|_{z=0} = \begin{cases} (2k)!\big(\frac{N\gamma\pi i}{\gamma\tau+\delta}\big)^k & \text{if } t = 2k \\ 0 & \text{if } t \neq 2k \end{cases}$$
$$(t = 1, 2, \dots, n;\ k = 1, 2, \dots, t).$$

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[1]Page 75 in the Russian version of [1] published in 1933.

Proof. From (7), according to Leibnitz's formula, we obtain

$$\frac{\partial^n}{\partial z^n}\vartheta_{gh}\Big(\frac{z}{\gamma\tau+\delta}\Big|\frac{\alpha\tau+\beta}{\gamma\tau+\delta};c,N\Big)=(\gamma\tau+\delta)^n\big(-i(\gamma\tau+\delta)\frac{\operatorname{sgn}\gamma}{N|\gamma|}\big)^{1/2}\times$$
$$\times\frac{\partial^n}{\partial z^n}\Big\{e\Big(\frac{N\gamma z^2}{2(\gamma\tau+\delta)}\Big)\sum_{H \bmod N}\varphi_{g'gh}(c,H;N)\vartheta_{g'h'}(z|\tau;H,N)\Big\}=$$
$$=(N|\gamma|i\operatorname{sgn}\gamma)^n\Big(-i(\gamma\tau+\delta)\frac{\operatorname{sgn}\gamma}{N|\gamma|}\Big)^{(2n+1)/2}\times$$
$$\times\sum_{H \bmod N}\varphi_{g'gh}(c,H;N)\Big\{e\Big(\frac{N\gamma z^2}{2(\gamma\tau+\delta)}\Big)\frac{\partial^n}{\partial z^n}\vartheta_{g'h'}(z|\tau;H,N)+$$
$$+\sum_{t=1}^{n}\binom{n}{t}\frac{\partial^t}{\partial z^t}e\Big(\frac{N\gamma z^2}{2(\gamma\tau+\delta)}\Big)\frac{\partial^{n-t}}{\partial z^{n-t}}\vartheta_{g'h'}(z|\tau;H,N)\Big\}. \tag{1.5}$$

As in Lemma 1, but with $e\big(\frac{N\gamma z^2}{2(\gamma\tau+\delta)}\big)$ instead of $e\big(\frac{Nz^2}{2\tau}\big)$, we have

$$\frac{\partial^t}{\partial z^t}\,e\Big(\frac{N\gamma z^2}{2(\gamma\tau+\delta)}\Big)\Big|_{z=0}=\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}\quad(t=1,2,\dots,n) \tag{1.6}$$

and

$$A_{tk}\Big|_{z=0}=\frac{\partial^t}{\partial z^t}\big(\frac{N\gamma\pi iz}{\gamma\tau+\delta}\big)^{2k}\Big|_{z=0}\quad(k=1,2,\dots,t). \tag{1.7}$$

Thus, according to the notations (4), the lemma follows from (1.5)-(1.7). □

Lemma 3. *If g is even, then for $n\geq 0$ and all substitutions from $\Gamma_0(4N)$ we have*

$$\vartheta^{(n)}_{gh}\big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N\big)=$$
$$=(\operatorname{sgn}\delta)^n\,i^{(2n+1)\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\,i^{(1-|\delta|)/2}\Big(\frac{2\beta N\operatorname{sgn}\delta}{|\delta|}\Big)\times$$
$$\times(\gamma\tau+\delta)^{(2n+1)/2}\,e\Big(-\frac{\alpha\gamma\delta^2h^2}{16N}\Big)\,e\Big(\frac{\beta\delta g^2}{4}\,\frac{\delta^{2\varphi(2N)-2}}{4N}\Big)\vartheta^{(n)}_{\alpha g,h}(\tau;0,2N).$$

Proof. 1) Let $\gamma\neq 0$. In [4] (p.18, formula (5.1)) it is shown that

$$S_{g0}\binom{\beta}{\delta};0,2N\Big)=$$
$$=e\Big(\frac{\beta\delta g^2}{4}\,\frac{\delta^{2\varphi(2n)-2}}{4N}\Big)i^{(1-|\delta|)/2}\Big(\frac{2\beta N\operatorname{sgn}\delta}{|\delta|}\Big)|\delta|^{1/2}. \tag{1.8}$$

Replacing $\alpha,\ \beta,\ \gamma,\ \delta,\ \tau,\ c,\ N$ by $\beta,\ -\alpha,\ \delta,\ -\gamma,\ \tau',\ 0,\ 2N$ in Lemma 2, we obtain

$$\vartheta_{gh}^{(n)}\big(\frac{\beta\tau'-\alpha}{\delta\tau'-\gamma};0,2N\big)=(2N|\delta|i\operatorname{sgn}\delta)^n\big(-i(\delta\tau'-\gamma)\frac{\operatorname{sgn}\delta}{2N|\delta|}\big)^{(2n+1)/2}\times$$
$$\times\sum_{H\bmod 2N}\varphi_{h'gh}(0,H;2N)\Big\{\vartheta_{h',\alpha g}^{(n)}(\tau';H,2N)+$$
$$+\sum_{t=1}^{n}\left(\begin{array}{c}n\\t\end{array}\right)\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}\cdot\vartheta_{h',\alpha g}^{(n-t)}(\tau';H,2N)\Big\},\tag{1.9}$$

where by (8),(9),(11) and (1)

$$h'=\beta g+\delta h+2\beta\delta N,\tag{1.10}$$
$$\varphi_{h'gh}(0,H;2N)=$$
$$=e\big(-\frac{\alpha\gamma h'^2}{16N}\big)e\Big(-\frac{\alpha g}{4N}\big(H+\frac{h'}{2}\big)\Big)S_{g0}\Big(\begin{array}{c}\beta\\\delta\end{array};0,2N\Big),\tag{1.11}$$
$$A_{tk}\Big|_{z=0}=\left\{\begin{array}{l}(2k)!\big(\frac{2N\delta\pi i}{\delta\tau'-\gamma}\big)^k\quad\text{if}\quad t=2k\\0\quad\text{if}\quad t\neq 2k\end{array}\right.$$
$$(k=1,2,\dots,t).\tag{1.12}$$

Writing $-\frac{1}{\tau}$ instead of τ' in (1.9) and (1.12), according to (1.11), we obtain

$$\vartheta_{gh}^{(n)}\big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N\big)=(2Ni)^n(|\delta|\operatorname{sgn}\delta)^n\Big(\frac{-i(-\frac{\delta}{\tau}-\gamma)\operatorname{sgn}\delta}{2N|\delta|}\Big)^{(2n+1)/2}\times$$
$$\times e\big(-\frac{\alpha\gamma h'^2}{16N}\big)e\Big(\frac{\beta\delta g^2}{4}\ \frac{\delta^{2\varphi(2N)-2}}{4N}\Big)i^{(1-|\delta|)/2}\Big(\frac{2\beta N\operatorname{sgn}\delta}{|\delta|}\Big)|\delta|^{1/2}\times$$
$$\times\sum_{H\bmod 2N}e\Big(-\frac{\alpha g}{4N}\big(H+\frac{h'}{2}\big)\Big)\Big\{\vartheta_{h',\alpha g}^{(n)}\big(-\frac{1}{\tau};H,2N\big)+$$
$$+\sum_{t=1}^{n}\left(\begin{array}{c}n\\t\end{array}\right)\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}\cdot\vartheta_{h',\alpha g}^{(n-t)}\big(-\frac{1}{\tau};H,2N\big)\Big\},\tag{1.13}$$

where

$$A_{tk}\Big|_{z=0}=\left\{\begin{array}{l}(2k)!\big(\frac{2N\delta\pi i\tau}{\gamma\tau+\delta}\big)^k\quad\text{if}\quad t=2k\\0\quad\text{if}\quad t\neq 2k\end{array}\right.\quad(k=1,2,\dots,t).$$

Writing αg, h', $-\frac{1}{\tau}$, 0, $2N$ instead of g, h, τ, c, N in Lemma 1, we obtain

$$\vartheta^{(n)}_{\alpha g,h'}(\tau;0,2N)=(2Ni)^n\Big(\frac{i}{2N\tau}\Big)^{(2n+1)/2}\sum_{H\bmod 2N}e\Big(-\frac{\alpha g}{4N}\big(H+\frac{h'}{2}\big)\Big)\times$$
$$\times\Big\{\vartheta^{(n)}_{h',\alpha g}\big(-\frac{1}{\tau};H,2N\big)+$$
$$+\sum_{t=1}^{n}\binom{n}{t}\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}\cdot\vartheta^{(n-t)}_{h',\alpha g}\big(-\frac{1}{\tau};H,2N\big)\Big\},\tag{1.14}$$

where

$$A_{tk}\Big|_{z=0}=\begin{cases}(2k)!(-2N\pi i\tau)^k & \text{if}\quad t=2k\\ 0\quad \text{if}\quad t\neq 2k\end{cases}\qquad (k=1,2,\dots,t).$$

From (1.14) it follows that

$$\sum_{H\bmod 2N}e\Big(-\frac{\alpha g}{4N}\big(H+\frac{h'}{2}\big)\Big)\Big\{\vartheta^{(n)}_{h',\alpha g}\big(-\frac{1}{\tau};H,2N\big)+$$
$$+\sum_{t=1}^{n}\binom{n}{t}\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}\cdot\vartheta^{(n-t)}_{h',\alpha g}\big(-\frac{1}{\tau};H,2N\big)\Big\}=$$
$$=(2\pi i)^{-n}(-2Ni\tau)^{(2n+1)/2}\vartheta^{(n)}_{\alpha g,h'}(\tau;0,2N).\tag{1.15}$$

From (1.13) and (1.15) we obtain

$$\vartheta^{(n)}_{gh}\big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N\big)=(-2Ni\tau)^{(2n+1)/2}(|\delta|\operatorname{sgn}\delta)^n\times$$
$$\times\Big(\frac{-i(-\frac{\delta}{\tau}-\gamma)\operatorname{sgn}\delta}{2N|\delta|}\Big)^{(2n+1)/2}i^{(1-|\delta|)/2}e\big(-\frac{\alpha\gamma h'^2}{16N}\big)\times$$
$$\times e\Big(\frac{\beta\delta g^2}{4}\frac{\delta^{2\varphi(2N)-2}}{4N}\Big)\Big(\frac{2\beta N\operatorname{sgn}\delta}{|\delta|}\Big)|\delta|^{1/2}\,\vartheta^{(n)}_{\alpha g,h'}(\tau;0,2N).\tag{1.16}$$

In [3] (p.85, formula (6.16) with $2N$ instead of N) it is shown that

$$\Big(\frac{-i(-\frac{\delta}{\tau}-\gamma)\operatorname{sgn}\delta}{2N|\delta|}\Big)^{1/2}(-2Ni\tau)^{1/2}=$$
$$=i^{\operatorname{sgn}\gamma(\operatorname{sgn}\delta-1)/2}\big(\frac{\gamma\tau+\delta}{|\delta|}\big)^{1/2}.\tag{1.17}$$

From (1.16) and (1.17) it follows that

$$\vartheta^{(n)}_{gh}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N\Big)=$$
$$=(|\delta|\operatorname{sgn}\delta)^n\big(i^{\operatorname{sgn}\gamma(\operatorname{sgn}\delta-1)}\frac{\gamma\tau+\delta}{|\delta|}\big)^{(2n+1)/2}i^{(1-|\delta|)/2}\times$$
$$\times e\big(-\frac{\alpha\gamma h^{'2}}{16N}\big)e\Big(\frac{\beta\delta g^2}{4}\frac{\delta^{2\varphi(2N)-2}}{4N}\Big)\Big(\frac{2\beta N\operatorname{sgn}\delta}{|\delta|}\Big)|\delta|^{1/2}\vartheta^{(n)}_{\alpha g,h'}(\tau;0,2N).$$

Thus, in view of (1.10) and (1.11), the lemma is proved for $\gamma\neq 0$.

2) Let $\gamma=0$. Then $\alpha=\delta=1$ or $\alpha=\delta=-1$ in (12). Putting $c=0$ in (12) and writing $2N$ instead of N, by (11), we obtain

$$\vartheta^{(n)}_{gh}(\tau+\beta;0,2N)=e\big(\frac{\beta g^2}{16N}\big)\vartheta^{(n)}_{gh}(\tau;0,2N),$$
$$\vartheta^{(n)}_{gh}(\tau-\beta;0,2N)=(-1)^n e\big(-\frac{\beta g^2}{16N}\big)\vartheta^{(n)}_{-g,h}(\tau;0,2N),$$

i.e., the lemma is also proved for $\gamma=0$. □

Lemma 4. *If $\gamma\neq 0$, then for $n\geq 0$*

$$(\gamma\tau+\delta)^{(2n+1)/2}\vartheta^{(n)}_{gh}(\tau;0,2N)=$$
$$=e\big((2n+1)\operatorname{sgn}\gamma/8\big)(2N|\gamma|)^{-1/2}(-i\operatorname{sgn}\gamma)^n\times$$
$$\times\sum_{H\bmod 2N}\varphi_{g'gh}(0,H;2N)\Big\{\vartheta^{(n)}_{g'h'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H,2N\Big)+$$
$$+\sum_{t=1}^{n}\binom{n}{t}\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}\cdot\vartheta^{(n-t)}_{g'h'}\big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H,2N\big)\Big\},$$

where

$$A_{tk}\Big|_{z=0}=\begin{cases}(2k)!(-2N\gamma\pi i(\gamma\tau+\delta))^k & \textit{if}\quad t=2k\\ 0\quad \textit{if}\quad t\neq 2k\end{cases}$$
$$(t=1,2,\dots;\ k=1,2,\dots,t).\tag{1.18}$$

Remark. From (1.18) it follows that

$$\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}=\begin{cases}\frac{A_{t,t/2}}{(t/2)!}\big|_{z=0}\quad\text{if}\quad 2|t\\ 0\quad\text{if}\quad 2\dagger t.\end{cases}\tag{1.19}$$

Proof. Replacing $\alpha,\ \beta,\ \gamma,\ \delta,\ \tau,\ c,\ N$ by $\delta,\ -\beta,\ -\gamma,\ \alpha,\ \tau',\ 0,\ 2N$ in Lemma 2, we obtain

$$\vartheta_{gh}^{(n)}\Big(\frac{\delta\tau'-\beta}{-\gamma\tau'+\alpha};0,2N\Big)=$$
$$=(-2N|\gamma|i\operatorname{sgn}\gamma)^n\Big(i(-\gamma\tau'+\alpha)\frac{\operatorname{sgn}\gamma}{2N|\gamma|}\Big)^{(2n+1)/2}\times$$
$$\times\sum_{H\,\mathrm{mod}\,2N}\varphi_{g'gh}(0,H;2N)\Big\{\vartheta_{g'h'}^{(n)}(\tau';H,2N)+$$
$$+\sum_{t=1}^{n}\binom{n}{t}\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}\cdot\vartheta_{g'h'}^{(n-t)}(\tau';H,2N)\Big\}, \tag{1.20}$$

where

$$g'=\delta g-\gamma h-2\gamma\delta N,\quad h'=-\beta g+\alpha h-2\alpha\beta N, \tag{1.21}$$
$$\varphi_{g'gh}(0,H;2N)=e\Big(-\frac{\alpha\beta}{4N}\big(H+\frac{g'}{2}\big)^2\Big)e\Big(-\frac{\beta g}{4N}\big(H+\frac{g'}{2}\big)\Big)\times$$
$$\times S_{g-\alpha g',h-\beta g'}\Big(\begin{matrix}\delta\\-\gamma\end{matrix};-\alpha H,2N\Big), \tag{1.22}$$
$$A_{tk}\Big|_{z=0}=\begin{cases}(2k)!\big(\frac{2N\gamma\pi i}{\gamma\tau'-\alpha}\big)^k & \text{if } t=2k,\\ 0 \quad \text{if } t\neq 2k.\end{cases} \tag{1.23}$$

In [3, p.87, formula (6.23)] (with $2N$ instead of N) it is shown that

$$\Big(\frac{i\operatorname{sgn}\gamma}{2N|\gamma|(\gamma\tau+\delta)}\Big)^{1/2}=\frac{e(\operatorname{sgn}\gamma/8)}{(2N|\gamma|)^{1/2}(\gamma\tau+\delta)^{1/2}}. \tag{1.24}$$

Taking $\frac{\alpha\tau+\beta}{\gamma\tau+\delta}$ instead of τ' in (1.20) and (1.23) and using (1.24), we complete the proof of the lemma. □

2.

Lemma 5. *For a given N let*

$$\Psi_1(\tau)=\Psi_1(\tau;g_1,g_2,h_1,h_2,c_1,c_2,N_1,N_2)=$$
$$=\frac{1}{N_1}\vartheta'''_{g_1h_1}(\tau;c_1,2N_1)\vartheta'_{g_2h_2}(\tau;c_2,2N_2)-$$
$$-\frac{1}{N_2}\vartheta'''_{g_2h_2}(\tau;c_2,2N_2)\vartheta'_{g_1h_1}(\tau;c_1,2N_1) \tag{2.1}$$

and

$$\Psi_2(\tau)=\Psi_2(\tau;g_1,g_2,h_1,h_2,c_1,c_2,N_1,N_2)=$$

$$= \frac{1}{N_1^2}\vartheta^{(4)}_{g_1h_1}(\tau;c_1,2N_1)\vartheta_{g_2h_2}(\tau;c_2,2N_2)+$$
$$+\frac{1}{N_2^2}\vartheta^{(4)}_{g_2h_2}(\tau;c_2,2N_2)\vartheta_{g_1h_1}(\tau;c_1,2N_1)-$$
$$-\frac{6}{N_1N_2}\vartheta''_{g_1h_1}(\tau;c_1,2N_1)\vartheta''_{g_2h_2}(\tau;c_2,2N_2), \tag{2.2}$$

where

$$2|g_1, 2|g_2, N_1|N, N_2|N, 4|N\Big(\frac{h_1}{N_1}+\frac{h_2}{N_2}\Big). \tag{2.3}$$

For all substitutions from Γ *in the neighborhood of each rational point* $\tau = -\frac{\delta}{\gamma}$ $(\gamma\neq 0, (\gamma,\delta)=1)$*, we then have*

$$(\gamma\tau+\delta)^5\Psi_j(\tau;g_1,g_2,h_1,h_2,0,0,N_1,N_2)=$$
$$=\sum_{n=0}^{\infty}C_n^{(j)}e\Big(\frac{n}{4N}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\quad (j=1,2). \tag{2.4}$$

Proof. I. From Lemma 4 for $n=3$ (with g_1, h_1, N_1, g_1', h_1', H_1 instead of g, h, N, g', h', H) and $n=1$ (with g_2, h_2, N_2, g_2', h_2', H_2 instead of g,h, N, g', h', H), according to (1.19), it follows that

$$\frac{1}{N_1}(\gamma\tau+\delta)^5\vartheta'''_{g_1h_1}(\tau;0,2N_1)\vartheta'_{g_2h_2}(\tau;0,2N_2)=$$
$$=\frac{1}{N_1}e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)(2|\gamma|(N_1N_2)^{1/2})^{-1}\times$$
$$\times\sum_{H_1 \bmod 2N_1}\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\Big\{\vartheta'''_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)+$$
$$+\frac{3\cdot 2}{2!}A_{21}\Big|_{z=0}\cdot\vartheta'_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big\}\times$$
$$\times\sum_{H_2 \bmod 2N_2}\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\vartheta'_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)=$$
$$=e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)\big(2|\gamma|(N_1N_2)^{1/2}\big)^{-1}\times$$
$$\times\sum_{\substack{H_1 \bmod 2N_1\\ H_2 \bmod 2N_2}}\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\times$$
$$\times\Big\{\frac{1}{N_1}\vartheta'''_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta'_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)-$$
$$-12\gamma\pi i(\gamma\tau+\delta)\vartheta'_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\times$$

$$\times\vartheta'_{g'_2h'_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\Big\}. \tag{2.5}$$

Replacing N_1, g_1, h_1, H_1, g'_1, h'_1 by N_2, g_2, h_2, H_2, g'_2, h'_2 in (2.5) and vice versa, we obtain

$$\begin{aligned}
&\frac{1}{N_2}(\gamma\tau+\delta)^5\vartheta'''_{g'_2h'_2}(\tau;0,2N_2)\vartheta'_{g'_1h'_1}(\tau;0,2N_1)=\\
&=e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)(2|\gamma|(N_1N_2)^{1/2})^{-1}\times\\
&\times\sum_{\substack{H_2\,\mathrm{mod}\,2N_2\\ H_1\,\mathrm{mod}\,2N_1}}\varphi_{g'_2g_2h_2}(0,H_2;2N_2)\varphi_{g'_1g_1h_1}(\tau;0,H_1;2N_1)\times\\
&\times\Big\{\frac{1}{N_2}\vartheta'''_{g'_2h'_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\,\vartheta'_{g'_1h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)-\\
&-12\gamma\pi i(\gamma\tau+\delta)\vartheta'_{g'_2h'_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\times\\
&\times\vartheta'_{g'_1h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big\}.
\end{aligned}\tag{2.6}$$

Subtracting (2.6) from (2.5), according to (2.1), we obtain

$$\begin{aligned}
&(\gamma\tau+\delta)^5\Psi_1(\tau;g_1,g_2,h_1,h_2,0,0,N_1,N_2)=\\
&=e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)(2|\gamma|(N_1N_2)^{1/2})^{-1}\times\\
&\times\sum_{\substack{H_1\,\mathrm{mod}\,2N_1\\ H_2\,\mathrm{mod}\,2N_2}}\varphi_{g'_1g_1h_1}(0,H_1;2N_1)\varphi_{g'_2g_2h_2}(0,H_2;2N_2)\times\\
&\times\Psi_1\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g'_1,g'_2,h'_1,h'_2,H_1,H_2,N_1,N_2\Big).
\end{aligned}\tag{2.7}$$

In (2.7) let γ be even. Then, by (1.21) and (2.3), g'_1 and g'_2 are also even. Therefore, according to (3) and the notations (4), we have

$$\begin{aligned}
&\vartheta'''_{g'_rh'_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_r,2N_r\Big)\vartheta'_{g'_sh'_s}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_s,2N_s\Big)=\\
&=\sum_{n_1=0}^{\infty}B_{n_1}e\Big(\frac{Nn_1/N_r}{4N}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\sum_{n_2=0}^{\infty}B_{n_2}e\Big(\frac{Nn_2/N_s}{4N}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)=\\
&=\sum_{n=0}^{\infty}B_ne\Big(\frac{n}{4N}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\\
&\text{for}\quad r=1,\ s=2\ \text{ and }\ r=2,\ s=1.
\end{aligned}\tag{2.8}$$

Hence, for even γ, (2.4) follows from (2.1), (2.7), and (2.8) if $j=1$.

Now let γ in (2.7) be odd. If h_1 and h_2 are both even, then by (1.21) and (2.3), g_1' and g_2' are also even, and we obtain the same result. But if h_r is odd, then by (1.21) g_r' will also be odd, and in (3) we have

$$\big(m+\frac{g_r'}{2}\big)^2=\big(m+\frac{1}{2}(g_r'-1)\big)^2+\big(m+\frac{1}{2}(g_r'-1)\big)+\frac{1}{4},$$

hence

$$\begin{aligned}\vartheta'''_{g_r'h_r'}\big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_r,2N_r\big)&=-\pi^3ie\big(\frac{1}{16N_r}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\big)\times\\ \times\sum_{m\equiv H_r \pmod{2N_r}}&(-1)^{h_r'(m-H_r)/2N_r}(2m+g_r')^3\times\\ \times e\Big\{\frac{1}{4N_r}\big((m+\frac{1}{2}(g_r'-1))^2&+(m+\frac{1}{2}(g_r'-1))\big)\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big\}=\\ &=e\big(\frac{h_r}{16N_r}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\big)\sum_{n_1=0}^{\infty}B'_{n_1}e\big(\frac{n_1}{4N_r}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\big),\end{aligned}$$

since (5) implies that $h_r=1$. Analogously,

$$\vartheta'_{g_s'h_s'}\big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_s,2N_s\big)=e\big(\frac{h_s}{16N_s}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\big)\sum_{n_2=0}^{\infty}B'_{n_2}e\big(\frac{n_2}{4N_s}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\big),$$

which implies that $h_s=1$ if h_s is odd and $h_s=0$ if h_s is even. Thus, if among h_1 and h_2 at least one is odd, then we have, for $r=1$, $s=2$ and $r=2$, $s=1$,

$$\begin{aligned}\vartheta'''_{g_r'h_r'}\big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_r,2N_r\big)\vartheta'_{g_s'h_s'}\big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_s,2N_s\big)=\\ =e\big(\frac{N/4(h_r/N_r+h_s/N_s)}{4N}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\big)\sum_{n_1=0}^{\infty}B'_{n_1}e\big(\frac{n_1}{4N_r}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\big)\times\\ \times\sum_{n_2=0}^{\infty}B'_{n_2}e\big(\frac{n_2}{4N_s}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\big)=\sum_{n=0}^{\infty}B'_{n}e\big(\frac{n}{4N}\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\big),\end{aligned}\tag{2.9}$$

since, by (2.3), $\frac{N}{4}(\frac{h_r}{N_r}+\frac{h_s}{N_s})$ is an integer. Hence for odd γ (2.4) follows from (2.1),(2.7), and (2.9) if $j=1$.

II. From Lemma 4 for $n=4$ and $n=0$, as in I, it follows that

$$\begin{aligned}\frac{1}{N_2^2}(\gamma\tau+\delta)^5\vartheta^{(4)}_{g_1h_1}(\tau;0,2N_1)\vartheta_{g_2h_2}(\tau;0,2N_2)=\\ =\frac{1}{N_1^2}e\big(\frac{5}{4}\operatorname{sgn}\gamma\big)(2|\gamma|(N_1N_2)^{1/2})^{-1}\times\end{aligned}$$

$$\times \sum_{H_1 \bmod 2N_1} \varphi_{g_1' g_1 h_1}(0, H_1; 2N_1) \Big\{ \vartheta^{(4)}_{g_1' h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_1, 2N_1\Big) +$$

$$+\frac{4\cdot 3}{2!} A_{21}\Big|_{z=0} \cdot \vartheta''_{g_1' h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_1, 2N_1\Big) +$$

$$+\frac{A_{42}}{2!}\Big|_{z=0} \cdot \vartheta_{g_1' h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_1, 2N_1\Big)\Big\} \times$$

$$\times \sum_{H_2 \bmod 2N_2} \varphi_{g_2' g_2 h_2}(0, H_2; 2N_2) \vartheta_{g_2' h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_2, 2N_2\Big) =$$

$$= e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)\big(2|\gamma|(N_1N_2)^{1/2}\big)^{-1} \times$$

$$\times \sum_{\substack{H_1 \bmod 2N_1 \\ H_2 \bmod 2N_2}} \varphi_{g_1' g_1 h_1}(0, H_1; 2N_1) \varphi_{g_2' g_2 h_2}(0, H_2; 2N_2) \times$$

$$\times \Big\{ \frac{1}{N_1^2} \vartheta^{(4)}_{g_1' h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_1, 2N_1\Big) \times$$

$$\times \vartheta_{g_2' h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_2, 2N_2\Big) - \frac{24\gamma\pi i}{N_1}(\gamma\tau+\delta)\vartheta''_{g_1' h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_1, 2N_1\Big) \times$$

$$\vartheta_{g_2' h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_2, 2N_2\Big) - 48\gamma^2\pi^2(\gamma\tau+\delta)^2 \vartheta_{g_1' h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_1, 2N_1\Big) \times$$

$$\times \vartheta_{g_2' h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_2, 2N_2\Big)\Big\}. \tag{2.10}$$

Replacing $N_1, g_1, h_1, H_1, g_1', h_1'$ by $N_2, g_2, H_2, g_2', h_2'$ in (2.10) and vice versa, we obtain

$$\frac{1}{N_1^2}(\gamma\tau+\delta)^5 \vartheta^{(4)}_{g_2 h_2}(\tau; 0, 2N_2)\vartheta_{g_1 h_1}(\tau; 0, 2N_1) =$$

$$= e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)(2|\gamma|(N_1N_2)^{1/2})^{-1} \times$$

$$\times \sum_{\substack{H_2 \bmod 2N_2 \\ H_1 \bmod 2N_1}} \varphi_{g_2' g_2 h_2}(0, H_2; 2N_2) \varphi_{g_1' g_1 h_1}(0, H_1; 2N_1) \times$$

$$\times \Big\{ \frac{1}{N_2^2} \vartheta^{(4)}_{g_2' h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_2, 2N_2\Big) \times$$

$$\times \vartheta_{g_1' h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_1, 2N_1\Big) - \frac{24\gamma\pi i}{N_2}(\gamma\tau+\delta)\vartheta''_{g_2' h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_2, 2N_2\Big) \times$$

$$\times \vartheta_{g_1' h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_1, 2N_1\Big) - 48\gamma^2\pi^2(\gamma\tau+\delta)^2 \vartheta_{g_2' h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_2, 2N_2\Big) \times$$

$$\times\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big\}. \tag{2.11}$$

Analogously, from Lemma 4 for $n=2$ it follows that

$$\begin{aligned}
&\frac{6}{N_1N_2}(\gamma\tau+\delta)^5\vartheta''_{g_1h_1}(\tau;0,2N_1)\vartheta''_{g_2h_2}(\tau;0,2N_2)=\\
&=e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)(2|\gamma|(N_1N_2)^{1/2})^{-1}\times\\
&\times\sum_{\substack{H_1 \bmod 2N_1\\ H_2 \bmod 2N_2}}\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\times\\
&\times\Big\{\frac{6}{N_1N_2}\vartheta''_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta''_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)-\\
&-\frac{24\gamma\pi i}{N_2}(\gamma\tau+\delta)\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)-\\
&-\frac{24\gamma\pi i}{N_1}(\gamma\tau+\delta)\vartheta''_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta''_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)-\\
&-96\gamma^2\pi^2(\gamma\tau+\delta)^2\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\times\\
&\times\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\Big\}.
\end{aligned} \tag{2.12}$$

Subtracting (2.12) from the sum of (2.10) and (2.11), according to (2.2), we obtain

$$\begin{aligned}
&(\gamma\tau+\delta)^5\Psi_2(\tau;g_1,g_2,h_1,h_2,0,0,N_1,N_2)=\\
&=e\Big(\frac{5}{4}\operatorname{sgn}\gamma\Big)\big(2|\gamma|(N_1N_2)^{1/2}\big)^{-1}\times\\
&\times\sum_{\substack{H_1 \bmod 2N_1\\ H_2 \bmod 2N_2}}\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\times\\
&\times\Psi_2\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_1',g_2',h_1',h_2',H_1,H_2,N_1,N_2\Big).
\end{aligned} \tag{2.13}$$

Further, reasoning as in I, from (2.2) and (2.13) we obtain (2.4) if $j=2$. □

Theorem 1. *For a given N the functions $\Psi_1(\tau)$ and $\Psi_2(\tau)$ with $c_1=c_2=0$ are entire modular forms of weight 10 and character $\chi(\delta)=\ sgn\,\delta(\frac{-\Delta}{|\delta|})$ (Δ is the determinant of an arbitrary positive quadratic form in 5 variables)* *for the congruence group* $\Gamma_0(4N)$ *if the following conditions hold:*

$$1)\quad 2|g_1, 2|g_2, N_1|N, N_2|N, \tag{2.14}$$

$$2)\quad 4\Big|N\Big(\frac{h_1^2}{N_1}+\frac{h_2^2}{N_2}\Big), 4\Big|\frac{g_1^2}{4N_1}+\frac{g_2^2}{4N_2}, \tag{2.15}$$

3) *for all* α *and* δ *with* $\alpha\equiv\delta\equiv 1 \pmod{4N}$

$$\Big(\frac{N_1N_2}{|\delta|}\Big)\Psi_j(\tau;\alpha g_1,\alpha g_2,h_1,h_2,0,0,N_1,N_2)=$$
$$=\Big(\frac{\Delta}{|\delta|}\Big)\Psi_j(\tau;g_1,g_2,h_1,h_2,0,0,N_1,N_2)\ \ (j=1,2). \tag{2.16}$$

Proof.

I. It is wellknown that the thetaseries (2)-(3) are regular on $\mathcal{H}$, hence the functions $\Psi_1(\tau)$ and $\Psi_2(\tau)$ satisfy condition 1 of the definition.

II. From (2.15), since $\delta^2\equiv 1 \pmod 4$, it follows that

$$4\Big|N\delta^2\Big(\frac{h_1^2}{N_1}+\frac{h_2^2}{N_2}\Big), 4\Big|\frac{g_1^2}{4N_1}\delta^{2\varphi(2N_1)-2}+\frac{g_2^2}{4N_2}\delta^{2\varphi(2N_2-2)}. \tag{2.17}$$

By Lemma 3 for $n=3$ and $n=1$ (with g_r,h_r,N_r and g_s,h_s,N_s instead of g,h,N), according to (2.14) and (2.17), for each substitution from $\Gamma_0(4N)$, we have

$$\vartheta'''_{g_rh_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_r\Big)\vartheta'_{g_sh_s}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_s\Big)=i^{\eta(\gamma)(sgn\delta-1)}i^{1-|\delta|}\times$$
$$\times(\gamma\tau+\delta)^5\Big(\frac{N_rN_s}{|\delta|}\Big)\vartheta'''_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta'_{\alpha g_s,h_s}(\tau;0,2N_s)=$$
$$=\operatorname{sgn}\delta\Big(\frac{-N_rN_s}{|\delta|}\Big)(\gamma\tau+\delta)^5\times$$
$$\times\vartheta'''_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta'_{\alpha g_s,h_s}(\tau;0,2N_s) \tag{2.18}$$

for $r=1,\ s=2$ and $r=2,\ s=1$.

Analogously, by Lemma 3, we have:

1) for $n=4$ and $n=0$

$$\vartheta^{(4)}_{g_rh_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_r\Big)\vartheta_{g_sh_s}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_s\Big)=i^{\eta(\gamma)(sgn\delta-1)}i^{1-|\delta|}\times$$
$$\times(\gamma\tau+\delta)^5\Big(\frac{N_rN_s}{|\delta|}\Big)\vartheta^{(4)}_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta_{\alpha g_s,h_s}(\tau;0,2N_s)=$$
$$=\operatorname{sgn}\delta\Big(\frac{-N_rN_s}{|\delta|}\Big)(\gamma\tau+\delta)^5\times$$
$$\times\vartheta^{(4)}_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta'_{\alpha g_s,h_s}(\tau;0,2N_s) \tag{2.19}$$

if $r=1,\ s=2$ and $r=2,\ s=1$;

2) for $n=2$

$$\vartheta''_{g_1h_1}\big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_1\big)\vartheta''_{g_2h_2}\big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_2\big)=$$
$$=\operatorname{sgn}\delta\big(\frac{-N_1N_2}{|\delta|}\big)(\gamma\tau+\delta)^5\times$$
$$\times\vartheta''_{\alpha g_1,h_1}(\tau;0,2N_1)\vartheta''_{\alpha g_2,h_2}(\tau;0,2N_2). \tag{2.20}$$

Hence, according to (2.16), for all α and δ with $\alpha\delta\equiv 1 \pmod{4N}$, we have

$$\Psi_j\big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_1,g_2,h_1,h_2,0,0,N_1,N_2\big)=$$
$$=\operatorname{sgn}\delta\big(\frac{-N_1N_2}{|\delta|}\big)(\gamma\tau+\delta)^5\Psi_j(\tau;\alpha g_1,\alpha g_2,h_1,h_2,0,0,N_1,N_2)=$$
$$=\operatorname{sgn}\delta\big(\frac{-\Delta}{|\delta|}\big)(\gamma\tau+\delta)^5\Psi_j(\tau;g_1,g_2,h_1,h_2,0,0,N_1,N_2)$$

for $j=1$, by (2.1) and (2.16), and for $j=2$, by (2.2),(2.19) and (2.20). Thus the functions $\Psi_1(\tau)$ and $\Psi_2(\tau)$ with $c_1=c_2=0$ satisfy condition 2 of the definition.

III. From (13) it follows that, for $r=1,\ s=2$ and $r=2,\ s=1$,

$$\vartheta'''_{g_rh_r}(\tau;0,2N_r)\vartheta'_{g_sh_s}(\tau;0,2N_s)=\pi^4\sum_{m_r,m_s=-\infty}^{\infty}(-1)^{h_rm_r+h_sm_s}\times$$
$$\times(4N_rm_r+g_r)^3(4N_sm_s+g_s)e(\Lambda\tau),$$
$$\vartheta^{(4)}_{g_rh_r}(\tau;0,2N_r)\vartheta_{g_sh_s}(\tau;0,2N_s)=\pi^4\sum_{m_r,m_s=-\infty}^{\infty}(-1)^{h_rm_r+h_sm_s}\times$$
$$\times(4N_rm_r+g_r)^4e(\Lambda\tau) \tag{2.21}$$

and also

$$\vartheta''_{g_1h_1}(\tau;0,2N_1)\vartheta''_{g_2h_2}(\tau;0,2N_2)=\pi^4\sum_{m_1,m_2=-\infty}^{\infty}(-1)^{h_1m_1+h_2m_2}\times$$
$$\times(4N_1m_1+g_1)^2(4N_2m_2+g_2)^2e(\Lambda\tau),$$

where

$$\Lambda=\sum_{k=1}^{2}\frac{1}{4N_k}(2N_km_k+g_k/2)^2=\sum_{k=1}^{2}(N_km_k^2+m_kg_k/2)+\frac{1}{4}\sum_{k=1}^{2}g_k^2/4N_k,$$

by (2.14) and (2.15), is an integer. Thus, the functions $\Psi_1(\tau)$ and $\Psi_2(\tau)$ with $c_1=c_2=0$ satisfy condition 3 of the definition.

IV. By Lemma 5 the functions $\Psi_1(\tau)$ and $\Psi_2(\tau)$ with $c_1 = c_2 = 0$ also satisfy condition 4 of the definition. □

3.

Lemma 6. *For a given N let*

$$\Phi_1(\tau) = \Phi_1(\tau; g_1, \dots, g_4, h_1, \dots, h_4, c_1, \dots, c_4, N_1, \dots, N_4) =$$
$$= \frac{1}{N_1}\vartheta'''_{g_1h_1}(\tau; c_1, 2N_1)\vartheta'_{g_2h_2}(\tau; c_2, 2N_2)\prod_{k=3}^{4}\vartheta_{g_kh_k}(\tau; c_k, 2N_k) -$$
$$-\frac{1}{N_2}\vartheta'''_{g_2h_2}(\tau; c_2, 2N_2))\vartheta'_{g_1h_1}(\tau; c_1, 2N_1)\prod_{k=3}^{4}\vartheta_{g_kh_k}(\tau; c_k, 2N_k) \quad (3.1)$$

and

$$\Phi_2(\tau) = \Phi_2(\tau; g_1, \dots, g_4, h_1, \dots, h_4, c_1, \dots, c_4, N_1, \dots, N_4) =$$
$$= \frac{1}{N_1^2}\vartheta^{(4)}_{g_1h_1}(\tau; c_1, 2N_1)\vartheta_{g_2h_2}(\tau; c_2, 2N_2)\prod_{k=3}^{4}\vartheta_{g_kh_k}(\tau; c_k, 2N_k) +$$
$$+\frac{1}{N_2^2}\vartheta^{(4)}_{g_2h_2}(\tau; c_2, 2N_2)\vartheta_{g_1h_1}(\tau; c_1, 2N_1)\prod_{k=3}^{4}\vartheta_{g_kh_k}(\tau; c_k, 2N_k) -$$
$$-\frac{6}{N_1N_2}\vartheta''_{g_1h_1}(\tau; c_1, 2N_1)\vartheta''_{g_2h_2}(\tau; c_2, 2N_2)\prod_{k=3}^{4}\vartheta_{g_kh_k}(\tau; c_k, 2N_k), \quad (3.2)$$

where

$$2\big|g_k,\ N_k\big|N\ (k = 1, 2, 3, 4),\ \ 4\Big|N\sum_{k=1}^{4}\frac{h_k}{N_k}. \quad (3.3)$$

For all substitutions from Γ in the neighborhood of each rational point $\tau = -\frac{\delta}{\gamma}$ $(\gamma \neq 0,\ (\gamma, \delta) = 1)$, we then have

$$(\gamma\tau + \delta)^6\Phi_j(\tau; g_1, \dots, g_4, h_1, \dots, h_4, 0, \dots, 0, N_1, \dots, N_4) =$$
$$= \sum_{n=0}^{\infty} D_n^{(j)} e\Big(\frac{n}{4N}\ \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\Big) \qquad (j = 1, 2). \quad (3.4)$$

Proof. From Lemma 4 for $n = 3$, $n = 1$, and $n = 0$, as in the proof of Lemma 5, it follows that

$$\frac{1}{N_1}(\gamma\tau + \delta)^6\vartheta'''_{g_1h_1}(\tau; 0, 2N_1)\vartheta'_{g_2h_2}(\tau; 0, 2N_2)\prod_{k=3}^{4}\vartheta_{g_kh_k}(\tau; 0, 2N_k) =$$

$$= e\Big(\frac{3}{2}\operatorname{sgn}\gamma\Big)\Big(4\gamma^2\Big(\prod_{k=1}^{4} N_k\Big)^{1/2}\Big)^{-1} \sum_{\substack{H_k \bmod 2N_k\\(k=1,2,3,4)}} \prod_{k=1}^{4} \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \times$$
$$\times\Big\{\Big(\frac{1}{N_1}\vartheta'''_{g'_1 h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_1, 2N_1\Big) -$$
$$-12\gamma\pi i(\gamma\tau+\delta)\vartheta'_{g'_1 h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_1, 2N_1\Big)\Big) \times$$
$$\times\vartheta'_{g'_2 h'_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_2, 2N_2\Big) \prod_{k=3}^{4} \vartheta_{g'_k h'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_k, 2N_k\Big)\Big\} \tag{3.5}$$

and

$$\frac{1}{N_2}(\gamma\tau+\delta)^6 \vartheta'''_{g_2 h_2}(\tau; 0, 2N_2)\vartheta'_{g_1 h_1}(\tau; 0, 2N_1) \prod_{k=3}^{4} \vartheta_{g_k h_k}(\tau; 0, 2N_k) =$$
$$= e\Big(\frac{3}{2}\operatorname{sgn}\gamma\Big)\Big(4\gamma^2\Big(\prod_{k=1}^{4} N_k\Big)^{1/2}\Big)^{-1} \sum_{\substack{H_k \bmod 2N_k\\(k=1,2,3,4)}} \prod_{k=1}^{4} \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \times$$
$$\times\Big\{\Big(\frac{1}{N_2}\vartheta'''_{g'_2 h'_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_2, 2N_2\Big) -$$
$$-12\gamma\pi i(\gamma\tau+\delta)\vartheta'_{g'_2 h'_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_2, 2N_2\Big)\Big) \times$$
$$\times\vartheta'_{g'_1 h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_1, 2N_1\Big) \prod_{k=3}^{4} \vartheta_{g'_k h'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_k, 2N_k\Big)\Big\}. \tag{3.6}$$

As in the proof of Lemma 5, from Lemma 4 it follows that
1) for $n = 4$ and $n = 0$

$$\frac{1}{N_1^2}(\gamma\tau+\delta)^6 \vartheta^{(4)}_{g_1 h_1}(\tau; 0, 2N_1) \prod_{k=2}^{4} \vartheta_{g_k h_k}(\tau; 0, 2N_k) =$$
$$= e\Big(\frac{3}{2}\operatorname{sgn}\gamma\Big)\Big(4\gamma^2\Big(\prod_{k=1}^{4} N_k\Big)^{1/2}\Big)^{-1} \sum_{\substack{H_k \bmod 2N_k\\(k=1,2,3,4)}} \prod_{k=1}^{4} \varphi_{g'_k g_k h_k}\big(0, H_k; 2N_k\big) \times$$
$$\times\Big\{\Big(\frac{1}{N_1^2}\vartheta^{(4)}_{g'_1 h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_1, 2N_1\Big) -$$
$$-\frac{24\gamma\pi i}{N_1}(\gamma\tau+\delta)\vartheta''_{g'_1 h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_1, 2N_1\Big) -$$
$$-48\gamma^2\pi^2(\gamma\tau+\delta)^2\vartheta_{g'_1 h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_1, 2N_1\Big)\Big) \times$$

$$\times\prod_{k=2}^{4}\vartheta_{g'_kh'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)\Big\} \tag{3.7}$$

and

$$\frac{1}{N_2^2}(\gamma\tau+\delta)^6\vartheta^{(4)}_{g_2h_2}(\tau;0,2N_2)\vartheta_{g_1h_1}(\tau;0,2N_1)\prod_{k=3}^{4}\vartheta_{g_kh_k}(\tau;0,2N_k)=$$

$$=e\Big(\frac{3}{2}\operatorname{sgn}\gamma\Big)\Big(4\gamma^2\Big(\prod_{k=1}^{4}N_k\Big)^{1/2}\Big)^{-1}\sum_{\substack{H_k \bmod 2N_k\\(k=1,2,3,4)}}\prod_{k=1}^{4}\varphi_{g'_kg_kh_k}\big(0,H_k;2N_k\big)\times$$

$$\times\Big\{\Big(\frac{1}{N_2^2}\vartheta^{(4)}_{g'_2h'_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)-$$

$$-\frac{24\gamma\pi i}{N_2}(\gamma\tau+\delta)\vartheta''_{g'_2h'_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\Big)\times$$

$$\times\vartheta_{g'_1h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\prod_{k=3}^{4}\vartheta_{g'_kh'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)-$$

$$-48\gamma^2\pi^2(\gamma\tau+\delta)^2\prod_{k=1}^{4}\vartheta_{g'_kh'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)\Big\}, \tag{3.8}$$

2) for $n=2$ and $n=0$

$$\frac{6}{N_1N_2}(\gamma\tau+\delta)^6\vartheta''_{g_1h_1}(\tau;0,2N_1)\vartheta''_{g_2h_2}(\tau;0,2N_2)\prod_{k=3}^{4}\vartheta_{g_kh_k}(\tau;0,2N_k)=$$

$$=e\Big(\frac{3}{2}\operatorname{sgn}\gamma\Big)\Big(4\gamma^2\Big(\prod_{k=1}^{4}N_k\Big)^{1/2}\Big)^{-1}\sum_{\substack{H_k \bmod 2N_k\\(k=1,2,3,4)}}\varphi_{g'_kg_kh_k}(0,H_k;2N_k)\times$$

$$\times\Big\{\Big(\frac{6}{N_1N_2}\vartheta''_{g'_1h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)-$$

$$-\frac{24\gamma\pi i}{N_2}(\gamma\tau+\delta)\vartheta_{g'_1h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big)\times$$

$$\times\vartheta''_{g'_2h'_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\prod_{k=3}^{4}\vartheta_{g'_kh'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Big)-$$

$$-\Big(\frac{24\gamma\pi i}{N_1}(\gamma\tau+\delta)\vartheta''_{g_1h_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)+$$

$$+96\gamma^2\pi^2(\gamma\tau+\delta)^2\vartheta_{g_1h_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big)\times$$

$$\times \prod_{k=2}^{4} \vartheta_{g'_k h'_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_k, 2N_k\Big)\Big\}. \tag{3.9}$$

Subtracting (3.6) from (3.5), and (3.9) from the sum of (3.7) and (3.8), according to (3.1) and (3.2) respectively, we obtain

$$\begin{aligned}(\gamma\tau+\delta)^6 \Phi_j(\tau; g_1, \ldots, g_4, h_1, \ldots, h_4, 0, \ldots, N_1, \ldots, N_4) =\\ = e\Big(\frac{3}{2}\operatorname{sgn}\gamma\Big)\Big(4\gamma^2\Big(\prod_{k=1}^{4} N_k\Big)^{1/2}\Big)^{-1} \sum_{\substack{H_k \bmod 2N_k\\(k=1,2,3,4)}} \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \times\\ \times \Phi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; g_1, \ldots, g_4, h_1, \ldots, h_4, H_1, \ldots, H_4, N_1, \ldots, N_4\Big)\\ (j = 1, 2).\end{aligned} \tag{3.10}$$

Further, reasoning as in Lemma 5, from (3.10) we obtain (3.4). □

Theorem 2. *For a given N the functions $\Phi_1(\tau)$ and $\Phi_2(\tau)$ with $c_1 = c_2 = c_3 = c_4 = 0$ are entire modular forms of weight 12 and character $\chi(\delta) = (\frac{\Delta}{|\delta|})$ (Δ is the determinant of an arbitrary positive quadratic form in 6 variables) for the congruence group $\Gamma_0(4N)$ if the following conditions hold:*

1) $2|g_k, N_k|N \qquad (k = 1, 2, 3, 4),$ (3.11)

2) $4\Big|N\sum_{k=1}^{4}(\frac{h_k^2}{N_k},\ 4\Big|\sum_{k=1}^{4}\frac{g_k^2}{4N_k},$ (3.12)

3) *for all α and δ with $\alpha\delta \equiv 1 \pmod{4N}$*

$$\begin{aligned}\Big(\frac{\prod_{k=1}^{4} N_k}{|\delta|}\Big)\Phi_j(\tau; \alpha g_1, \ldots, \alpha g_4, h_1, \ldots, h_4, 0, \ldots, 0, N_1, \ldots, N_4) =\\ = \Big(\frac{\Delta}{|\delta|}\Big)\Phi_j(\tau; g_1, \ldots, g_4, h_1, \ldots, h_4, 0, \ldots, 0, N_1, \ldots, N_4)\\ (j = 1, 2).\end{aligned} \tag{3.13}$$

Proof.

I. As in the case of Theorem 1, the functions $\Phi_1(\tau)$ and $\Phi_2(\tau)$ with $c_1 = c_2 = c_3 = c_4 = 0$ satisfy condition 1 and, by Lemma 6, also condition 4 of the definition.

II. From (3.12), since $\delta^2 \equiv 1 \pmod 4$, it follows that

$$4\Big|N\delta^2\sum_{k=1}^{4}\frac{h_k^2}{N_k},\ 4\Big|\sum_{k=1}^{4}\frac{g_k^2}{4N_k}\delta^{2\varphi(2N_k)-2}. \tag{3.14}$$

By Lemma 3, for $n=3,\ n=1$ and $n=0$, according to (3.11) and (3.14), for each substitution from $\Gamma_0(4N)$, we have

$$\vartheta'''_{g_rh_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_r\Big)\vartheta'_{g_sh_s}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_s\Big)\times$$
$$\times\prod_{k=3}^{4}\vartheta_{g_kh_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_k\Big)=$$
$$=(\gamma\tau+\delta)^6\Big(\frac{\prod_{k=1}^4 N_k}{|\delta|}\Big)\vartheta'''_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta'_{\alpha g_s,h_s}(\tau;0,2N_s)\times$$
$$\times\prod_{k=3}^{4}\vartheta_{\alpha g_k,h_k}(\tau;0,2N_k) \tag{3.15}$$

for $r=1,\ s=2$ and $r=2,\ s=1$.

Analogously, by Lemma 3, we have:

1) for $n=4$ and $n=0$

$$\vartheta^{(4)}_{g_rh_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_r\Big)\vartheta_{g_sh_s}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_s\Big)\times$$
$$\times\prod_{k=3}^{4}\vartheta_{g_kh_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_k\Big)=$$
$$=(\gamma\tau+\delta)^6\Big(\frac{\prod_{k=1}^4 N_k}{|\delta|}\Big)\vartheta^{(4)}_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta_{\alpha g_s,h_s}(\tau;0,2N_s)\times$$
$$\times\prod_{k=3}^{4}\vartheta_{\alpha g_k,h_k}(\tau;0,2N_k) \tag{3.16}$$

for $r=1,\ s=2$ and $r=2,\ s=1$,

2) for $n=2$ and $n=0$

$$\prod_{k=1}^{2}\vartheta''_{g_kh_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_k\Big)\prod_{k=3}^{4}\vartheta_{g_kh_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_k\Big)=$$
$$=(\gamma\tau+\delta)^6\Big(\frac{\prod_{k=1}^4 N_k}{|\delta|}\Big)\prod_{k=1}^{2}\vartheta''_{\alpha g_k,h_k}(\tau;0,2N_k)\prod_{k=3}^{4}\vartheta_{\alpha g_k,h_k}(\tau;0,2N_k).$$

Hence, according to (3.13), for all α and δ with $\alpha\delta\equiv 1 \pmod{4N}$, we have

$$\Phi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_1,\dots,g_4,h_1,\dots,h_4,0,\dots,N_1,\dots,N_4\Big)=$$
$$=\Big(\frac{\prod_{k=1}^4 N_k}{|\delta|}\Big)(\gamma\tau+\delta)^6\times$$

$$\times\Phi_j(\tau;\alpha g_1,\dots,\alpha g_4,h_1,\dots,h_4,0,\dots,0,N_1,\dots,N_4)=$$
$$=\Big(\frac{\Delta}{|\delta|}\Big)(\gamma\tau+\delta)^6\Phi_j(\tau;g_1,\dots,g_4,h_1,\dots,h_4,0,\dots,N_1,\dots,N_4)$$

for $j=1$, by (3.1) and (3.15), and for $j=2$, by (3.2), (3.16), and (3.17). Thus the functions $\Phi_1(\tau)$ and $\Phi_2(\tau)$ satisfy condition 2 of the definition.

III. From (13) it follows that, for $r=1,\ s=2$ and $r=2,\ s=1$,

$$\vartheta'''_{g_rh_r}(\tau;0,2N_r)\vartheta'_{g_sh_s}(\tau;0,2N_s)\prod_{k=3}^{4}\vartheta_{g_kh_k}(\tau;0,2N_k)=$$
$$=\pi^4\sum_{m_r,m_s,m_3,m_4=-\infty}^{\infty}(-1)^{m_r+m_s+m_3+m_4}(4N_rm_r+g_r)^3(4N_sm_s+g_s)e(\Lambda\tau),$$
$$\vartheta^{(4)}_{g_rh_r}(\tau;0,2N_r)\vartheta_{g_sh_s}(\tau;0,2N_s)\prod_{k=3}^{4}\vartheta_{g_kh_k}(\tau;0,2N_k)=$$
$$=\pi^4\sum_{m_r,m_s,m_3,m_4=-\infty}^{\infty}(-1)^{m_r+m_s+m_3+m_4}(4N_rm_r+g_r)^4e(\Lambda\tau),$$

and also

$$\vartheta''_{g_1h_1}(\tau;0,2N_1)\vartheta''_{g_2h_2}(\tau;0,2N_2)\prod_{k=3}^{4}\vartheta_{g_kh_k}(\tau;0,2N_k)=$$
$$=\pi^4\sum_{m_1,\dots,m_4=-\infty}^{\infty}(-1)^{\sum_{k=1}^{4}h_km_k}(4N_1m_1+g_1)^2(4N_2m_2+g_2)^2e(\Lambda\tau),$$

where

$$\Lambda=\sum_{k=1}^{4}\frac{1}{4N_k}\big(2N_km_k+\frac{g_k}{2}\big)^2=\sum_{k=1}^{4}\big(N_km_k^2+\frac{1}{2}m_kg_k\big)+\frac{1}{4}\sum_{k=1}^{4}\frac{g_k^2}{4N_k},$$

by (3.11) and (3.12), is an integer. Thus the functions $\Phi_1(\tau)$ and $\Phi_2(\tau)$ with $c_1=c_2=c_3=c_4=0$ satisfy condition 3 of the definition. □

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