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ON TWO-POINT BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF HIGHER-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH SINGULARITIES

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Abstract. The sufficient conditions of solvability and unique solvability of the two-point boundary value problems of Vallèe-Poussin and Cauchy-Niccoletti have been found for a system of ordinary differential equations of the form

$$u^{(n)} = f(t, u, u', \dots, u^{(n-1)}),$$

where the vector function $f :]a, b[\times \mathbb{R}^{nl} \to \mathbb{R}^l$ has nonintegrable singularities with respect to the first argument at the points a and b.

§ 1. Statement of the main results

In this paper for an l-dimensional system of differential equations

$$u^{(n)} = f(t, u, u', \dots, u^{(n-1)}) \tag{1.1}$$

we consider the boundary value problem of Vallèe-Poussin

$$\begin{gathered} u(a+) = \dots = u^{(m-1)}(a+) = 0, \\ u(b-) = \dots = u^{(n-m-1)}(b-) = 0 \end{gathered} \tag{1.2}$$

and that of Cauchy-Niccoletti

$$\begin{gathered} u(a+) = \dots = u^{(m-1)}(a+) = 0, \\ u^{(m)}(b-) = \dots = u^{(n-1)}(b-) = 0, \end{gathered} \tag{1.3}$$

where $l \geq 1$, $n \geq 2$, m is an integer part of the number $\frac{n}{2}$, $-\infty < a < b < +\infty$, and the vector function $f :]a, b[\times \mathbb{R}^{nl} \to \mathbb{R}^l$ satisfies the Caratheodory conditions on each compact contained in $]a, b[\times \mathbb{R}^{nl}$. We are interested mainly in the singular case when f is nonintegrable with respect to the first argument on $[a, b]$, having singularities at the ends of this interval. The above problems were investigated for $l = 1$ in [2-6].

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The following notations will be used:

$$I_n(a,b) = \begin{cases} \;]a,b[& \text{for} \quad n = 2m \\ \;]a,b] & \text{for} \quad n = 2m+1 \end{cases};$$

$$\mu_n = \begin{cases} 1 & \text{for} \quad n = 2m \\ \frac{n}{2} & \text{for} \quad n = 2m+1 \end{cases};$$

$$\lambda_{im}(a,b;t) = \frac{\min\{(t-a)^{2m-i},\,(b-t)^{2m-i}\}}{(m-1)!(m-i)!\sqrt{(2m-1)(2m-2i+1)}}$$
$$(i = 1,\dots,m);$$

$\mathbb{R}$ is a set of real numbers, $\mathbb{R}_+ = [0,+\infty[$;

$\xi = (\xi_j)_{j=1}^l \in \mathbb{R}^l$ and $A = (a_{kj})_{k,j=1}^l \in \mathbb{R}^{l\times l}$ are respectively an l-dimensional column vector and an $l\times l$ matrix with real components ξ_j $(j = 1,\dots,l)$ and a_{kj} $(k,j = 1,\dots,l)$,

$$|\xi| = (|\xi_j|)_{j=1}^l, \quad \|\xi\| = \sum_{j=1}^l |\xi_j|, \quad \|A\| = \sum_{k,j=1}^l |a_{kj}|,$$

$$S(\xi) = \begin{pmatrix} \operatorname{sign}\xi_1 & 0 & \dots & 0 \\ 0 & \operatorname{sign}\xi_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \operatorname{sign}\xi_l \end{pmatrix};$$

$r(A)$ is the spectral radius of the matrix A;

$\mathbb{R}_+^l$ and $\mathbb{R}_+^{l\times l}$ are sets of l-dimensional vectors and $l\times l$ matrices with nonnegative components;

the inequalities $\xi \le \bar\xi$ and $A \le \bar A$, where ξ and $\bar\xi \in \mathbb{R}^l$ and A and $\bar A \in \mathbb{R}^{l\times l}$, imply respectively $\bar\xi - \xi \in \mathbb{R}_+^l$ and $\bar A - A \in \mathbb{R}_+^{l\times l}$;

$L_{loc}(I;\mathbb{R}_+)$, where $I \subset \mathbb{R}$ is an interval, is a set of functions $x : I \to \mathbb{R}_+$ which are Lebesgue integrable on each segment contained in I;

$K_{loc}(I\times\mathbb{R}^p;\mathbb{R}^l)$, where p is a natural number, is a set of vector functions mapping $I\times\mathbb{R}^p$ into $\mathbb{R}^l$ and satisfying the Caratheodory conditions on each compact contained in $I\times\mathbb{R}^p$;

$\widetilde{C}^p_{loc}(I;\mathbb{R}^l)$ is a set of vector functions $u : I \to \mathbb{R}^l$ which are absolutely continuous together with all their derivatives up to order p inclusive on each segment contained in I;

$\widetilde{C}^{n-1,m}(I;\mathbb{R}^l)$ is a set of vector functions $u \in \widetilde{C}^{n-1}_{loc}(I;\mathbb{R}^l)$ satisfying the condition

$$\int_I \|u^{(m)}(\tau)\|^2 d\tau < +\infty.$$

As mentioned above, throughout this paper it is assumed that

$$f \in K_{loc}(]a,b[\times\mathbb{R}^{nl};\mathbb{R}^l).$$

Theorem 1.1. *Let the following inequalities be fulfilled on* $]a,b[\times\mathbb{R}^{nl}$*:*

$$(-1)^{n-m-1}S(x_1)f(t,x_1,\dots,x_n)\geq-\sum_{i=1}^{m}H_i(t)|x_i|-h(t) \tag{1.4}$$

and

$$\|f(t,x_1,\dots,x_n)\|\leq q(t,x_1,\dots,x_m)\sum_{i=m+1}^{n}(1+\|x_i\|)^{\frac{2n-2m-1}{2i-2m-1}}, \tag{1.5}$$

where

$$q\in K_{loc}(I_n(a,b)\times\mathbb{R}^{ml};\mathbb{R}_+), \tag{1.6}$$

and $H_i:]a,b[\to\mathbb{R}_+^{l\times l}$ $(i=1,\dots,m)$ *and* $h:]a,b[\to\mathbb{R}_+^l$ *are respectively measurable matrix and vector functions satisfying the conditions*

$$\int_a^b(\tau-a)^{n-m-\frac{1}{2}}(b-\tau)^{m-\frac{1}{2}}\|h(\tau)\|d\tau<+\infty, \tag{1.7}$$

$$\int_a^b(\tau-a)^{n-i}(b-\tau)^{2m-i}\|H_i(\tau)\|d\tau<+\infty\ \ (i=1,\dots,m), \tag{1.8}$$

$$r\Big(\sum_{i=1}^{m}\int_a^b(\tau-a)^{n-2m}\lambda_{im}(a,b;\tau)H_i(\tau)d\tau\Big)<\mu_n. \tag{1.9}$$

Then the problem (1.1),(1.2) *is solvable in the class* $\widetilde{C}^{n-1,m}(I_n(a,b);\mathbb{R}^l)$.

Theorem 1.2. *Let on* $]a,b[\times\mathbb{R}^{nl}$ *the inequalities* (1.4) *and* (1.5) *be fulfilled, where* $q\in K_{loc}(]a,b]\times\mathbb{R}^{ml};\mathbb{R}_+)$*, and* $H_i:]a,b]\to\mathbb{R}_+^{l\times l}$ $(i=1,\dots,m)$ *and* $h:]a,b]\to\mathbb{R}_+^l$ *are respectively measurable matrix and vector functions satisfying the conditions*

$$\int_a^b(\tau-a)^{n-m-\frac{1}{2}}\|h(\tau)\|d\tau<+\infty, \tag{1.10}$$

$$\int_a^b(\tau-a)^{n-i}\|H_i(\tau)\|d\tau<+\infty\quad(i=1,\dots,m), \tag{1.11}$$

$$r\Big(\sum_{i=1}^{m}\frac{1}{(m-1)!(m-i)!\sqrt{(2m-1)(2m-2i+1)}}\times\\ \times\int_a^b(\tau-a)^{n-i}H_i(\tau)d\tau\Big)<\mu_n. \tag{1.12}$$

Then the problem (1.1), (1.3) *is solvable in the class* $\widetilde{C}^{n-1,m}(]a,b];\mathbb{R}^l)$.

For a differential system

$$u^{(n)} = f(t, u, u', \dots, u^{(m-1)}), \tag{1.1'}$$

not containing intermediate derivatives of order higher than $(m-1)$, Theorems 1.1 and 1.2 can be formulated as follows:

Theorem 1.1′. *Let* $f \in K_{loc}(I_n(a,b) \times \mathbb{R}^{ml}; \mathbb{R}^l)$ *and on* $]a,b[\times\mathbb{R}^{ml}$

$$(-1)^{n-m-1} S(x_1) f(t, x_1, \dots, x_m) \geq -\sum_{i=1}^{m} H_i(t)|x_i| - h(t), \tag{1.4'}$$

where $H_i :]a,b[\to \mathbb{R}_+^{l\times l}$ $(i = 1, \dots, m)$ *and* $h :]a,b[\to \mathbb{R}_+^l$ *are measurable matrix and vector functions satisfying the conditions* (1.7)-(1.9). *Then the problem* (1.1′), (1.2) *is solvable in the class* $\widetilde{C}^{n-1,m}(I_n(a,b);\mathbb{R}^l)$.

Theorem 1.2′. *Let* $f \in K_{loc}(]a,b] \times \mathbb{R}^{ml}; \mathbb{R}^l)$ *and on* $]a,b[\times\mathbb{R}^{ml}$ *the inequality* (1.4′) *be fulfilled, where* $H_i :]a,b] \to \mathbb{R}_+^{l\times l}$ $(i = 1, \dots, m)$ *and* $h :]a,b] \to \mathbb{R}_+^l$ *are measurable matrix and vector functions satisfying the conditions* (1.10)-(1.12). *Then the problem* (1.1′), (1.3) *is solvable in the class* $\widetilde{C}^{n-1,m}(]a,b];\mathbb{R}^l)$.

Theorem 1.3. *Let*

$$\begin{gathered} f \in K_{loc}(I_n(a,b) \times \mathbb{R}^{ml}; \mathbb{R}^l), \\ \int_a^b (\tau - a)^{n-m-\frac{1}{2}} (b-\tau)^{m-\frac{1}{2}} \|f(\tau, 0, \dots, 0)\| d\tau < +\infty \end{gathered} \tag{1.13}$$

and on $]a,b[\times\mathbb{R}^{ml}$

$$\begin{aligned} (-1)^{n-m-1} S(x_1 - y_1)[f(t, x_1, \dots, x_m) - f(t, y_1, \dots, y_m)] \geq \\ \geq -\sum_{i=1}^{m} H_i(t)|x_i - y_i|, \end{aligned} \tag{1.14}$$

where $H_i :]a,b[\to \mathbb{R}_+^{l\times l}$ $(i = 1, \dots, m)$ *are measurable matrix functions satisfying the conditions* (1.8) *and* (1.9). *Then the problem* (1.1′), (1.2) *is uniquely solvable in the class* $\widetilde{C}^{n-1,m}(I_n(a,b);\mathbb{R}^l)$.

Theorem 1.4. *Let*

$$f \in K_{loc}(]a,b] \times \mathbb{R}^{ml}; \mathbb{R}^l), \quad \int_a^b (\tau - a)^{n-m-\frac{1}{2}} \|f(\tau, 0, \dots, 0)\| d\tau < +\infty$$

and on $]a,b[\times\mathbb{R}^{ml}$ *the inequality* (1.14) *be fulfilled, where* $H_i :]a,b] \to \mathbb{R}_+^{l\times l}$ $(i = 1, \dots, m)$ *are measurable matrix functions satisfying the conditions* (1.11) *and* (1.12). *Then the problem* (1.1′), (1.3) *is uniquely solvable in the class* $\widetilde{C}^{n-1,m}(]a,b];\mathbb{R}^l)$.

§ 2. Auxiliary propositions

Lemma 2.1. *Let $I \subset \mathbb{R}$ be some interval, k be a natural number, $\rho_0 \in]0,+\infty[$ and*

$$\varphi \in L_{loc}(I;\mathbb{R}_+). \tag{2.1}$$

Then there exists a continuous function $\rho : I \to \mathbb{R}_+$ such that for any vector function $v \in \widetilde{C}^k_{loc}(I;\mathbb{R}^l)$ satisfying almost everywhere on I the differential inequality

$$\|v^{(k+1)}(t)\| \le \varphi(t)\left[1+\sum_{i=0}^{k}\|v^{(i)}(t)\|^{\frac{2k+1}{2i+1}}\right] \tag{2.2}$$

and the condition

$$\int_I \|v(\tau)\|^2\,d\tau \le \rho_0^2, \tag{2.3}$$

the estimates

$$\|v^{(i)}(t)\| < \rho(t) \quad \textit{for } t \in I \ \ (i=0,\dots,k) \tag{2.4}$$

hold.

Proof. In the case $I=[a,b]$ it is not difficult to verify by Lemma 2.2 from [6] that there exists a positive constant $\widetilde{\rho}$ such that the estimates $\|v^{(i)}(t)\| < \widetilde{\rho}$ for $a \le t \le b$ $(i=0,\dots,k)$ hold for any vector function $v \in \widetilde{C}^k_{loc}(I;\mathbb{R}^l)$ satisfying the conditions (2.2) and (2.3); in other words, we have (2.4), where $\rho(t) \equiv \widetilde{\rho}$.

Now consider the case $I=]a,b]$. Choose any decreasing sequence $a_j \in]a,b]$ $(j=0,1,2,\dots)$ such that $a_0=b$ and $\lim\limits_{j\to+\infty} a_j = a$. Then, by virtue of the above reasoning, for any natural number j there exists a positive constant ρ_j such that any vector function $v \in \widetilde{C}^k_{loc}(I;\mathbb{R}^l)$ satisfying the conditions (2.2) and (2.3) admits the estimates

$$\|v^{(i)}(t)\| < \rho_j \quad \text{for } a_j \le t \le b \ \ (i=0,\dots,k). \tag{2.5}$$

Without loss of generality the sequence $(\rho_j)_{j=1}^{+\infty}$ can be assumed to be non-decreasing. Then (2.5) yields the estimates (2.4), where

$$\rho(t) = \rho_j + \frac{t-a_{j-1}}{a_j-a_{j-1}}\,(\rho_{j+1}-\rho_j) \quad \text{for } a_j < t \le a_{j-1} \ \ (j=1,2,\dots)$$

with $\rho : I \to \mathbb{R}_+$ being continuous and independent of v.

The cases $I=[a,b[$ and $I=]a,b[$ are considered similarly. $\square$

Lemma 2.2. *Let $H_i :]a,b[\to \mathbb{R}_+^{l\times l}$ $(i = 1,\dots,m)$ and $h :]a,b[\to \mathbb{R}_+^l$ be measurable matrix and vector functions satisfying the conditions* (1.7)-(1.9) *and*

$$H = \sum_{i=1}^{m} \int_a^b (\tau - a)^{n-2m} \lambda_{im}(a,b;\tau) H_i(\tau)\, d\tau. \tag{2.6}$$

Then for any vector function $u \in \widetilde{C}^{n-1,m}(]a,b[;\mathbb{R}^l)$ satisfying a system of differential inequalities

$$(-1)^{n-m-1} S(u(t)) u^{(n)}(t) \geq -\sum_{i=1}^{m} H_i(t) |u^{(i-1)}(t)| - h(t) \tag{2.7}$$
$$\text{for} \quad a < t < b$$

and the boundary conditions (1.2) *we have the estimates*

$$\int_a^b \|u^{(m)}(\tau)\|^2\, d\tau \leq \rho_0^2 \tag{2.8}$$

and

$$\|u^{(i-1)}(t)\| \leq \rho_0 \sigma_{im}(a,b;t) \quad \text{for} \ a < t < b \ \ (i = 1,\dots,m), \tag{2.9}$$

where

$$\sigma_{im}(a,b;t) = \frac{\min\{(t-a)^{m-i+\frac12},\, (b-t)^{m-i+\frac12}\}}{(m-i)!\sqrt{2m-2i+1}},$$
$$\rho_0 = \sqrt{l}\,\|(\mu_n E - H)^{-1}\| \times$$
$$\times \int_a^b (\tau - a)^{n-2m} \sigma_{1m}(a,b;\tau) \|h(\tau)\|\, d\tau \tag{2.10}$$

and E is the unit $l \times l$ matrix.

To prove this lemma we need

Lemma 2.3. *Let*

$$w(t) = \sum_{i=1}^{n-m} \sum_{k=i}^{n-m} c_{ik}(t) v^{(n-k)}(t) v^{(i-1)}(t),$$

where

$$\begin{gathered} v \in \widetilde{C}^{n-1,m}(]a,b[;\mathbb{R}), \quad v^{(i-1)}(a+) = 0 \ \ (i = 1,\dots,m), \\ v^{(j-1)}(b-) = 0 \quad (j = 1,\dots,n-m) \end{gathered} \tag{2.11}$$

and each $c_{ik} : [a,b] \to \mathbb{R}$ *is a* $(n-k-i+1)$*-times continuously differentiable function; in that case there exists a positive constant* c_0 *such that*

$$|c_{ii}(t)| \le c_0 (t-a)^{n-2m} \quad \textit{for } \ a \le t \le b \qquad (i = 1, \dots, n-m). \tag{2.12}$$

Then

$$\lim_{t \to a+} \inf |w(t)| = 0, \qquad \lim_{t \to b-} \inf |w(t)| = 0.$$

Proof. In the first place it will be shown that

$$\lim_{t \to a+} \inf |w(t)| = 0. \tag{2.13}$$

Let the opposite be true. Then without loss of generality one may assume that the inequality $w(t) \ge \delta$ for $a < t \le a + 2\varepsilon_0$ is fulfilled for some $\delta \in]0, +\infty[$ and $\varepsilon_0 \in \big]0, \frac{b-a}{4}\big[\cap]0,1[$.

Therefore

$$\sum_{i=1}^{n-m} \sum_{k=i}^{n-m} q_{ik}(t;\varepsilon) v^{(n-k)}(t) v^{(i-1)}(t) \ge \delta (t-a-\varepsilon)^n (a+2\varepsilon - t)^n \quad \text{for} \quad a+\varepsilon \le t \le a+2\varepsilon, \quad 0 < \varepsilon \le \varepsilon_0, \tag{2.14}$$

where $q_{ik}(t;\varepsilon) = (t-a-\varepsilon)^n (a+2\varepsilon-t)^n c_{ik}(t)$. After integrating the latter inequality from $a+\varepsilon$ to $a+2\varepsilon$ according to Lemma 4.1 from [7], we obtain

$$\sum_{i=1}^{n-m} \sum_{k=i}^{n-m} \sum_{j=0}^{m_{ik}} \nu_{ikj} \int_{a+\varepsilon}^{a+2\varepsilon} q_{ik}^{(n-k-i-2j+1)}(\tau;\varepsilon) [v^{(i+j-1)}(\tau)]^2 d\tau \ge \ge \delta \int_{a+\varepsilon}^{a+2\varepsilon} (\tau - a - \varepsilon)^n (a + 2\varepsilon - \tau)^n \, d\tau, \tag{2.15}$$

where m_{ik} is the integer part of the number $\frac{1}{2}(n-k-i+1)$ and ν_{ikj} $(i = 1, \dots, n-m;\ k = i, \dots, n-m;\ j = 0, \dots, m_{ik})$ are positive constants independent of a, ε and v.

If $k \in \{i+1, \dots, n-m\}$, then we have $i+j-1 \le m-1$, $2n - (n-k-i-2j+1) \ge 2i+2j+n$ for any $j \in \{0, \dots, m_{ik}\}$.

Therefore, taking into account (2.11) and (2.14), we find

$$[v^{(i+j-1)}(t)]^2 = \left[\frac{1}{(m-i-j)!} \int_a^t (t-\tau)^{m-i-j} v^{(m)}(\tau)\, d\tau \right]^2 \le \le \alpha(\varepsilon) \varepsilon^{2m-2i-2j+1} \quad \text{for} \quad a < t \le a + 2\varepsilon \tag{2.16}$$

and $|q_{ik}^{(n-k-i-2j+1)}(t;\varepsilon)| \le \alpha_1 \varepsilon^{2i+2j+n}$ for $a \le t \le a+2\varepsilon$, where

$$\alpha(\varepsilon) = 2^{2m-1} \int_a^{a+2\varepsilon} [v^{(m)}(\tau)]^2\, d\tau \to 0 \quad \text{for} \quad \varepsilon \to 0 \tag{2.17}$$

and α_1 is a positive constant independent of ε. Therefore

$$\left| \int_{a+\varepsilon}^{a+2\varepsilon} q_{ik}^{(n-k-i-2j+1)}(\tau;\varepsilon)[v^{(i+j-1)}(\tau)]^2\, d\tau \right| \le \alpha_1 \alpha(\varepsilon) \varepsilon^{2m+2+n} \le$$
$$\le \alpha_1 \alpha(\varepsilon) \varepsilon^{2n+1}.$$

Consider now the case $k = i$. By virtue of (2.12) and (2.14) we have

$$|q_{ik}^{(n-k-i-2j+1)}(t;\varepsilon)| = |q_{ii}^{(n-2i-2j+1)}(t;\varepsilon)| \le \alpha_2 \varepsilon^{2n-2m+2i+2j-1}$$
$$\text{for} \quad a \le t \le a+2\varepsilon,$$

where α_2 is a positive constant independent of ε. Therefore if $i+j-1 = m$, then

$$\left| \int_{a+\varepsilon}^{a+2\varepsilon} q_{ii}^{(n-2i-2j+1)}(\tau;\varepsilon)[v^{(i+j-1)}(\tau)]^2\, d\tau \right| =$$
$$= \left| \int_{a+\varepsilon}^{a+2\varepsilon} q_{ii}(\tau;\varepsilon)[v^{(m)}(\tau)]^2\, d\tau \right| \le \alpha_2 \alpha(\varepsilon) \varepsilon^{2n+1},$$

if however $i+j-1 < m$, then, taking into account (2.16), we obtain

$$\left| \int_{a+\varepsilon}^{a+2\varepsilon} q_{ii}^{(n-2i-2j+1)}(\tau;\varepsilon)[v^{(i+j-1)}(\tau)]^2\, d\tau \right| \le \alpha_2 \alpha(\varepsilon) \varepsilon^{2n+1}.$$

Thus

$$\left| \int_{a+\varepsilon}^{a+2\varepsilon} q_{ik}^{(n-k-i-2j+1)}(\tau;\varepsilon)[v^{(i+j-1)}(\tau)]^2\, d\tau \right| \le \alpha_0 \alpha(\varepsilon) \varepsilon^{2n+1}, \tag{2.18}$$
$$(i = 1, \dots, n-m; \quad k = i, \dots, n-m, \quad j = 0, \dots, m_{ik}),$$

where $\alpha_0 = \max\{\alpha_1, \alpha_2\}$.

On the other hand,

$$\int_{a+\varepsilon}^{a+2\varepsilon} (\tau - a - \varepsilon)^n (a + 2\varepsilon - \tau)^n\, d\tau \ge \frac{\varepsilon^n}{2^n} \int_{a+\varepsilon}^{a+\frac{3\varepsilon}{2}} (\tau - a - \varepsilon)^n d\tau =$$
$$= \frac{1}{2^{2n+1}(n+1)} \varepsilon^{2n+1}.$$

Due to (2.18) and the latter inequality we find from (2.15) that $\alpha(\varepsilon) \ge \delta_0$ for $0 < \varepsilon \le \varepsilon_0$, where δ_0 is a positive constant independent of ε. But the latter inequality contradicts the condition (2.17). This contradiction proves that (2.13) holds.

The equality $\liminf\limits_{t\to b-}|w(t)|=0$ is proved similarly, the only difference being that for $n=2m+1$ instead of (2.12) the condition $v^{(m)}(b-)=0$ is used. □

Proof of Lemma 2.2. For each component u_j $(j=1,\dots,l)$ of the solution u of the problem (2.7), (1.2) we have

$$|u_j^{(i-1)}(t)|=\left|\frac{1}{(m-i)!}\int_a^t(t-\tau)^{m-i}u_j^{(m)}(\tau)\,d\tau\right|\le$$
$$\le\frac{1}{(m-i)!\sqrt{2m-2i+1}}(t-a)^{m-i+\frac12}\left(\int_a^b[u_j^{(m)}(\tau)]^2\,d\tau\right)^{\frac12}$$
$$\text{for}\ \ a<t<b\quad(i=1,\dots,m)$$

and

$$|u_j^{(i-1)}(t)|=\left|\frac{1}{(m-i)!}\int_t^b(\tau-t)^{m-i}u_j^{(m)}(\tau)\,d\tau\right|\le$$
$$\le\frac{1}{(m-i)!\sqrt{2m-2i+1}}(b-t)^{m-i+\frac12}\left(\int_a^b[u_j^{(m)}(\tau)]^2\,d\tau\right)^{\frac12}$$
$$\text{for}\ \ a<t<b\quad(i=1,\dots,m).$$

Therefore

$$|u_j^{(i-1)}(t)|\le\sigma_{im}(a,b;t)\rho_j\quad\text{for}\ \ a<t<b\ \ (i=1,\dots,m),\tag{2.19}$$

where

$$\rho_j=\left(\int_a^b[u_j^{(m)}(\tau)]^2\,d\tau\right)^{\frac12}.$$

Let $H_i(t)=\big(h_{ijk}(t)\big)_{j,k=1}^{l}$ $(i=1,\dots,m)$, $h(t)=(h_j(t))_{j=1}^{l}$. Rewrite (2.7) in terms of components as

$$(-1)^{n-m-1}u_j^{(n)}(t)\operatorname{sign}u_j(t)\ge$$
$$\ge-\sum_{i=1}^m\sum_{k=1}^l h_{ijk}(t)|u_k^{(i-1)}(t)|-h_j(t)\ \ (j=1,\dots,l).\tag{2.7$'$}$$

After multiplying both sides of $(2.7')$ by $(t-a)^{n-2m}|u_j(t)|$ and integrating from s to t, we obtain

$$(-1)^{n-m}\int_s^t(\tau-a)^{n-2m}u_j^{(n)}(\tau)u_j(\tau)\,d\tau\le$$
$$\le\sum_{i=1}^m\sum_{k=1}^l\int_s^t(\tau-a)^{n-2m}h_{ijk}(\tau)|u_k^{(i-1)}(\tau)||u_j(\tau)|\,d\tau+$$
$$+\int_s^t(\tau-a)^{n-2m}h_j(\tau)|u_j(\tau)|\,d\tau\quad\text{for }\ a<s\le t<b.\tag{2.20}$$

By virtue of (2.19)

$$\sum_{k=1}^l\int_s^t(\tau-a)^{n-2m}h_{ijk}(\tau)|u_k^{(i-1)}(\tau)||u_j(\tau)|\,d\tau\le$$
$$\le\rho_j\sum_{k=1}^l\rho_k\int_s^t(\tau-a)^{n-2m}\sigma_{1m}(a,b;\tau)\sigma_{im}(a,b;\tau)h_{ijk}(\tau)\,d\tau=$$
$$=\rho_j\sum_{k=1}^l\rho_k\int_s^t(\tau-a)^{n-2m}\lambda_{im}(a,b;\tau)h_{ijk}(\tau)\,d\tau\tag{2.21}$$
$$(i=1,\dots,m),$$
$$\int_s^t(\tau-a)^{n-2m}h_i(\tau)|u_j(\tau)|\,d\tau\le$$
$$\le\rho_j\int_s^t(\tau-a)^{n-2m}\sigma_{1m}(a,b;\tau)h_j(\tau)\,d\tau.\tag{2.22}$$

On the other hand, by Lemma 4.1 from [7]

$$\int_s^t(\tau-a)^{n-2m}u_j^{(n)}(\tau)u_j(\tau)\,d\tau=$$
$$=w_j(t)-w_j(s)+(-1)^{n-m}\mu_n\int_s^t[u_j^{(m)}(\tau)]^2\,d\tau,\tag{2.23}$$

where

$$w_j(t)=\begin{cases}\sum\limits_{p=1}^{n-m}(-1)^{p-1}u_j^{(n-p)}(t)u_j^{(p-1)}(t) & \text{for }\ n=2m,\\ \sum\limits_{p=1}^{n-m-1}(-1)^{p-1}[(t-a)u_j^{(n-p)}(t)- & \\ -pu_j^{(n-p-1)}(t)]u_j^{(p-1)}(t)+(-1)^m\frac{t-a}{2}[u_j^{(m)}(t)]^2 & \text{for }\ n=2m+1.\end{cases}$$

As one may readily verify, the functions w_j $(j = 1, \dots, l)$ satisfy the conditions of Lemma 2.3 and therefore

$$\lim_{s\to a+} \inf |w_j(s)| = 0, \quad \lim_{t\to b-} \inf |w_j(t)| = 0 \quad (j = 1, \dots, l).$$

Taking into account the latter equalities and conditions (1.7) and (1.8) from (2.20)-(2.23) we obtain

$$\mu_n \rho_j^2 \le \rho_j \sum_{i=1}^{m} \sum_{k=1}^{l} \rho_k \int_a^b (\tau - a)^{n-2m} \lambda_{im}(a, b; \tau) h_{ijk}(\tau) d\tau +$$
$$+\rho_j \int_a^b (\tau - a)^{n-2m} \sigma_{1m}(a, b; \tau) h_j(\tau) d\tau \quad (j = 1, \dots, l).$$

Hence by virtue of (2.6) we have

$$\mu_n \rho \le H\rho + \int_a^b (\tau - a)^{n-2m} \sigma_{1m}(a, b; \tau) h(\tau) d\tau,$$

where $\rho = (\rho_j)_{j=1}^{l}$. In view of (1.9) and the notation (2.10) from the latter inequality we find

$$\rho \le (\mu_n E - H)^{-1} \int_a^b (\tau - a)^{n-2m} \sigma_{1m}(a, b; \tau) h(\tau) d\tau$$

and

$$\|\rho\| \le \|(\mu_n E - H)^{-1}\| \int_a^b (\tau - a)^{n-2m} \sigma_{1m}(a, b; \tau) \|h(\tau)\| d\tau = l^{-\frac{1}{2}} \rho_0.$$

Hence

$$\int_a^b \|u^{(m)}(\tau)\|^2 d\tau \le l\|\rho\|^2 \le \rho_0^2.$$

On the other hand, in view of (2.19)

$$\|u^{(i-1)}(t)\| \le \sigma_{im}(a, b; t)\|\rho\| \le \rho_0 \sigma_{im}(a, b; t) \quad \text{for} \quad a < t < b$$
$$(i = 1, \dots, m).$$

Therefore, the estimates (2.8) and (2.9) hold. □

In a similar manner we prove

Lemma 2.4. *Let $H_i :]a, b] \to \mathbb{R}_+^{l\times l}$ $(i = 1, \dots, m)$, and let $h :]a, b] \to \mathbb{R}_+^l$ be measurable matrix and vector functions satisfying the conditions* (1.10)-(1.12). *Then for any solution $u \in \widetilde{C}^{n-1,m}(]a, b]; \mathbb{R}^l)$ of the problem* (2.7), (1.3) *we have the estimates*

$$\int_a^b \|u^{(m)}(\tau)\|^2 d\tau \le \rho_0^2$$

and $\|u^{(i-1)}(t)\| \le \rho_0(t-a)^{m-i+\frac{1}{2}}$ *for* $a<t<b$ $(i=1,\dots,m)$, *where*

$$\rho_0 = \frac{\sqrt{l}}{(m-1)!\sqrt{2m-1}}\|(\mu_n E - H)^{-1}\| \int_a^b (\tau-a)^{n-m-\frac{1}{2}}\|h(\tau)\|\,d\tau,$$

$$H = \sum_{i=1}^{m} \frac{1}{(m-1)!(m-i)!\sqrt{(2m-1)(2m-2i+1)}} \int_a^b (\tau-a)^{n-i} H_i(\tau)d\tau,$$

and E *is the unit* $l \times l$ *matrix.*

§ 3. Proof of the main results

Proof of Theorem 1.1. Let ρ_0 and $\sigma_{im}(a,b;t)$ $(i=1,\dots,m)$ be respectively the number and functions from Lemma 2.2 and

$$\varphi(t) = 4^n \sup\{q(t,x_1,\dots,x_m) : \|x_i\| \le \rho_0\sigma_{im}(a,b;t) \\ (i=1,\dots,m)\}. \tag{3.1}$$

Then due to (1.6), (2.1) holds with $I = I_n(a,b)$.

For $k=n-m-1$, ρ_0, and φ, by virtue of Lemma 2.1 there exists a continuous function $\rho : I_n(a,b) \to \mathbb{R}_+$ such that estimates (2.4) are valid for any vector function $v \in \widetilde{C}^k_{loc}(I_n(a,b);\mathbb{R}^l)$ satisfying the conditions (2.2) and (2.3).

Let

$$\rho_i(t) = \begin{cases} \rho_0\sigma_{im}(a,b;t) & \text{for } \ i \in \{1,\dots,m\} \\ \rho(t) & \text{for } \ i \in \{m+1,\dots,n\} \end{cases} \tag{3.2}$$

and $f^*(t) = \sup\{\|f(t,x_1,\dots,x_n)\| : \|x_i\| \le \rho_i(t) \ (i=1,\dots,n)\}$. For any $i \in \{1,\dots,n\}$ and $\xi = (\xi_p)_{p=1}^l$ we set

$$\chi_{ip}(t,\xi) = \begin{cases} \xi_p & \text{for } \ |\xi_p| \le \rho_i(t) \\ \rho_i(t)\operatorname{sign}\xi_p & \text{for } \ |\xi_p| > \rho_i(t) \end{cases}, \\ \chi_i(t,\xi) = \big(\chi_{ip}(t,\xi)\big)_{p=1}^l. \tag{3.3}$$

Let j be an arbitrary natural number,

$$I_{nj}(a,b) = \begin{cases} [a+\frac{b-a}{3j},\, b-\frac{b-a}{3j}] & \text{for } \ n=2m \\ [a+\frac{b-a}{3j},\, b] & \text{for } \ n=2m+1 \end{cases},$$

$$f_j(t,x_1,\dots,x_n) = \\ = \begin{cases} f(t,\chi_1(t,x_1),\dots,\chi_n(t,x_n)) & \text{for } \ t \in I_{nj}(a,b) \\ 0 & \text{for } \ t \in [a,b]\backslash I_{nj}(a,b) \end{cases}, \tag{3.4}$$

$$f_j^*(t) = \begin{cases} f^*(t) & \text{for} \quad t \in I_{nj}(a,b) \\ 0 & \text{for} \quad t \in [a,b] \backslash I_{nj}(a,b) \end{cases}.$$

Clearly, that $f_j^* : [a,b] \to \mathbb{R}_+$ is the Lebesgue integrable function and on the $[a,b] \times \mathbb{R}^{nl}$ the inequality $\|f_j(t, x_1, \ldots, x_n)\| \leq f_j^*(t)$ holds. On the other hand the homogeneous differential system $u^{(n)} = 0$ by boundary conditions (1.2) has only the trivial solution. Therefore by virtue of the Conti theorem [1] [1] the differential system $u^{(n)} = f_j(t, u, \ldots, u^{(n-1)})$ has a solution $u_j \in \widetilde{C}_{loc}^{n-1}([a,b]; \mathbb{R}^l)$ satisfying the boundary conditions (1.2). It is obvious that $u_j \in \widetilde{C}^{n-1,m}([a,b]; \mathbb{R}^l)$. Simultaneously, from (1.4), (3.3), and (3.4) it follows that u_j is the solution of the system of the differential inequalities (2.7). Therefore by virtue of Lemma 2.2

$$\int_a^b \|u_j^{(m)}(\tau)\|^2 \, d\tau \leq \rho_0^2 \tag{3.5}$$

and

$$\|u_j^{(i-1)}(t)\| \leq \rho_0 \sigma_{im}(a,b;t) \quad \text{for} \;\; a < t < b \;\; (i = 1, \ldots, m). \tag{3.6}$$

From conditions (1.5) and (3.1)-(3.6) it is clear that the vector function $v_j(t) = u_j^{(m)}(t)$ satifies the inequalities (2.2) and (2.3). Hence by Lemma 2.1 $\|u_j^{(i-1)}(t)\| < \rho(t)$ for $t \in I_n(a,b)$ $(i = m+1, \ldots, n)$. Therefore

$$\|u_j^{(i-1)}(t)\| < \rho_i(t) \quad \text{for} \;\; a < t < b \;\; (i = 1, \ldots, n) \tag{3.7}$$

and

$$\|u_j^{(n)}(t)\| \leq f^*(t) \quad \text{for} \quad a < t < b. \tag{3.8}$$

Moreover, in view of (3.3),(3.4) and (3.7) it is clear that

$$u_j^{(n)}(t) = f(t, u_j(t), \ldots, u_j^{(n-1)}(t)) \quad \text{for } t \in I_{nj}(a,b). \tag{3.9}$$

Since $f^* \in L_{loc}(I_n(a,b); \mathbb{R}_+)$, the estimates (3.7) and (3.8) imply that the sequences $(u_j^{(i-1)})_{j=1}^{+\infty}$ $(i = 1, \ldots, n)$ are uniformly bounded and equicontinuous on each segment contained in $I_n(a,b)$. Therefore, by virtue of the Arcela-Ascoli lemma these sequences can be regarded without loss of generality as uniformly converging on each segment from $I_n(a,b)$.

If we set $\lim_{j \to +\infty} u_j(t) = u(t)$ for $t \in I_n(a,b)$, then

$$\lim_{j \to +\infty} u_j^{(i-1)}(t) = u^{(i-1)}(t) \quad \text{for } t \in I_n(a,b) \;\; (i = 1, \ldots, n) \tag{3.10}$$

[1] See also [8], Corollary 2.1.

uniformly on each segment contained in $I_n(a,b)$. Therefore from (3.5) and (3.6) we obtain

$$\int_a^b \|u^{(m)}(\tau)\|^2 \, d\tau \le \rho_0^2, \tag{3.11}$$

$$\|u^{(i-1)}(t)\| \le \rho_0 \sigma_{im}(a,b;t) \quad \text{for} \quad t \in I_n(a,b) \tag{3.12}$$
$$(i=1,\dots,m).$$

In view of (3.9) for arbitrary fixed s and $t \in I_n(a,b)$ there exists a natural number j_0 such that

$$u_j^{(n-1)}(t) - u_j^{(n-1)}(s) = \int_s^t f(\tau, u_j(\tau), \dots, u_j^{(n-1)}(\tau)) d\tau$$
$$(j = j_0, j_0+1, \dots)$$

and $s, t \in I_{nj}(a,b)$ for $j \ge j_0$. Passing to the limit in the latter equality by $j \to +\infty$, we obtain

$$u^{(n-1)}(t) - u^{(n-1)}(s) = \int_s^t f(\tau, u(\tau), \dots, u^{(n-1)}(\tau)) d\tau.$$

Therefore u is the solution of the system (1.1). Simultaneously, (3.10)-(3.12) imply that $u \in \widetilde{C}^{n-1,m}(I_n(a,b);\mathbb{R}^l)$ and satisfies the boundary conditions (1.2). $\square$

Theorem 1.1$'$ immediately follows from Theorem 1.1, since in the case where $f(t,x_1,\dots,x_n) \equiv f(t,x_1,\dots,x_m)$ and $f \in K_{loc}(I_n(a,b) \times \mathbb{R}^{ml};\mathbb{R}^l)$, inequality (1.5) is fulfilled automatically and the function $q(t,x_1,\dots,x_m) \equiv \|f(t,x_1,\dots,x_m)\|$ satisfies the condition (1.6).

Proof of Theorem 1.3. (1.13) and (1.14) yield the conditions (1.4$'$) and (1.7), where $h(t) = |f(t,0,\dots,0)|$. Therefore by virtue of Theorem 1.1$'$ the problem (1.1$'$), (1.2) is solvable in the class $\widetilde{C}^{n-1,m}(I_n(a,b);\mathbb{R}^l)$.

To complete the proof of the theorem it remains for us to verify that the problem under consideration has at most one solution in the class $\widetilde{C}^{n-1,m}(I_n(a,b);\mathbb{R}^l)$.

Let $u, \bar{u} \in \widetilde{C}^{n-1,m}(I_n(a,b);\mathbb{R}^l)$ be two arbitrary solutions of the problem (1.1$'$), (1.2). We set $v(t) = u(t) - \bar{u}(t)$ for $t \in I_n(a,b)$.

It is clear that $v \in \widetilde{C}^{n-1,m}(I_n(a,b);\mathbb{R}^l)$ and $v(a+) = \dots = v^{(m-1)}(a+) = 0$, $v(b-) = \dots = v^{(n-m-1)}(b-) = 0$.

On the other hand, by the condition (1.14) from the equality

$$v^{(n)}(t) = f(t,u(t),\dots,u^{(n-1)}(t)) - f(t,\bar{u}(t),\dots,\bar{u}^{(n-1)}(t))$$

we have

$$(-1)^{n-m-1}S(v(t))v^{(n)}(t) \geq -\sum_{i=1}^{m} H_i(t)|v^{(i-1)}(t)|.$$

Therefore due to Lemma 2.2 $v(t) \equiv 0$, i.e., $u(t) \equiv \bar{u}(t)$. $\square$

Theorems 1.2 and 1.2$'$ are proved similarly to Theorems 1.1 and 1.1$'$, while Theorem 1.4 is proved similarly to Theorem 1.3 with the only difference being that Lemma 2.4 is used instead of Lemma 2.2.

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