

ON CONJUGACY OF HIGH-ORDER LINEAR ORDINARY  
DIFFERENTIAL EQUATIONS

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ABSTRACT. It is shown that the differential equation

$$u^{(n)} = p(t)u,$$

where  $n \geq 2$  and  $p : [a, b] \rightarrow \mathbb{R}$  is a summable function, is not conjugate in the segment  $[a, b]$ , if for some  $l \in \{1, \dots, n-1\}$ ,  $\alpha \in ]a, b[$  and  $\beta \in ]\alpha, b[$  the inequalities

$$n \geq 2 + \frac{1}{2}(1 + (-1)^{n-l}), \quad (-1)^{n-l}p(t) \geq 0 \quad \text{for } t \in [a, b],$$
$$\int_{\alpha}^{\beta} (t-a)^{n-2}(b-t)^{n-2}|p(t)|dt \geq l!(n-l)! \frac{(b-a)^{n-1}}{(b-\beta)(\alpha-a)},$$

hold.

Consider the differential equation

$$u^{(n)} = p(t)u, \tag{1}$$

where  $n \geq 2$ ,  $p \in L_{loc}(I)$ , and  $I \subset \mathbb{R}$  is an interval.

The following definitions will be used below.

Equation (1) is said to be conjugate in  $I$  if there exists a nontrivial solution of this equation with at least  $n$  zeroes (each zero counted accordingly to its multiplicity) in  $I$ .

Let  $l \in \{1, \dots, n-1\}$ . Equation (1) is said to be  $(l, n-l)$  conjugate in  $I$  if there exists a nontrivial solution  $u$  of this equation satisfying

$$u^{(i)}(t_1) = 0 \quad (i = 0, \dots, l-1),$$
$$u^{(i)}(t_2) = 0 \quad (i = 0, \dots, n-l-1),$$

with  $t_1, t_2 \in I$  and  $t_1 < t_2$ .

Suppose first that  $-\infty < a < b < +\infty$  and  $p \in L([a, b])$ .

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**Lemma.** *Let  $a < \alpha < \beta < b$ . Then the Green's function  $G$  of the problem*

$$\begin{aligned} u^{(n)}(t) &= 0 & \text{for } t \in [a, b], \\ u^{(j)}(a) &= 0 & (j = 0, \dots, l-1), \\ u^{(j)}(b) &= 0 & (j = 0, \dots, n-l-1), \end{aligned}$$

*satisfies the inequality*

$$\begin{aligned} &(-1)^{n-l}G(t, s) > \\ &> \frac{(b-\beta)(\alpha-a)(s-a)^{n-l-1}(b-s)^{l-1}(t-a)^{l-1}(b-t)^{n-l-1}}{(b-a)^{n-1}} \times \\ &\times \sum_{i=1}^{n-l} \frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} \quad \text{for } \alpha \leq t < s \leq \beta. \end{aligned} \quad (2)$$

*Proof.* The function  $G$  can be written in the form

$$G(t, s) = \begin{cases} \sum_{i=n-l+1}^n (-1)^{i-1} x_i(t) x_{n-i+1}(s) & \text{for } a \leq s < t \leq b, \\ -\sum_{i=1}^{n-l} (-1)^{i-1} x_i(t) x_{n-i+1}(s) & \text{for } a \leq t \leq s \leq b, \end{cases}$$

where

$$x_i(t) = \frac{(t-a)^{n-i}(b-t)^{i-1}}{(i-1)!(b-a)^{n-i}}.$$

It is easy to verify that for any fixed  $s \in ]a, b[$  the function  $\frac{(-1)^{n-l}G(\cdot, s)}{x_{n-l}(\cdot)x_{l+1}(s)}$  decreases on  $]a, b[$  and the function  $\frac{(-1)^{n-l}G(\cdot, s)}{x_{n-l+1}(\cdot)x_l(s)}$  increases on  $]a, b[$ . Thus

$$(-1)^{n-l}G(t, s) \geq (-1)^{n-l}G(s, s) \frac{x_{n-l}(t)}{x_{n-l}(s)} \quad \text{for } t \leq s. \quad (3)$$

Taking into account that

$$\begin{aligned} (-1)^{n-l}G(s, s) &= (-1)^{n-l-1} \sum_{i=1}^{n-l} (-1)^{i-1} x_i(s) x_{n-i+1}(s) = \\ &= \frac{(s-a)^{n-1}(b-s)^{n-1}}{(b-a)^{n-1}} \sum_{i=1}^n \frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} \end{aligned}$$

and

$$\frac{x_{n-l}(t)}{x_{n-l}(s)} = \frac{(t-a)^l(b-t)^{n-l-1}}{(s-a)^l(b-s)^{n-l-1}},$$

from the inequality (3) we deduce

$$\begin{aligned}
& (-1)^{n-l}G(t, s) \geq \\
& \geq \frac{(s-a)^{n-l-1}(b-s)^l(t-a)^l(b-t)^{n-l-1}}{(b-a)^{n-1}} \sum_{i=1}^{n-l} \frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} > \\
& > (b-\beta)(\alpha-a) \frac{(s-a)^{n-l-1}(b-s)^{l-1}(t-a)^{l-1}(b-t)^{n-l-1}}{(b-a)^{n-1}} \times \\
& \times \sum_{i=1}^{n-l} \frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} \quad \text{for } \alpha \leq t < s \leq \beta. \quad \square
\end{aligned}$$

**Theorem 1.** Let  $l \in \{1, \dots, n-1\}$ ,

$$n \geq 2 + \frac{1 + (-1)^{n-l}}{2} \quad \text{and} \quad (-1)^{n-l}p(t) \geq 0 \quad \text{for } t \in [a, b]. \quad (4)$$

If, in addition, there exist  $\alpha, \beta \in ]a, b[$  such that  $a < \alpha < \beta < b$  and

$$\int_{\alpha}^{\beta} (t-a)^{n-2}(b-t)^{n-2}|p(t)|dt \geq l!(n-l)! \frac{(b-a)^{n-1}}{(b-\beta)(\alpha-a)}, \quad (5)$$

then Eq. (1) is  $(l, n-l)$  conjugate in  $[a, b]$ .

Note that analogous results are given in [3,5] for the case where  $n = 2$ .

*Proof.* Put  $p(t) = 0$  for  $t > b$  and consider Eq. (1) in the interval  $[a, +\infty[$ . For any  $\gamma > a$ , let  $u_{\gamma}$  be the solution of (1) satisfying

$$\begin{aligned}
u_{\gamma}^{(i)}(a) &= 0 & (i = 0, \dots, l-1), \\
u_{\gamma}^{(i)}(\gamma) &= 0 & (i = 0, \dots, n-l-2), \\
\sum_{i=0}^{n-1} |u_{\gamma}^{(i)}(a)| &= 1, \quad \max\{u_{\gamma}(t) : a \leq t \leq \gamma\} > 0.
\end{aligned}$$

Suppose now that in spite of the statement of the theorem Eq. (1) is not  $(l, n-l)$  conjugate in  $[a, b]$ .

Note that if  $\gamma \in ]a, b]$ , then  $u_{\gamma}(t) > 0$  for  $t \in ]a, \gamma[$  and  $(-1)^{n-l-1} \times u_{\gamma}^{(n-l-1)}(\gamma) > 0$ . Indeed, if it is not so, there exists  $t_0 \in ]a, \gamma[$  such that  $u_{\gamma}(t_0) = 0$ . Let  $\gamma_0 = \inf\{\gamma > a : u_{\gamma}(t) = 0 \text{ for a certain } t \in ]a, \gamma[ \}$ . Then  $u_{\gamma_0}(t) > 0$  for  $t \in ]a, \gamma_0[$  and

$$\begin{aligned}
u_{\gamma_0}^{(i)}(a) &= 0 & (i = 0, \dots, l-1), \\
u_{\gamma_0}^{(i)}(\gamma_0) &= 0 & (i = 0, \dots, n-l-1),
\end{aligned}$$

which contradicts our assumption.

Let  $\gamma^0 = \sup\{\gamma > b : u_\gamma(t) > 0 \text{ for } t \in ]a, \gamma[ \}$ . Consider first the case where  $\gamma^0 = +\infty$ . There exists the sequence  $\{\gamma_k\}_{k=1}^{+\infty}$  such that

$$\lim_{k \rightarrow +\infty} \gamma_k = +\infty, \quad \lim_{k \rightarrow +\infty} u_{\gamma_k}(t) = u_0(t)$$

where  $u_0$  is the solution of Eq. (1). Show that

$$u_0(t) > 0 \quad \text{for } t > a. \quad (6)$$

It is clear that  $u_0(t) \geq 0$  for  $t > a$ . If now  $u_0(t_*) = 0$  for some  $t_* > a$ , then for any  $k$  large enough the function  $u'_{\gamma_k}$  will have at least one zero in  $]a, \gamma_k[$ . Taking into account the multiplicities of zeroes of  $u_{\gamma_k}$  in  $a$  and  $\gamma_k$ , it is easy to show that  $u_{\gamma_k}^{(n-1)}$  has at least two zeroes in  $]a, \gamma_k[$ . Hence  $u_{\gamma_k}^{(n)}$  changes sign in this interval and this is impossible.

Thus inequality (6) is proved. This inequality and the results of [1] imply that there exist  $l_0 \in \{1, \dots, n\}$  ( $l - l_0$  is even) and  $t_1 > b$  such that

$$\begin{aligned} u_0^{(i)}(t) &> 0 \quad \text{for } t \geq t_1 \quad (i = 0, \dots, l_0 - 1), \\ (-1)^{i+l_0} u_0^{(i)}(t) &\geq 0 \quad \text{for } t \geq t_1 \quad (i = l_0, \dots, n). \end{aligned} \quad (7)$$

Clearly,

$$\begin{aligned} (-1)^{i+l_0} u_0^{(i)}(t) &\geq 0 \quad \text{for } t \geq a \quad (i = l_0, \dots, n), \\ (-1)^{i+l_0} u_0^{(i)}(a) &> 0 \quad (i = l_0, \dots, n - 1). \end{aligned} \quad (8)$$

Hence  $l \in \{1, \dots, l_0\}$ .

Suppose that  $l < l_0$ . Then for any  $k$  large enough we have  $\gamma_k > t_1$ ,  $u_{\gamma_k}^{(i)}(t_1) > 0$  ( $i = 0, \dots, l_0 - 1$ ). This means that the function  $u_{\gamma_k}^{(i)}$  has at least one zero in  $]t_1, \gamma_k[$ . Taking into account the multiplicity of zero in  $\gamma_k$ , it is easy to see that  $u_{\gamma_k}^{(n-1)}$  has at least two zeroes in  $]t_1, \gamma_k[$ , and  $u_{\gamma_k}^{(n)}$  changes sign in this interval. But this is impossible. Thus  $l = l_0$ .

As  $l = l_0$ , inequalities (7) and (8) imply

$$\begin{aligned} (-1)^{i+l} u_0^{(i)}(t) &\geq 0 \quad \text{for } t \geq a \quad (i = l, \dots, n), \\ u_0^{(i)} &> 0 \quad \text{for } t > a \quad (i = 0, \dots, l - 1). \end{aligned}$$

Let

$$v(t) = u_0^{(l-1)}(t) - \sum_{j=l}^{n-1} \frac{(-1)^{j-l}}{(j-l+1)!} (t-a)^{j-l+1} u_0^{(j)}(t);$$

then

$$v'(t) = \frac{(-1)^{n-l}}{(n-l)!} (t-a)^{n-l} u_0^{(n)}(t) \geq 0 \quad \text{for } t \geq a.$$

Hence

$$\begin{aligned} u_0^{(l-1)}(t) &\geq \sum_{j=l}^{n-1} \frac{(-1)^{j-l}(t-a)^{j-l+1}u_0^{(j)}(t)}{(j-l+1)!} \geq \\ &\geq \frac{(t-a)(-1)^{n-l}}{(n-l)!} \int_t^{+\infty} (s-a)^{n-l-1}p(s)u_0(s)ds \quad \text{for } t \geq a. \end{aligned} \quad (9)$$

Denote  $\rho_i(t) = iu_0^{(l-i)}(t) - (t-a)u_0^{(l-i+1)}(t)$  for  $t \geq a$  ( $i = 0, 1, \dots, l$ ). Then  $\rho'_i(t) = \rho_{i-1}(t)$  for  $t \geq a$  ( $i = 1, \dots, l$ ). Since  $\rho_0(t) = -(t-a)u_0^{(l+1)}(t) \geq 0$  for  $t \geq a$  and  $\rho_i(a) = 0$  ( $i = 1, \dots, l$ ), we have  $\rho_i(t) \geq 0$  for  $t \geq a$  ( $i = 0, 1, \dots, l$ ). This implies

$$u_0(t) \geq \frac{(t-a)^{l-1}}{l!} u_0^{(l-1)}(t). \quad (10)$$

From (9) and (10) we obtain

$$1 \geq \frac{(t-a)}{l!(n-l)!} \int_t^\beta (s-a)^{n-2}|p(s)|ds \quad \text{for } t \geq a,$$

which contradicts (5). The case  $\gamma^0 = +\infty$  is thus eliminated.

Now consider the case where  $\gamma^0 < +\infty$ . As we have already noted,  $\gamma^0 > b$ ,  $u_{\gamma^0}(t) > 0$  for  $t \in ]a, \gamma^0[$  and

$$\begin{aligned} u_{\gamma^0}^{(i)}(a) &= 0 \quad (i = 0, \dots, l-1), \\ u_{\gamma^0}^{(i)}(\gamma^0) &= 0 \quad (i = 0, \dots, n-l-1). \end{aligned} \quad (11)$$

Hence

$$u_{\gamma^0}(t) = \int_a^{\gamma^0} G(t,s)p(s)u_{\gamma^0}(s)ds,$$

where  $G$  is the Green's function of the boundary value problem (11) for the equation  $u^{(n)} = 0$ .

Let  $t_0 \in ]\alpha, \beta[$  be such that

$$\begin{aligned} \frac{u_{\gamma^0}(t)}{(t-a)^{l-1}(\gamma^0-t)^{n-l-1}} &\geq \frac{u_{\gamma^0}(t_0)}{(t_0-a)^{l-1}(\gamma^0-t_0)^{n-l-1}} \\ &\text{for } t \in [\alpha, \beta]. \end{aligned} \quad (12)$$

Then from the lemma and the inequality (12) it follows that

$$\begin{aligned}
u_{\gamma^0}(t_0) &\geq (\gamma^0 - \beta)(\alpha - a) \sum_{i=1}^{n-l} \frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} \times \\
&\quad \times \int_{\alpha}^{\beta} \frac{(s-a)^{n-2}(\gamma^0-s)^{n-2}}{(\gamma^0-a)^{n-1}} |p(s)| ds u_{\gamma^0}(t_0) > \\
&> \frac{(b-\beta)(\alpha-a)}{(b-a)^{n-1}} \sum_{i=1}^{n-l} \frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} \times \\
&\quad \times \int_{\alpha}^{\beta} (s-a)^{n-2}(b-s)^{n-2} |p(s)| ds u_{\gamma^0}(t_0).
\end{aligned} \tag{13}$$

Since  $\sum_{i=1}^{n-l} \frac{(-1)^{n-l-i}}{(i-1)!(n-i)!} \geq \frac{1}{l!(n-l)!}$  the inequality (13) contradicts (5).  $\square$

Denote

$$\begin{aligned}
\mu_n^- &= \min\{l!(n-l)! : l \in \{1, \dots, n-1\}, n-l \text{ odd}\}, \\
\mu_n^+ &= \min\{l!(n-l)! : l \in \{1, \dots, n-1\}, n-l \text{ even}\}.
\end{aligned}$$

It is clear that

$$\mu_n^- = \begin{cases} \left(\frac{n}{2}-1\right)!\left(\frac{n}{2}+1\right)! & \text{for } n \equiv 0 \pmod{4}, \\ \left[\left(\frac{n}{2}\right)!\right]^2 & \text{for } n \equiv 2 \pmod{4}, \\ \left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)! & \text{for } n \equiv 1 \pmod{2}, \end{cases}$$

$$\mu_n^+ = \begin{cases} \left[\left(\frac{n}{2}\right)!\right]^2 & \text{for } n \equiv 0 \pmod{4}, \\ \left(\frac{n}{2}-1\right)!\left(\frac{n}{2}+1\right)! & \text{for } n \equiv 2 \pmod{4}, \\ \left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)! & \text{for } n \equiv 1 \pmod{2}. \end{cases}$$

**Corollary 1.** *Let either  $n \geq 2$ ,  $\mu_n = \mu_n^-$ , and  $p(t) \leq 0$  for  $t \in [a, b]$  or  $n \geq 3$ ,  $\mu_n = \mu_n^+$ , and  $p(t) \geq 0$  for  $t \in [a, b]$ . Let, moreover,  $\alpha, \beta \in ]a, b[$  exist such that  $a < \alpha < \beta < b$  and*

$$\int_{\alpha}^{\beta} |p(t)| dt \geq \mu_n \left( \frac{b-a}{(b-\beta)(\alpha-a)} \right)^{n-1}.$$

Then Eq. (1) is conjugate in  $[a, b]$ .

Note that  $\max\{l!(n-l)! : l \in \{1, \dots, n-1\}\} = (n-1)!$ . Thus from Theorem 1 easily follows

**Corollary 2.** Let  $l \in \{1, \dots, n-1\}$ , let the conditions (4) be fulfilled, and let  $\alpha, \beta \in ]a, b[$  exist such that  $a < \alpha < \beta < b$  and

$$\int_{\alpha}^{\beta} |p(t)| dt \geq (n-1)! \left( \frac{b-a}{(b-\beta)(\alpha-a)} \right)^{n-1}. \quad (14)$$

Then Eq. (1) is  $(l, n-l)$  conjugate in  $[a, b]$ .

Note that in the inequality (14) the factor  $(n-1)!$  cannot be replaced by  $(n-1)! - \varepsilon$  with  $\varepsilon \in ]0, 1[$ . This is shown by the following

**Example.** Let  $\varepsilon \in ]0, 1[$  be given beforehand and choose  $\alpha \in ]a, b[$  and  $\beta \in ]\alpha, b[$  such that

$$(n-1)! \left( \frac{\alpha-a}{\beta-a} \right)^{n-1} \left( \frac{b-\beta}{b-a} \right)^{n-1} > (n-1)! - \varepsilon. \quad (15)$$

Put

$$v(t) = \begin{cases} t-a & \text{for } t \in [a, \alpha], \\ \frac{\alpha+\beta}{2} - a - \frac{1}{2(\beta-\alpha)}(t-\beta)^2 & \text{for } t \in ]\alpha, \beta[, \\ \frac{\alpha+\beta}{2} - a & \text{for } t \in [\beta, b], \end{cases}$$

$$u_0(t) = \frac{1}{(n-3)!} \int_a^t (t-s)^{n-3} v(s) ds \quad \text{for } t \in [a, b], n \geq 3,$$

$$u_0(t) = v(t) \quad \text{for } t \in [a, b], n = 2,$$

and

$$p(t) = \frac{v''(t)}{u_0(t)} \quad \text{for } a < t < b.$$

Then the function  $u_0$  is non-decreasing for  $t \in [a, \beta]$ , and the inequality

$$u_0(t) \leq u_0(\beta) \leq \frac{1}{(n-3)!} \int_a^{\beta} (\beta-s)^{n-3} (s-a) ds = \frac{(\beta-a)^{n-1}}{(n-1)!}$$

is valid. Taking into account the inequality (15), we obtain

$$\begin{aligned} \int_{\alpha}^{\beta} p(t) dt &= \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \frac{dt}{u_0(t)} > \frac{(n-1)!}{(\beta-a)^{n-1}} > \\ &> ((n-1)! - \varepsilon) \left( \frac{b-a}{(b-\beta)(\alpha-a)} \right)^{n-1}. \end{aligned}$$

On the other hand, in the case considered, Eq. (1) is not conjugate in  $[a, b]$  because it has a solution  $u_0$  satisfying the following conditions:

$$u_0^{(i)}(a) = 0 \quad (i = 0, \dots, n-2), \quad u_0^{(n-1)}(a) = 1, \quad u_0(t) > 0 \quad \text{for } a < t \leq b.$$

This example shows that in Corollary 2 inequality (14) cannot be replaced by the inequality

$$\int_{\alpha}^{\beta} |p(t)| dt \geq ((n-1)! - \varepsilon) \left( \frac{b-a}{(b-\beta)(\alpha-a)} \right)^{n-1}$$

no matter how small  $\varepsilon > 0$  is.

Now consider Eq. (1) on the whole axis  $\mathbb{R}$  with  $p \in L_{loc}(\mathbb{R})$ . From Corollary 2 easily follows

**Corollary 3.** *Let  $l \in \{1, \dots, n-1\}$ ,  $p$  is not zero on the set of the positive measure and*

$$n \geq 2 + \frac{1 + (-1)^{n-l}}{2}, \quad (-1)^{n-l} p(t) \geq 0 \quad \text{for } t \in \mathbb{R}.$$

*Then Eq. (1) is  $(l, n-l)$  conjugate in  $\mathbb{R}$ .*

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