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On the n-uniformly close to convex functions with respect to a convex domain

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Abstract

Using the differential operator $D^n f$ introduced by G. Sălăgean and the results given in [2] and [3] on the several types of close-to-convex functions, in this paper I define new sets of univalent functions called n–uniformly close-to-convex with respect to a convex domain. In the definition of this functions it is specified which is the convex domain, symmetrical to the real axis, for example: eliptic, parabolic, hiperbolic region, or half plane. A certain analogy with the uniformly convex functions and with the functions of the class S_p defined by Goodman and by Frode Ronning respectively, justifies the name "uniformly close-to-convex".

In the second part, the intermediate classes of Mocanu type are also being defined.

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1 Introduction

Let

$$A = \{f/f \in \mathcal{H}(U)\ ,\ f(0) = 0\ ,\ f'(0) = 1\}$$
$$S = \{f \in A\ ,\ \text{and } f \text{ is univalent}\}\ .$$

Let D^n the Sălăgean differential operator defined as:

Definition 1 $D^n : \mathcal{H}(U) \longrightarrow \mathcal{H}(U)$ *and*

(i) $D^0 f(z) = f(z)$

(ii) $D^1 f(z) = Df(z) = zf(z)$

(iii) $D^n f(z) = D(D^{n-1} f(z))$.

We denote by $C(\alpha)$, $S^*(\alpha)$ and $CC(\alpha)$ the well-known subclass of S: convex, starlike and close-to-convex functions of order α, in other words with respect to a half-plane ($Re\, w > \alpha$). For example

$$CC(\alpha) = \left\{ f \in A \,/\, Re\frac{f'(z)}{g'(z)} > \alpha\ ,\ g \in C(0)\,,\, \alpha > 0\ z \in U \right\}\ .$$

Using the operator D^n, Gr. Sălăgean [15, 16] defines the set of n–starlike of order α functions noted $S_n(\alpha)$

$$S_n(\alpha) = \left\{ f \in A \,/\, Re\frac{D^{n+1}f(z)}{D^n f(z)} > \alpha\ ,\ z \in U\ ,\ \alpha \in [0,1)\ ,\ n \in N_0 \right\}$$

where $N_0 = \{0, 1, 2, \ldots\}$.

Remark 1 *If* $f \in S_n(\alpha)$ *then according to the Definition 1 we can write*

$$Re\frac{z(D^n f(z))'}{D^n f(z)} > \alpha \quad , \quad z \in U$$

therefore the function $F(z) = D^n f(z)$ *belongs to* $S^*(\alpha)$, $\alpha \in [0, 1)$.

The main results of this paper have been obtained by using the well-known "admissible functions method" introduced by S.S. Miller and P.T. Mocanu. I need the following special cases included in the theorems:

Theorem A [11, 12] *Let q be the convex in U and let $P : U \longrightarrow C$ with $Re\, P(z) > 0$. If p is analytic in U, then*

$$p(z) + P(z)zp'(z) \prec q(z) \Longrightarrow p(z) \prec q(z)\ .$$

Theorem B [4] *Let q be convex in U with $Re\,[\beta q(z)+\gamma]>0$. If p is analytic in U with $p(0)=q(0)$ then*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec q(z)\Longrightarrow p(z)\prec q(z)\ .$$

2 Preliminary results

In [1, 2, 3] the results are being given on so-called n-close-to-convex functions on order α with respect to a half-plane (or of Kaplan type) $CCK_n(\alpha)$ and n-close-to-convex functions of order α with respect to a sector (or of Rény type, or named later, strongly close-to-convex functions) noted $CCR_n(\alpha)$.

Definition 2 *The function $f\in A$ belongs to set $CCK_n(\alpha)$ if the differential expression $D^{n+1}f(z)/D^n g(z)$ take values in the halfplane $Re\,w>0$, that is*

$$CCK_n(\alpha)=\left\{f\in A/Re\frac{D^{n+1}f(z)}{D^n g(z)}>\alpha, g\in S_n(0), n\in N_0, \alpha\in[0,1), z\in U\right\}.$$

Remark 2 *For $n=0$, $CCK_0(\alpha)=CC(\alpha)$.*

Definition 3 *The function f is called n-close-to-convex of order γ with respect to a sector if it verifies the following conditions*

$$CCR_n(\gamma)=\left\{f\in A\ /\ \left|arg\frac{D^{n+1}f(z)}{D^n g(z)}\right|\le\frac{\pi}{2}\gamma\ ,\ \gamma\in[0,1),\ g\in SR_n(0)\right\}\ .$$

For $n=0$ the above definition can by expressed also in the form: $f(z)\in CCR_0(\gamma)$ if for every $0\le\theta_1<\theta_2\le 2\pi$

$$\int_{\theta_1}^{\theta_2}\left(1+Re\frac{zf''(z)}{f'(z)}\right)d\theta>-\pi\gamma\quad,\quad z=re^{i\theta}\ ,\ r\in(0,1)\ ,\ \gamma\in[0,1)\ ,$$

which has got the well-known geometric characterization.

Let now have an univalent function $q(z)$, $q(0) = 1$, $q'(0) > 0$ which maps the unit disc U into a symmetrical domain with respect to real axis.

Let $\mathcal{P}(q)$ the family of holomorphic functions p in U so that $p(0) = 1$ and $p(U) \subseteq q(U)$, in other words, $p \prec q$.

We note by $S^C(q)$ and $S^*(q)$ the classes of all univalent functions for which we have got

$$1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}(q) \quad \text{respectively} \quad \frac{zf'(z)}{f(z)} \in \mathcal{P}(q) \ .$$

The connection between the functions of $S^C(q)$ and $S^*(q)$ is given by a theorem of Alexander-type.

Theorem 1 *The function $f(z) \in S^C(q)$ if and only if $zf'(z) \in S^*(q)$ where $q(z)$, $S^C(q)$ and $S^*(q)$ verify the above conditions.*

Proof. By a simple classical calculus the conclusion of Theorem 1 follows.

Definition 4 *The function $f \in A$ is n–starlike with respect to convex domain $\mathcal{D}$ if the differential expression $D^{n+1}f(z) \,/\, D^n f(z)$ takes values in the domain $\mathcal{D}$ or*

$$\frac{D^{n+1}f(z)}{D^n f(z)} \prec q(z) \quad ; \quad q(U) = \mathcal{D} \ .$$

We can note by $S_n^*(q)$ the set of all these functions.

Remark 3 The special set $S_n^*\left(\dfrac{1+(1-2\alpha)z}{1-z}\right)$ called n–starlike of order α was studied by Gr. Sălăgean [15, 16]. The set $S_n^*\left[\left(\dfrac{1+z}{1-z}\right)^{\alpha}\right]$ n–starlike of Rény type was defined in [1]. Let be the set noted $S_n^*(Q(z))$ where $Q(z) = 1 + \dfrac{2}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$ which maps the unit disk U in the domain Ω bounded by a parabola

$$\Omega = \{w : |w-1| < Re\, W\} = \{W = u + iv \,,\, v^2 = 2u - 1\} \ .$$

Frode Ronning [13, 14], Ma and Minda [9] have independently introduced the class S_p, where $S_p = S_0^*[Q(z)]$. For $n = 1$ $S_1^*[Q(z)] = UCV$ the well-known set of uniformly convex functions, which was introduced by Goodman [5]. I.C. Magdaş [10] has studied the classes $S_n^*[Q(z)]$ for general n.

I. Stankiewiez, S. Kanas and A. Wisniowska [6, 7] have introduced and studied in detail the classes of K–uniformly convex and related classes of K–starlike functions ($0 \leq K < \infty$) denoted $K-UCV$ and $K-ST$ for which the values of expressions $\dfrac{zf'(z)}{f(z)}$ and $1 + \dfrac{zf''(z)}{f'(z)}$ lie inside the conic regions respectively, using the Sălăgean differential operator $D''f$, mentioned before, S. Kanas and Teuro Yaguchi have then introduced and extensively studied some subclones of $K - UCV$ and $K - ST$ [8].

3 Main results

1. A general family of close-to-convex functions.

Definition 5 *Let $q(z)$ be an univalent function $q(0) = 1$ $Re\, q(z) > 0$, $q'(0) > 0$ which maps the unit disc U into convex domain $\mathcal{D}$ symmetrical with respect to the real axis.*

Let be $f \in A$, we say that f is n–close-to-convex with respect to $\mathcal{D}$, or n–close-to-convex subordinated to function q, if there exists a function $q \in S_n(q)$ such that

$$\frac{D^{n+1}f(z)}{D^n g(z)} \prec q(z) \quad , \quad z \in U \ , \ n \in \mathbb{N}$$

We can note by $CC_n(q)$ the set of all these functions.

Remark 4 *From the above definition it easily results that $q_1(z) \prec q_2(z)$ implies $CC_n(q_1) \subset CC_n(q_2)$.*

Theorem 2 *If $n \in \mathbb{N}_0$ and $f \in CC_{n+1}(q)$ then $f \in CC_n(q)$. That is*

$$CC_{n+1}(q) \subset CC_n(q) \ .$$

Proof. With notation $\dfrac{D^{n+1}f(z)}{D^n g(z)} = p(z)$ we have $\dfrac{D^{n+2}f(z)}{D^{n+1}g(z)} = p(z) + \dfrac{1}{h(z)}zp'(z)$ where $h(z) = \dfrac{D^{n+1}g(z)}{D^n g(z)}$.

According to the Definition 4 and 5 it comes that $Re\, h(z) \geq 0$. It is easy to observe that

$$\psi(r, s) = r + \frac{1}{h(z)}s = p(z) + \frac{1}{h(z)}zp'(z)$$

is an admissible function according to the definition given by P.T. Mocanu and S.S. Miller [11, 12]. By dint of admissible function theory, see Theorem A, it follows that

$$p(z) + \frac{1}{h(z)}zp'(z) \prec q(z) \quad \text{implies} \quad p(z) \prec q(z)$$

that means the conclusion of Theorem 2.

Corollary 2 *Exists the following inclusions:*

$$CC_{n+1}(q) \subset CC_n(q) \subset CC_0(q) \subset CC$$

where CC - is the set of classical close-to-convex functions defined by Kaplan which are univalent. Hence we can assert that the sets $CC_n(q)$ contains only univalent functions.

2. If we choose for $q(z)$ some functions which maps the unit disc U into a domain bounded by a parabola, or elipsa, or hyperbola respectively, we obtain the special close-to-convex function which improves several previous results.

Since these functions have connections with the uniformly convex functions USC and the uniformly starlike functions UST and with the set SP, we call them n–uniformly close-to-convex functions.

Definition 6 *A function $f \in A$ is n-uniformly close-to-convex of order γ and type α where $\alpha \geq 0$, $\gamma \in [-1, 1)$, $\alpha + \gamma \geq 0$, and $n \in \mathbb{N}_0$ if there* *exists a function* $g \in US_n(\alpha, \gamma)$ *so that*

$$Re\,\frac{D^{n+1}f(z)}{D^n g(z)} \geq \alpha \left| \frac{D^{n+1}f(z)}{D^n g(z)} - 1 \right| + \gamma \qquad \forall\, z \in U\ .$$

We can note by $UCC_n(\alpha, \gamma)$ the sets of these functions.

Remark 5 *The geometric interpretation of the relation from Definition 6 it is that* $f \in UCC_n(\alpha, \gamma)$ *if and only if the differential expression* $D^{n+1}f(z)\,/\,D^n g(z)$ *takes all values into the region* $D_{\alpha,\gamma}$*, when it is:*

(i) *elliptic region*

$$\frac{\left(u - \dfrac{\alpha^2 - \gamma}{\alpha^2 - 1}\right)^2}{\left[\dfrac{\alpha(1-\gamma)^2}{\alpha^2 - 1}\right]^2} + \frac{v^2}{\left(\dfrac{1-\gamma}{\sqrt{\alpha^2 - 1}}\right)^2} < 1 \quad for \quad \alpha > 1\ ;$$

(ii) *parabolic region*

$$v^2 < 2(1-\gamma)u - (1-\gamma^2) \quad for \quad \alpha = 1\ ;$$

(iii) *hiperbolic region*

$$\frac{\left(u - \dfrac{\gamma - \alpha^2}{1 - \alpha^2}\right)^2}{\left[\dfrac{\alpha(1-\gamma)}{1 - \alpha^2}\right]^2} - \frac{v^2}{\left(\dfrac{1-\gamma}{\sqrt{1 - \alpha^2}}\right)^2} > 1 \qquad \begin{array}{l} and \quad u > 0 \\ for \quad \alpha \in (0,1) \end{array} \ ;$$

(iv) *half plane*

$$u > \gamma \quad for \quad \alpha = 0.$$

In all this cases we have

$$Re\left\{\frac{D^{n+1}f(z)}{D^n g(z)}\right\} > \frac{\alpha + \gamma}{\alpha + 1}\ .$$

That is $UCC_n(\alpha, \gamma) \subset CCK_n\left(\dfrac{\alpha+\gamma}{\alpha+1}\right) \subset CC$.

a) *If* $f(z) \equiv g(z)$ $\quad UCC_n(\alpha,\gamma) = US_n(\alpha,\gamma)$ *defined by I.C. Magdaş [9].*

b) *If* $g(z) \equiv f(z)$ *and* $\alpha = k,\ \gamma = 0,\qquad UCC_n(k,0) = (k,n) - UCV$ *the subclones introduced and studied by Stansislawa Kanas and Teuro Yaguchi* [8].

c) *For* $n = 1,\ f(z) = g(z),\ \alpha = 1,\ \gamma = 0$ *we obtain the classical definition of uniformly convex functions* USC *introduced by Goodman* [5].

d) *For* $n = 0,\ f(z) = g(z),\ \alpha = 1,\ \gamma = 0$ *we rediscovered the definition for the function which belongs to the set* SP*, introduced by Frode Ronning [13, 14].*

Remark 6 *All the functions of the sets* $UCC_n(\alpha,\gamma)$ *verify the conclusions of the Remark 4, Theorem 2 and Corollary 2.*

4 Intermediate classes

Definition 7 *For* $\beta \in \mathbb{R},\ n \in \mathbb{N}_0,\ g \in US_n(\alpha)$ *we denote by*

$$J(n,\beta,g;f(z)) = (1-\beta)\frac{D^{n+1}f(z)}{D^n g(z)} + \beta\frac{D^{n+2}f(z)}{D^{n+1}g(z)} \qquad z \in U\ .$$

We say that f *is* n*–uniform by* β *close-to-convex Mocanu function iff*

$$J(n,\beta,g,f) \prec Q(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 \qquad z \in U$$

and we denote by $UCCM_n(\beta)$ *the set of all these functions.*

Theorem 4 *If* $n \in \mathbb{N}_0,\ \beta > 0,\ UCCM_n \subset UCC_n$.

Proof. If we denote $D^{n+1}f(z)\ /\ D^n g(z) = p(z)$ we obtain

$$J(n,\beta,g,f) = p(z) + \frac{\beta}{h(z)}\cdot zp'(z)\ .$$

But $Re\,\beta\,/\,h(z) > 0$ follows that

$$\psi(p(z), zp'(z)) = p(z) + \frac{\beta}{h(z)}zp'(z)$$

is an "admissible function" and according to the "admissible functions method" it follows that

$$\beta(z) + \frac{\beta}{h(z)} zp'(z) \prec Q(z) \longrightarrow p(z) \prec Q(z)$$

is the conclusion of the last theorem.

Remark 7 *The function $id\, z = z$ belongs to all these classes.*

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