General Mathematics Vol. 9, No. 1–2 (2001), 11–14

Some Inequalities for Certain Means in Two Arguments

Dumitru Acu

Abstract

In this paper the inequalities (1) are established.

2000 Mathematical Subject Classification: 26D20

1

The means in two arguments are important for a variety of reasons (see [3]). They have been intensively studied. Power means are among the most extensively investigated means. The theory of Gini means is closely connected to the theory of generalized entropies, data compression and Information Theory (see [1], [4], [5]).

Let $0 < a < b$. The power mean is

$$M_p = \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{\frac{1}{p}} & , \ p \neq 0 \ , \\ (ab)^{\frac{1}{2}} & , \ p = 0 \ ; \end{cases}$$

and the Gini mean, is

$$S_p = \begin{cases} \left(\dfrac{a^{p-1} + b^{p-1}}{a+b}\right)^{\frac{1}{p-2}} & , \ p \neq 2 \ , \\ (a^a b^b)^{\frac{1}{a+b}} & , \ p = 2 \ . \end{cases}$$

In the paper [4] the authors propose the following conjecture: the inequalities

$$\frac{S_p}{M_p}\begin{cases} <1 & , \quad p\in(0,1) \\ =1 & , \quad p\in\{0,1\} \\ >1 & , \quad p\in(-\infty,0)\cup(1,\infty) \end{cases} \tag{1}$$

are satisfied.

In this paper we study this conjecture.

2

We prove the following results.

Theorem. *The following inequalities are satisfied*

$$\frac{S_p}{M_p}\begin{cases} <1 & , \quad p\in(0,1\cup(1,2)) \\ =1 & , \quad p\in\{0,1\} \\ >1 & , \quad p\in(-\infty,0)\cup[2,\infty)\ . \end{cases}$$

Proof. It is easy to verify this theorem for $p\in\{0,1\}$. The inequality $S_2\,/\,M_2>1$ has been proved in [6] and [4].

For $p\in\mathbb{R}-\{0,1,2\}$ we denote $t:=b\,/\,a>1$ and we have

$$\frac{S_p}{M_p}=\frac{\left(\dfrac{1+t^{p-1}}{1+t}\right)^{\frac{1}{p-2}}}{\left(\dfrac{1+t^p}{2}\right)^{\frac{1}{p}}}\ .$$

We consider the function

$$f(t)=\frac{p}{p-2}\ln\frac{1+t^{p-1}}{1+t}-\ln\frac{1+t^p}{2}\quad , \quad t>1\ .$$

Simple calculation give

$$f'(t)=\frac{p}{p-2}\cdot\frac{t^{2p-2}-(p-1)t^p+(p-1)t^{p-2}-1}{(1+t)(1+t^{p-1})(1+t^p)}$$

Denoting

$$g(t) = t^{2p-2} - (p-1)t^p + (p-1)t^{p-2} - 1 \quad , \quad t > 1 \; ,$$

we have

$$g'(t) = (p-1)t^{p-3}(2t^p - pt^2 + p - 2) \; .$$

If

$$h(t) = 2t^p - pt^2 + p - 2 \quad , \quad t > 1 \; ,$$

then we obtain

(2) $$h'(t) = 2p(t^{p-1} - 1) \; .$$

We consider four cases.

2.1. If $p > 2$ we have $h'(t) > 0$ for $t > 1$. This relations implies that the function h is increasing. From $h(1) = 0$ we obtain $h(t) > 0$ for $t > 1$. Then $g'(t) > 0$ for $t > 1$, where it results that the functions g is increasing. From $g(1) = 0$ we obtain $g(t) > 0$ for $t > 1$. Hence, $f'(t) > 0$ for $t > 1$ which implies that $f(t) > f(1) = 0$. From $f(t) > 0$ for $t > 1$, we obtain $S_p \,/\, M_p > 1$.

2.2. If $p \in (1, 2)$ then $h'(t) > 0$ for $t > 1$. We infer that h is an increasing function on $(1, \infty)$. From $h(1) = 0$ we find $h(t) > 0$ for $t > 1$. Then $g'(t) > 0$ for $t > 1$, where we find $g(t) > g(1) = 0$. Now, we obtain $f'(t) < 0$ for $t > 1$ which implies that $f(t) < f(1) = 0$. From $f(t) < 0$ for $t > 1$, we find $S_p \,/\, M_p < 1$.

2.3. If $p \in (0, 1)$, then we have $h'(t) < 0$, which implies that $h(t) < h(1) = 0$. From this inequality we conclude that $g'(t) > 0$ for $t > 1$. As above we obtain that $g(t) > g(1) = 0$. It is easy to observe that $f'(t) < 0$ which gives $f(t) < f(1) = 0$ for $t > 1$. Thus $S_p \,/\, M_p < 1$ is verified.

2.4. Finally, we consider $p < 0$. As above from (2) we obtain that $h'(t) > 0$ for $t > 1$. This implies that $h(t) > h(1) = 0$ which gives $g'(t) < 0$ for $t > 1$. From this inequality we have $g(t) < g(1) = 0$. Of course, it follows that the function f is increasing, where we have

$f(t) > f(1) = 0$ for $t > 1$. From this inequality we conclude that $S_p \,/\, M_p < 1$ holds.

This completes the proof of the theorem.

References

[1] D. Farnsworth, R. Orr, *Gini means*, Amer. Math. Monthly 93 (1986), 603–607.

[2] F. Qi, *Generalized weighted mean values with two parameters*, Proc. R. Soc. Lond. 454 (1998), 2723–2732.

[3] G. Lorenzen, *Why means in two arguments are special*, Elem. Math. 49 (1994), 32–37.

[4] M. Ivan, I. Raşa, *Some Inequalities for means II*, Proceedings of the Tiberiu Popoviciu Itinerant Seminar of Functional Equations, Approximation and Convexity, Cluj–Napoca, May 22 – May 2001, 75–79.

[5] L.L. Persson, S. Sjöstrand, *On generalized Gini means and scales of means*, Results Math. 18 (1990), 320–332.

[6] I. Sándor, I. Raşa, *Inequalities for certain means in two arguments*, Nieuw Archief voor Wiskunde 15, No.1–2 (1997), 51–55.

Department of Mathematics
Faculty of Sciences
University "Lucian Blaga" of Sibiu
Str.I.Raţiu, No.7,
2400 Sibiu, Romania
E-mail: *depmath@ulbsibiu.ro*