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On Certain Subclass of Meromorphic p-valent Functions with Negative Coefficients [1]

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Abstract

We introduce a class $\Sigma^*_{\lambda,p,\alpha}$ of meromorphic p-valent functions with negative coefficients. The main object of this paper is to investigate various important properties and characteristics of this class. Further many interesting results for the Hadamard product of functions belonging to the class $\Sigma^*_{\lambda,p,\alpha}$ is obtained.

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1 Introduction

Let Σ_p denote the class of functions of the form

$$f(z) = \frac{1}{z^p} - \sum_{k=p}^{\infty} a_k z^k \quad (a_k \geq 0;\ p \in \mathbb{N} = \{1, 2, \dots\}). \tag{1}$$

Various subclasses of Σ_p were studied by Aouf et al. [1, 2, 3], Srivastava et al. [8], Uralegaddi and Ganigi [10].

A function $f \in \Sigma_p$ is said to be in the class $\Sigma^*_p(\alpha)$ of meromorphic p-valent starlike of order α if

$$\Re\left[\frac{-zf'(z)}{f(z)}\right] > \alpha \quad (z \in \Delta;\ 0 \leq \alpha < p).$$

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Motivated by the works of Chen et al. [4] and Aouf [1], we here define the class $\Sigma^*_{\lambda,p,\alpha}$ as follows.

A function $f(z) \in \Sigma_p$ is said to be in the class $\Sigma^*_{\lambda,p,\alpha}$, if it satisfies the following inequality:

$$\Re\left[\frac{(\lambda(p+1)-1)zf'(z)+\lambda z^2 f''(z)}{f(z)}\right] > \alpha \quad (0 \leq \alpha < p;\ p \in \mathbb{N}) \tag{2}$$

for some suitably restricted real parameter λ.

We see that $\Sigma^*_{0,p,\alpha} = \Sigma^*_p(\alpha)$.

In this paper, we investigate various properties and characteristics of the class $\Sigma^*_{\lambda,p,\alpha}$ by restricting the real parameter λ as $\lambda > \frac{1}{p}$. Further many interesting results for the Hadamard products of functions belonging to the class $\Sigma^*_{\lambda,p,\alpha}$ $\left(\lambda > \frac{1}{p}\right)$ is also obtained.

2 Coefficient Estimates and Closure Theorem

We first establish a necessary and sufficient condition for a function $f(z)$, given by (1), to be in the class $\Sigma^*_{\lambda,p,\alpha}$ and obtain the coefficient estimates.

Theorem 1 *A function $f(z)$ given by (1) is in $\Sigma^*_{\lambda,p,\alpha}$ if and only if,*

$$\sum_{k=p}^{\infty}[k(\lambda(p+k)-1)-\alpha]a_k \leq p-\alpha \quad (0 \leq \alpha < p;\ \lambda > \frac{1}{p};\ p \in \mathbb{N}) \tag{3}$$

Proof. Suppose $f(z)$ is in $\Sigma^*_{\lambda,p,\alpha}$. Then from (1) and (2),

$$\Re\left[\frac{(\lambda(p+1)-1)zf'(z)+\lambda z^2 f''(z)}{f(z)}\right] > \alpha \quad (0 \leq \alpha < p;\ \lambda > \frac{1}{p})$$

or equivalently,

$$\Re\left[\frac{p-\sum_{k=p}^{\infty}[k(\lambda(p+1)-1)+k(k-1)\lambda]a_k z^{k+p}}{1-\sum_{k=p}^{\infty} a_k z^{k+p}}\right] > \alpha.$$

Choosing values of z on the real axis and letting $z \to 1^-$, we obtain

$$\frac{p-\sum_{k=p}^{\infty}[k(\lambda(p+1)-1)+k(k-1)\lambda]a_k}{1-\sum_{k=p}^{\infty} a_k} \geq \alpha$$

which implies

$$\sum_{k=p}^{\infty}[k(\lambda(p+k)-1)-\alpha]a_k \leq p-\alpha,$$

proving (3).

Conversely, assume that (3) is true.

Then, if we let $z \in \partial U = \{z : z \in \mathbb{C} \text{ and } |z| = 1\}$, we find that

$$\begin{aligned}
&\left|\frac{(\lambda(p+1)-1)zf'(z)+\lambda z^2 f''(z)}{f(z)} - p\right| \\
&\quad \leq \frac{\sum_{k=p}^{\infty}[k(\lambda(p+1)-1)+k(k-1)\lambda-p]a_k|z|^{k+p}}{1-\sum_{k=p}^{\infty}a_k|z|^{k+p}} \\
&\quad = \frac{\sum_{k=p}^{\infty}[k(\lambda(p+1)-1)+k(k-1)\lambda-p]a_k}{1-\sum_{k=p}^{\infty}a_k} \\
&\quad \leq (p-\alpha); \quad (0 \leq \alpha < p;\ \lambda > \frac{1}{p};\ p \in \mathbb{N}).
\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f(z) \in \Sigma^*_{\lambda,p,\alpha}$. This completes the proof of the Theorem 1.

Corollary 1 *Let the function $f(z)$ defined by* (1) *be in the class $\Sigma^*_{\lambda,p,\alpha}$. Then*

$$a_k \leq \frac{p-\alpha}{k(\lambda(p+k)-1)-\alpha}, \quad (k \geq p;\ p \in \mathbb{N}). \tag{4}$$

The equality in (4) *is attained for the function $f(z)$ given by*

$$f(z) = \frac{1}{z^p} - \frac{p-\alpha}{k(\lambda(p+k)-1)-\alpha}z^k, \quad (k \geq p;\ p \in \mathbb{N}). \tag{5}$$

Theorem 2 *The class $\Sigma^*_{\lambda,p,\alpha}$ is closed under convex linear combinations.*

Proof. Let each of the functions

$$f(z) = \frac{1}{z^p} - \sum_{k=p}^{\infty} a_k z^k$$

and

$$g(z) = \frac{1}{z^p} - \sum_{k=p}^{\infty} b_k z^k$$

be in the class $\Sigma^*_{\lambda,p,\alpha}$.

It is sufficient to show that the function $h(z)$ defined by

$$h(z) = (1-\gamma)f(z) + \gamma g(z) \quad (0 \leq \gamma \leq 1)$$

is also in the class $\Sigma^*_{\lambda,p,\alpha}$. Since

$$h(z) = \frac{1}{z^p} - \sum_{k=p}^{\infty}[(1-\gamma)a_k + \gamma b_k]z^k \quad (0 \leq \gamma \leq 1),$$

with the aid of Theorem 1, we have

$$\begin{aligned}
&\sum_{k=p}^{\infty}[k(\lambda(p+k)-1)-\alpha]\{(1-\gamma)a_k + \gamma b_k\} \\
&\quad = (1-\gamma)\sum_{k=p}^{\infty}[k(\lambda(p+k)-1)-\alpha]a_k + \gamma\sum_{k=p}^{\infty}[k(\lambda(p+k)-1)-\alpha]b_k \\
&\quad \leq (1-\gamma)(p-\alpha) + \gamma(p-\alpha) \\
&\quad = p-\alpha; \quad (0 \leq \alpha < p;\ \lambda > \frac{1}{p};\ p \in \mathbb{N}),
\end{aligned}$$

which shows that $h(z) \in \Sigma^*_{\lambda,p,\alpha}$. Hence Theorem 2 is proved.

3 Distortion Theorem

In this section, we prove the following growth and distortion theorem for the class $\Sigma^*_{\lambda,p,\alpha}$.

Theorem 3 *If a function $f(z)$ defined by* (1) *is in the class $\Sigma^*_{\lambda,p,\alpha}$, then*

$$\begin{aligned}
&\left\{\frac{(p+m-1)!}{(p-1)!} - \frac{p!}{(p-m)!}\left[\frac{p-\alpha}{p(2\lambda p-1)-\alpha}\right]r^{2p}\right\}r^{-(p+m)} \leq |f^{(m)}(z)| \\
&\qquad \leq \left\{\frac{(p+m-1)!}{(p-1)!} + \frac{p!}{(p-m)!}\left[\frac{p-\alpha}{p(2\lambda p-1)-\alpha}\right]r^{2p}\right\}r^{-(p+m)}, \qquad (6)
\end{aligned}$$

$$(0 < |z| = r < 1;\ 0 \leq \alpha < p;\ \lambda > \tfrac{1}{p};\ p \in \mathbb{N};\ p > m).$$

The result is sharp for the function given by

$$f(z) = \frac{1}{z^p} - \frac{p-\alpha}{p(2\lambda p-1)-\alpha}z^k, \quad (k \geq p;\ p \in \mathbb{N}) \qquad (7)$$

Proof. In view of Theorem 1, we have

$$\frac{p(2\lambda p-1)-\alpha}{p!}\sum_{k=p}^{\infty} k!a_k \leq \sum_{k=p}^{\infty}[k(\lambda(p+k)-1)-\alpha]a_k \leq p-\alpha$$

which gives

$$\sum_{k=p}^{\infty} k!a_k \leq \frac{(p-\alpha)p!}{p(2\lambda p-1)-\alpha} \quad (p \in \mathbb{N}). \tag{8}$$

Now, by differentiating both sides of (1) m times with respect to z, we have

$$f^{(m)}(z) = (-1)^m \frac{(p+m-1)!}{(p-1)!} z^{-(p+m)} - \sum_{k=p}^{\infty} \frac{k!}{(k-m)!} a_k z^{k-m} \quad (p \in \mathbb{N}; p > m) \tag{9}$$

and Theorem 3 now follows easily from (8) and (9).

Further, it is easy to see that the equality in (6) are attained for the function $f(z)$ given by (7).

4 Radii of Meromorphically p-valent Starlikeness and Convexity

In this section, we determine the radii of meromorphically p-valent starlikeness of order δ $(0 \leq \delta < p)$ and meromorphically p-valent convexity of order δ $(0 \leq \delta < p)$ for functions in the class $\Sigma^*_{\lambda,p,\alpha}$.

Theorem 4 *Let the function* $f(z)$ *defined by* (1) *be in the class* $\Sigma^*_{\lambda,p,\alpha}$, *then*

(i) $f(z)$ *is meromorphically* p*-valent starlike of order* δ $(0 \leq \delta < p)$ *in the disc* $|z| < r_1$. *That is,*

$$\Re\left\{\frac{-zf'(z)}{f(z)}\right\} > \delta \quad (|z| < r_1;\ 0 \leq \delta < p;\ p \in \mathbb{N}),$$

where

$$r_1 = \inf_{k \geq p}\left\{\frac{(p-\delta)[k(\lambda(p+k)-1)-\alpha]}{(k+\delta)(p-\alpha)}\right\}^{\frac{1}{k+p}}. \tag{10}$$

(ii) *$f(z)$ is meromorphically p-valent convex of order δ $(0 \le \delta < p)$ in the disc $|z| < r_2$. That is*

$$\Re\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \delta \quad (|z| < r_2;\ 0 \le \delta < p;\ p \in \mathbb{N}),$$

where

$$r_2 = \inf_{k \ge p}\left\{\frac{p(p-\delta)[k(\lambda(p+k)-1)-\alpha]}{k(k+\delta)(p-\alpha)}\right\}^{\frac{1}{k+p}}. \tag{11}$$

Each of these results are sharp for the function $f(z)$ given by (5).

Proof.

(i) From the definition of $f(z)$ given by (1), we get

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\delta}\right| \le \frac{\sum_{k=p}^{\infty}(k+p)a_k|z|^{k+p}}{2(p-\delta)-\sum_{k=p}^{\infty}(k-p+2\delta)a_k|z|^{k+p}}.$$

Thus we have the desired inequality:

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\delta}\right| \le 1 \quad (0 \le \delta < p;\ p \in \mathbb{N})$$

if

$$\sum_{k=p}^{\infty}\left(\frac{k+\delta}{p-\delta}\right)a_k|z|^{k+p} \le 1. \tag{12}$$

Hence, by Theorem 1, (12) will be true if

$$\left(\frac{k+\delta}{p-\delta}\right)|z|^{k+p} \le \frac{k(\lambda(p+k)-1)-\alpha}{p-\alpha}, \quad (k \ge p; \lambda > \frac{1}{p};\ p \in \mathbb{N}). \tag{13}$$

The inequality (13) lead us immediately to the disc $|z| < r_1$, where r_1 is given by (10).

(ii) In order to prove the second assertion of Theorem 4, we find from the definition of $f(z)$ given by (1) that

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p+2\delta}\right| \le \frac{\sum_{k=p}^{\infty}k(k+p)a_k|z|^{k+p}}{2p(p-\delta)-\sum_{k=p}^{\infty}k(k-p+2\delta)a_k|z|^{k+p}}.$$

Thus we have

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p+2\delta}\right| \leq 1 \quad (0 \leq \delta < p;\ p \in \mathbb{N}),$$

if

$$\sum_{k=p}^{\infty} \frac{k(k+\delta)}{p(p-\delta)} a_k |z|^{k+p} \leq 1. \tag{14}$$

Hence, by Theorem 1, (14) will be true if

$$\frac{k(k+\delta)}{p(p-\delta)}|z|^{k+p} \leq \frac{k(\lambda(p+k)-1)-\alpha}{p-\alpha}, \quad (k \geq p; \lambda > \frac{1}{p};\ p \in \mathbb{N}). \tag{15}$$

This inequality (15) readily yields the disc $|z| < r_2$, where r_2 is given by (11). Clearly the results are sharp for the function $f(z)$ defined by (5), which completes the proof of Theorem 4.

5 Convolution Properties

Let

$$f_j(z) = \frac{1}{z^p} - \sum_{k=p}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geq 0;\ j = 1, 2;\ p \in \mathbb{N}). \tag{16}$$

Then the convolution (or Hadamard product) $(f_1 * f_2)(z)$ of the functions $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = \frac{1}{z^p} - \sum_{k=p}^{\infty} a_{k,1} a_{k,2} z^k.$$

Theorem 5 *Let $f_j(z)$ $(j = 1, 2)$ defined by* (16) *be in the class $\Sigma^*_{\lambda,p,\alpha}$. Then $(f_1 * f_2)(z) \in \Sigma^*_{\lambda,p,\gamma}$ where*

$$\gamma = p - \frac{2p(p-\alpha)^2(\lambda p - 1)}{(p(2\lambda p - 1) - \alpha)^2 - (p-\alpha)^2}.$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_j(z) = \frac{1}{z^p} - \frac{p-\alpha}{p(2\lambda p - 1) - \alpha} z^p \quad (j = 1, 2;\ p \in \mathbb{N}). \tag{17}$$

Proof. Since $f_j(z) \in \Sigma^*_{\lambda,p,\alpha}$ $(j = 1, 2)$, we have

$$\sum_{k=p}^{\infty} \left(\frac{k(\lambda(p+k)-1)-\alpha}{p-\alpha} \right) a_{k,j} \leq 1.$$

Using Cauchy-Schwarz inequality, we obtain

$$\sum_{k=p}^{\infty} \left(\frac{k(\lambda(p+k)-1)-\alpha}{p-\alpha} \right) \sqrt{a_{k,1}a_{k,2}} \leq 1.$$

We have to find a largest γ such that

$$\sum_{k=p}^{\infty} \left(\frac{k(\lambda(p+k)-1)-\gamma}{p-\gamma} \right) a_{k,1}a_{k,2} \leq 1. \tag{18}$$

(18) is satisfied if

$$\left(\frac{k(\lambda(p+k)-1)-\gamma}{p-\gamma} \right) a_{k,1}a_{k,2} \leq \left(\frac{k(\lambda(p+k)-1)-\alpha}{p-\alpha} \right) \sqrt{a_{k,1}a_{k,2}}$$

or if

$$\sqrt{a_{k,1}a_{k,2}} \leq \left(\frac{p-\gamma}{p-\alpha} \right) \left[\frac{k(\lambda(p+k)-1)-\alpha}{k(\lambda(p+k)-1)-\gamma} \right].$$

But

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{p-\alpha}{k(\lambda(p+k)-1)-\alpha}.$$

Thus (18) will be satisfied if

$$\frac{p-\alpha}{k(\lambda(p+k)-1)-\alpha} \leq \left(\frac{p-\gamma}{p-\alpha} \right) \left[\frac{k(\lambda(p+k)-1)-\alpha}{k(\lambda(p+k)-1)-\gamma} \right].$$

or if

$$\gamma \leq p - \frac{(p-\alpha)^2[k(\lambda(p+k)-1)-p]}{[k(\lambda(p+k)-1)-\alpha]^2-(p-\alpha)^2}, \quad (k \geq p).$$

The right side of this inequality is an increasing function of k.
On setting $k = p$, we get

$$\gamma \leq p - \frac{(p-\alpha)^2(p(2\lambda p-1)-p)}{(p(2\lambda p-1)-\alpha)^2-(p-\alpha)^2}.$$

Thus $p - \dfrac{2p(p-\alpha)^2(\lambda p-1)}{(p(2\lambda p-1)-\alpha)^2-(p-\alpha)^2}$ is the largest γ such that (18) is true, which implies that $(f_1 * f_2)(z) \in \Sigma^*_{\lambda,p,\gamma}$, thereby completing the proof of Theorem 5.

Theorem 6 *Let $f_1(z)$ be in the class $\Sigma^*_{\lambda,p,\alpha}$. Suppose if $f_2(z) \in \Sigma^*_{\lambda,p,\beta}$. Then $(f_1 * f_2)(z) \in \Sigma^*_{\lambda,p,\xi}$ where*

$$\xi = p - \frac{2p(p-\alpha)^2(p-\beta)^2(\lambda p - 1)}{(p(2\lambda p-1)-\alpha)^2(p(2\lambda p-1)-\beta)^2-(p-\alpha)^2(p-\beta)^2}.$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_1(z) = \frac{1}{z^p} - \frac{p-\alpha}{p(2\lambda p-1)-\alpha} z^p$$

and

$$f_2(z) = \frac{1}{z^p} - \frac{p-\beta}{p(2\lambda p-1)-\beta} z^p.$$

Proof. If $f_1(z) \in \Sigma^*_{\lambda,p,\alpha}$ and $f_z(z) \in \Sigma^*_{\lambda,p,\beta}$, then

$$\sum_{k=p}^{\infty} \left[\frac{k(\lambda(k+p)-1)-\alpha}{p-\alpha}\right] a_{k,1} \leq 1$$
$$\text{and } \sum_{k=p}^{\infty} \left[\frac{k(\lambda(k+p)-1)-\beta}{p-\beta}\right] a_{k,2} \leq 1.$$

By Cauchy-Schwarz inequality

$$\sum_{k=p}^{\infty} \left[\frac{k(\lambda(k+p)-1)-\alpha}{p-\alpha}\right]\left[\frac{k(\lambda(k+p)-1)-\beta}{p-\beta}\right] \sqrt{a_{k,1}a_{k,2}} \leq 1.$$

We have to find a largest ξ so that

$$\sum_{k=p}^{\infty} \left(\frac{k(\lambda(k+p)-1)-\xi}{p-\xi}\right) a_{k,1}a_{k,2} \leq 1. \tag{19}$$

(19) is satisfied if,

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(p-\xi)}{(p-\alpha)(p-\beta)} \left[\frac{(k(\lambda(k+p)-1)-\alpha)(k(\lambda(k+p)-1)-\beta)}{(k(\lambda(k+p)-1)-\xi)}\right].$$

But

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(p-\alpha)(p-\beta)}{(k(\lambda(k+p)-1)-\alpha)(k(\lambda(k+p)-1)-\beta)}.$$

Thus (19) will be satisfied if

$$\frac{(p-\alpha)(p-\beta)}{(k(\lambda(k+p)-1)-\alpha)(k(\lambda(k+p)-1)-\beta)} \leq \frac{(p-\xi)}{(p-\alpha)(p-\beta)}\left[\frac{(k(\lambda(k+p)-1)-\alpha)(k(\lambda(k+p)-1)-\beta)}{(k(\lambda(k+p)-1)-\xi)}\right]$$

or if,

$$\xi \leq p - \frac{(p-\alpha)^2(p-\beta)^2(k(\lambda(p+k)-1)-p)}{(k(\lambda(p+k)-1)-\alpha)^2(k(\lambda(p+k)-1)-\beta)^2-(p-\alpha)^2(p-\beta)^2}.$$

We see that the right side of this inequality is an increasing function of k. On setting $k = p$, we get

$$\xi \leq p - \frac{2p(p-\alpha)^2(p-\beta)^2(\lambda p-1)}{(p(2\lambda p-1)-\alpha)^2(p(2\lambda p-1)-\beta)^2-(p-\alpha)^2(p-\beta)^2},$$

which is the largest ξ such that (19) is true, which implies $(f_1 * f_2)(z) \in \Sigma^*_{\lambda,p,\xi}$.

Theorem 7 *Let the functions* $f_j(z)$ $(j = 1, 2)$ *defined by* (16) *be in the class* $\Sigma^*_{\lambda,p,\alpha}$. *Then the function* $h(z)$ *defined by* $h(z) = \frac{1}{z^p} - \sum_{k=p}^{\infty}\left(a^2_{k,1} + a^2_{k,2}\right) z^k$ *belongs to the class* $\Sigma^*_{\lambda,p,\zeta}$ *where*

$$\zeta = p\left[1 - \frac{2(p-\alpha)^2(2\lambda p-2)}{(p(2\lambda p-1)-\alpha)^2-2(p-\alpha)^2}\right].$$

The result is sharp for the functions
$f_j(z)$ $(j = 1, 2)$ *given by* (17).

Proof. Since

$$\sum_{k=p}^{\infty}\left(\frac{k(\lambda(k+p)-1)-\alpha}{p-\alpha}\right)^2 a^2_{k,j} \leq \left[\sum_{k=p}^{\infty}\left(\frac{k(\lambda(k+p)-1)-\alpha}{p-\alpha}\right)a_{k,j}\right]^2 \leq 1,$$

for $f_j(z) \in \Sigma^*_{\lambda,p,\alpha}$ $(j = 1, 2)$, we have,

$$\sum_{k=p}^{\infty}\frac{[k(\lambda(k+p)-1)-\alpha]^2}{2(p-\alpha)^2}[a^2_{k,1} + a^2_{k,1}] \leq 1.$$

We have to find a largest ζ so that

$$\sum_{k=p}^{\infty}\left[\frac{k(\lambda(k+p)-1)-\zeta}{p-\zeta}\right](a_{k,1}^2+a_{k,1}^2)\leq 1. \tag{20}$$

(20) is satisfied if

$$\frac{k(\lambda(k+p)-1)-\zeta}{p-\zeta}\leq\frac{[k(\lambda(k+p)-1)-\alpha]^2}{2(p-\alpha)^2}$$

or if

$$\zeta\leq p-\frac{2(p-\alpha)^2[k(\lambda(p+k)-1)-p]}{(k(\lambda(p+k)-1)-\alpha)^2-2(p-\alpha)^2}.$$

We see that the right side of this inequality is an increasing function of k. On setting $k=p$, we get

$$\zeta\leq p\left[1-\frac{2(p-\alpha)^2(2\lambda p-2)}{(p(2\lambda p-1)-\alpha)^2-2(p-\alpha)^2}\right],$$

which is the largest ζ such that (20) is true, which implies that $h(z)\in\Sigma^*_{\lambda,p,\zeta}$.

6 Integral Operators

Now, we consider integral transforms of functions in the class $\Sigma^*_{\lambda,p,\alpha}$.

Theorem 8 *Let the function $f(z)$ given by (1) be in $\Sigma^*_{\lambda,p,\alpha}$. Then the integral operator*

$$F(z)=(c-p+1)\int_0^1 u^c f(uz)du,\quad (0<u\leq 1, p-1<c<\infty) \tag{21}$$

is in $\Sigma^*_{\lambda,p,\beta}$, where

$$\beta=p-\frac{2p(\lambda p-1)(c-p+1)(p-\alpha)}{[p(2p\lambda-1)-\alpha](c+p+1)-(c-p+1)(p-\alpha)}. \tag{22}$$

The result is sharp for the function

$$f(z)=\frac{1}{z^p}-\frac{(p-\alpha)}{p(2p\alpha-1)-\alpha}z^p.$$

Proof. Let $f(z) = \frac{1}{z^p} - \sum_{k=p}^{\infty} a_k z^k$ be in $\Sigma^*_{\lambda,p,\alpha}$. Then

$$\begin{aligned} F(z) &= (c-p+1)\int_0^1 u^c f(uz)du \\ &= (c-p+1)\int_0^1 u^c \left[\frac{1}{(uz)^p} - \sum_{k=p}^{\infty} a_k (uz)^k\right] du \\ &= (c-p+1)\int_0^1 \left[\frac{u^{c-p}}{z^p} - \sum_{k=p}^{\infty} a_k u^{c+k} z^k\right] du \\ &= \frac{1}{z^p} - \sum_{k=p}^{\infty}\left(\frac{c-p+1}{c+k+1}\right) a_k z^k. \end{aligned}$$

It is sufficient to show that

$$\sum_{k=p}^{\infty}\left(\frac{c-p+1}{c+k+1}\right)\frac{k[\lambda(p+k)-1]-\beta}{(p-\beta)} a_k \le 1. \tag{23}$$

Since $f \in \Sigma^*_{\lambda,p,\alpha}$, we have

$$\sum_{k=p}^{\infty}\frac{k[\lambda(p+k)-1]-\alpha}{(p-\alpha)} a_k \le 1.$$

Note that (23) is satisfied if

$$\left(\frac{c-p+1}{c+k+1}\right)\frac{k[\lambda(p+k)-1]-\beta}{(p-\beta)} \le \frac{k[\lambda(p+k)-1]-\alpha}{(p-\alpha)}.$$

Rewriting the inequality, we have

$$\{k[\lambda(p+k)-1]-\beta\}(p-\alpha)(c-p+1) \le \{k[\lambda(p+k)-1]-\alpha\}(p-\beta)(c+k+1).$$

Solving for β, we have

$$\beta \le \frac{[k(\lambda(p+k)-1)-\alpha](c+k+1)p-(c-p+1)(p-\alpha)k(\lambda(p+k)-1)}{[k(\lambda(p+k)-1)-\alpha](c+k+1)-(c-p+1)(p-\alpha)}$$

or

$$\beta \le p - \frac{[k(\lambda(p+k)-1)-p](c-p+1)(p-\alpha)}{[k(\lambda(p+k)-1)-\alpha](c+k+1)-(c-p+1)(p-\alpha)}. \tag{24}$$

The right side of this inequality is an increasing function of k. On setting $k = p$, we get

$$\beta = p - \frac{[p(\lambda(p+p)-1)-p](c-p+1)(p-\alpha)}{[p(\lambda(p+p)-1)-\alpha](c+p+1)-(c-p+1)(p-\alpha)}$$
$$= p - \frac{2p(\lambda p-1)(c-p+1)(p-\alpha)}{[p(2p\lambda-1)-\alpha](c+p+1)-(c-p+1)(p-\alpha)}.$$

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