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On double gai difference sequence space defined by a sequence of Orlicz functions [1]

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Abstract

In this paper we define double gai difference sequence spaces by a sequence of Orlicz functions and establish some inclusion relations.

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1 Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces are due to Bromwich[4]. Later on, the double sequence spaces were studied by Hardy[5], Moricz[9], Moricz and Rhoades[10], Basarir and Solankan[2], Tripathy[17], Turkmenoglu[17], and many others.

Let us define the following sets of double sequences:

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$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : \sup_{m,n \in N} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n \to \infty} |x_{mn} - l|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n \to \infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n \to \infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [21,22] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zeltser [23] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [24] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [25] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Basar [27] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$

of double series. Quite recently Basar and Sever [28] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [29] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.
We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see[1]).

A sequence $x = (x_{mn})$is said to be double analytic if $\sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. By ϕ, we denote the set of all finite sequences.

Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=1}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is $\frac{1}{(i+j)!}$ in the $(i, j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn}) (m, n \in \mathbb{N})$ are also continuous.

Orlicz[13] used the idea of Orlicz function to construct the space $\left(L^M\right)$. Lindenstrauss and Tzafriri [7] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p\,(1 \leq p < \infty)$. subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [14], Mursaleen et al. [11], Bektas and Altin [3], Tripathy et al. [18], Rao and Subramanian [15], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [6].

Recalling [13] and [6], an Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano [12] and further discussed by Ruckle [16] and Maddox [8], and many others.

An Orlicz function M is said to satisfy the Δ_2- condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)\,(u \geq 0)$. The Δ_2- condition is equivalent to $M(\ell u) \leq K\ell M(u)$, for all values of u and for $\ell > 1$.

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

The space ℓ_M with the norm

$$\|x\| = inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p. If X is a sequence space, we give the following definitions:

(i)X'= the continuous dual of X;

(ii)$X^\alpha = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X \right\};$

(iii)$X^\beta = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \text{ is convegent, for each } x \in X \right\};$

(iv)$X^\gamma = \left\{ a = (a_{mn}) : \sup_{MN} \geq 1 \left| \sum_{m,n=1}^{M,N} a_{mn}x_{mn} \right| < \infty, \text{ for each } x \in X \right\};$

(v)let X be an FK-space $\supset \phi$; then $X^f = \left\{ f(\Im_{mn}) : f \in X' \right\};$

(vi)$X^\delta = \left\{ a = (a_{mn}) : \sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \text{ for each } x \in X \right\};$

$X^{\alpha}, X^{\beta}, X^{\gamma}$ and X^{δ} are called $\alpha -$ (or Köthe-Toeplitz) dual of $X, \beta -$ (or generalized-Köthe-Toeplitz) dual of $X, \gamma \ -$ dual of $X, \delta \ -$ dual of X respectively. X^{α} is defined by Gupta and Kamptan [20]. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\alpha} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces of single sequences was introduced by Kizmaz [30] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here c, c_0 and ℓ_{∞} denote the classes of convergent,null and bounded scalar valued single sequences respectively. The above difference spaces are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k|$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$
$\Delta^m x_{mn} = \Delta\Delta^{m-1} x_{mn} = \left(\Delta^{m-1} x_{mn} - \Delta^{m-1} x_{mn+1} - \Delta^{m-1} x_{m+1n} + \Delta^{m-1} x_{m+1n+1}\right)$

2 Definitions and preliminaries

Let w^2 denote the set of all complex double sequences $x = (x_{mn})_{m,n=1}^{\infty}$ and $M : [0, \infty) \to [0, \infty)$ be an Orlicz function, or a modulus function. Let $p = (p_{mn})$ be any sequence of strictly positive real numbers.

$\chi^2_M = \left\{x \in w^2 : \left(M\left(\frac{((m+n)!|x_{mn}|)^{1/m+n}}{\rho}\right)\right) \to 0 \text{ as } m, n \to \infty \text{ for some } \rho > 0\right\}$
and

$$\Lambda^2_M = \left\{x \in w^2 : \sup_{m,n \geq 1} \left(M\left(\frac{|x_{mn}|^{1/m+n}}{\rho}\right)\right) < \infty \text{ for some } \rho > 0\right\}.$$

A sequence $x \in \Lambda^2$ is said to be almost convergent if all Banach limits of x coincide. Then

$$\hat{c} = \left\{ x = (x_{mn}) : \tfrac{1}{\mu\gamma} \sum_{m,n=1}^{\mu\gamma} x_{m+s,n+s} \to 0, \, as \, \mu, \gamma \to \infty, \text{ uniformly in } s \right\}$$

Let $M = (M_{mn})$ be a sequence of Orlicz function and $p = (p_{mn})$ be any sequence of strictly positive real numbers. We define the following sequence sets

$$\chi^2_M [\hat{c}, \Delta^m, p] =$$

$$\left\{ x = (x_{mn}) : \lim_{\mu\gamma \to \infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} \left[M \left(\frac{((m+n)! \, |\Delta^m x_{m+s,n+s}|)^{1/m_s+n_s}}{\rho} \right) \right]^{p_{mn}} = 0 \right\},$$

uniformly in s, for some $\rho > 0$,

$$\Lambda^2_M [\hat{c}, \Delta^m, p] =$$

$$\left\{ x = (x_{mn}) : \sup_{s,(\mu\gamma)} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} \left[M \left(\frac{(|\Delta^m x_{m+s,n+s}|)^{1/m_s+n_s}}{\rho} \right) \right]^{p_{mn}} = 0 \right\},$$

uniformly in s, for some $\rho > 0$.

If $M_{mn}(x) = x$ for every m, n; then $\chi^2_M [\hat{c}, \Delta^m, p] = \chi^2 [\hat{c}, \Delta^m, p]$. We denote $\chi^2_M [\hat{c}, \Delta^m, p]$ and $\Lambda^2_M [\hat{c}, \Delta^m, p]$ by $\chi^2 [\hat{c}, \Delta^m, p]$ and $\Lambda^2 [\hat{c}, \Delta^m, p]$, respectively, when $p_{mn} = 1$ for all m, n.

3 Main results

Theorem 1 *Let $M = (M_{mn})$ be a sequence of Orlicz functions. Then the following statements are equivalent*
(i) $\Lambda^2 [\hat{c}, \Delta^m, p] \subseteq \Lambda^2_M [\hat{c}, \Delta^m, p]$;
(ii) $\chi^2 [\hat{c}, \Delta^m, p] \subseteq \Lambda^2_M [\hat{c}, \Delta^m, p]$;
(iii) $\sup_{\mu,\gamma} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} \left[M_{mn} \left(\frac{((m_s+n_s)! \, |\Delta^m x_{m+s,n+s}|)^{1/m_s+n_s}}{\rho} \right) \right]^{p_{m_s+n_s}} < \infty,$
for some $\rho > 0$.

Proof. (i)⇒(ii) is obvious, since $\chi^2 [\hat{c}, \Delta^m, p] \subseteq \Lambda^2 [\hat{c}, \Delta^m, p]$.

(ii)⇒(iii)Let $\chi^2 [\hat{c}, \Delta^m, p] \subseteq \chi^2_M [\hat{c}, \Delta^m, p]$. Suppose that (iii) is not satisfied. Then for some $\rho > 0$

$$\sup_{\mu,\gamma} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} \left[M_{mn} \left(\frac{((m_s + n_s)! \, |\Delta^m x_{m+s,n+s}|)^{1/m_s+n_s}}{\rho} \right) \right]^{p_{m_s+n_s}} = \infty$$

and therefore there is sequence $(\mu_i\gamma_i)$ of positive integers such that

$$\frac{1}{\mu_i\gamma_i}\sum_{m,n=1}^{\mu_i\gamma_i}\left[M_{mn}\left(\frac{i^{-1}}{\rho}\right)\right]^{p_{mn}}>i, i=1,2,\cdots \tag{2}$$

Define $x=(x_{mn})$ by

$$(((m+n)!x_{mn}))^{1/m+n}=\begin{cases} i^{-1}, & \text{if } 1\leq m,n\leq \mu_i\gamma_i, i=1,2,\cdots \\ 0, & \text{if } m>\mu_i, n>\gamma_i\end{cases}$$

Then $x\in\chi^2[\hat{c},\Delta^m,p]$, but by (2), $x\notin\Lambda^2_M[\hat{c},\Delta^m,p]$ which contradicts (ii). Hence (iii) must hold.

(iii)⇒(i) Let (iii) be satisfied and $x\in\Lambda^2[\hat{c},\Delta^m,p]$. Suppose that $x\notin\Lambda^2[\hat{c},\Delta^m,p]$. Then

$$\sup_{s,(\mu,\gamma)}\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma}\left[M_{mn}\left(\frac{(|\Delta^m x_{m+s,n+s}|)^{1/m_s+n_s}}{\rho}\right)\right]^{p_{m_s+n_s}}=\infty \tag{3}$$

Let $t=|\Delta^m x_{m+s,n+s}|^{1/m_s+n_s}$ for each m,n and fixed s, then by (3)

$$\sup_{\mu\gamma}\frac{1}{\mu\gamma}\sum_{m,n=1}^{\mu\gamma}\left[M_{mn}\left(\frac{t}{\rho}\right)\right]^{p_{m_s+n_s}}=\infty$$

which contradicts (iii). Hence (i) must hold. This completes the proof.

Theorem 2 *Let $1\leq p_{mn}\leq\sup_{mn}p_{mn}<\infty$. Then the following statements are equivalent for a sequence of Orlicz functions $M=(M_{mn})$.*
(i) $\chi^2_M[\hat{c},\Delta^m,p]\subseteq\chi^2[\hat{c},\Delta^m,p]$;
(ii) $\chi^2_M[\hat{c},\Delta^m,p]\subseteq\Lambda^2[\hat{c},\Delta^m,p]$;
(iii) $inf_{\mu\gamma}\frac{1}{\mu\gamma}\sum_{m,n=1}^{\mu\gamma}\left[M_{mn}\left(\frac{t}{\rho}\right)\right]^{p_{mn}}>0\,(t,\rho>0)$.

Proof. (i)⇒ (ii) is obvious.

(ii)⇒ (iii) Let $\chi^2_M[\hat{c},\Delta^m,p]\subseteq\Lambda^2[\hat{c},\Delta^m,p]$. Suppose that (iii) does not hold. Then

$$inf_{\mu\gamma}\frac{1}{\mu\gamma}\sum_{m,n=1}^{\mu\gamma}\left[M_{mn}\left(\frac{t}{\rho}\right)\right]^{p_{mn}}=0\,(t,\rho>0). \tag{4}$$

We can choose an index sequence $(\mu_i\gamma_i)$ such that

$$\tfrac{1}{\mu_i\gamma_i}\sum_{m,n=1}^{\mu_i\gamma_i}\left[M_{mn}\left(\frac{i}{\rho}\right)\right]^{p_{mn}} < i^{-1}, i = 1, 2, 3, \cdots$$

Define the sequence $x = (x_{mn})$ by

$$(((m+n)!x_{mn}))^{1/m+n} = \begin{cases} i, & \text{if } 1 \leq m, n \leq \mu_i\gamma_i, i = 1, 2, \cdots \\ 0, & \text{if } m, n > \mu_i, \gamma_i \end{cases}$$

Thus by (4), $x \in \chi^2_M[\hat{c}, \Delta^m, p]$ but $x \notin \Lambda^2[\hat{c}, \Delta^m, p]$ which contradicts (ii). Hence (iii) must hold.

(iii)⇒(i) Let (iii) hold and $x \in \chi^2_M[\hat{c}, \Delta^m, p]$,

(5)

$$(i.e) \lim_{\mu\gamma\to\infty}\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma}\left[M\left(\frac{((|\Delta^m x_{m+s,n+s}|)^{1/m_s+n_s}}{\rho}\right)\right]^{p_{mn}} = 0, \text{ uniformly in } s.$$

Suppose that $x \notin \chi^2[\hat{c}, \Delta^m, p]$. Then for some number $\epsilon_0 > 0$ and index $\mu_0\gamma_0$, we have

$((m_s+n_s)!\,|\Delta^m x_{m+s,n+s}|)^{1/m_s+n_s} \geq \epsilon_0$, for some $s > s'$ and $1 \leq m, n \leq \mu_0\gamma_0$. Therefore

$$\left[M_{mn}\left(\tfrac{\epsilon_0}{\rho}\right)\right]^{p_{mn}} \leq \left[M_{mn}\left(\tfrac{((m_s+n_s)!|\Delta^m x_{m+s,n+s}|)^{1/m_s+n_s}}{\rho}\right)\right]^{p_{mn}}$$

and consequently by equ (5). Hence

$$\lim_{\mu\gamma\to\infty}\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma}\left[M\left(\frac{\epsilon_0}{\rho}\right)\right]^{p_{mn}} = 0$$

which contradicts (iii). Hence $\chi^2_M[\hat{c}, \Delta^m, p] \subseteq \chi^2[\hat{c}, \Delta^m, p]$. This completes the proof.

Theorem 3 *Let* $1 \leq p_{mn} \leq \sup_{mn} p_{mn} < \infty$. *The inclusion* $\Lambda^2_M[\hat{c}, \Delta^m, p] \subseteq \chi^2[\hat{c}, \Delta^m, p]$ *hold if*

$$(6) \qquad \frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma}\left[M_{mn}\left(\frac{t}{\rho}\right)\right]^{p_{mn}} = \infty\,(t, \rho > 0).$$

Proof. Let $\Lambda^2_M\left[\hat{c},\Delta^m,p\right] \subseteq \chi^2\left[\hat{c},\Delta^m,p\right]$. Suppose that (6) does not satisfied. Therefore there is a number $t_0 > 0$ and an index sequence $(\mu_i\gamma_i)$ such that

$$(7) \qquad \frac{1}{\mu_i\gamma_i}\sum_{mn=1}^{\mu_i\gamma_i}\left[M_{mn}\left(\frac{t_0}{\rho}\right)\right]^{p_{mn}} \leq N < \infty, i = 1,2,3,\cdots$$

Define the sequence $x = (x_{mn})$ by

$$x_{mn} = \begin{cases} t_0, & \text{if } 1 \leq m,n \leq \mu_i\gamma_i, i = 1,2,\cdots, \\ 0, & \text{if } m,n > \mu_i,\gamma_i \end{cases}$$

Thus by (7), $x \in \Lambda^2_M\left[\hat{c},\Delta^m,p\right]$, but $x \notin \chi^2\left[\hat{c},\Delta^m,p\right]$. Hence (6) must hold.

Conversely, let (6) be satisfied. If $x \in \Lambda^2_M\left[\hat{c},\Delta^m,p\right]$, then for each s and $\mu\gamma$

$$(8) \qquad \frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma}\left[M_{mn}\left(\frac{\left|\Delta^m x_{m+s,n+s}\right|^{1/m_s+n_s}}{\rho}\right)\right]^{p_{mn}} \leq N < \infty.$$

Suppose that $x \notin \chi^2\left[\hat{c},\Delta^m,p\right]$. Then for some number $\epsilon_0 > 0$ there is a number s_0 and index $\mu_0\gamma_0$

$$\left|\Delta^m x_{m+s,n+s}\right|^{1/m_s+n_s} \geq \epsilon_0,\ for\, s \geq s_0.$$

Therefore

$$\left[M_{mn}\left(\frac{\epsilon_0}{\rho}\right)\right]^{p_{mn}} \leq \left[M_{mn}\left(\frac{(\left|\Delta^m x_{m+s,n+s}\right|)^{1/m_s+n_s}}{\rho}\right)\right]^{p_{mn}},$$

and hence for each m,n and s we get

$$\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma}\left[M_{mn}\left(\frac{\epsilon_0}{\rho}\right)\right]^{p_{mn}} \leq N < \infty,$$

for some $N > 0$, by (8) which contradicts (6). Hence $\Lambda^2_M\left[\hat{c},\Delta^m,p\right] \subseteq \chi^2\left[\hat{c},\Delta^m,p\right]$ This completes the proof.

Theorem 4 *Let* $1 \leq p_{mn} \leq \sup_{mn} p_{mn} < \infty$. *Then the inclusion* $\Lambda^2\left[\hat{c},\Delta^m,p\right] \subseteq \chi^2_M\left[\hat{c},\Delta^m,p\right]$ *hold if*

$$(9) \qquad \lim_{\mu\gamma\to\infty}\frac{1}{\mu\gamma}\sum_{mn=1}^{\mu\gamma}\left[M_{mn}\left(\frac{t_0}{\rho}\right)\right]^{p_{mn}} = 0,\ (t,\rho).$$

Proof. Let $\Lambda^2[\hat{c}, \Delta^m, p] \subseteq \chi^2_M[\hat{c}, \Delta^m, p]$. Suppose that (9) does not hold. Then for some $t_0 > 0$,

$$\lim_{\mu\gamma\to\infty} \frac{1}{\mu\gamma} \sum_{mn=1}^{\mu\gamma} \left[M_{mn}\left(\frac{t_0}{\rho}\right)\right]^{p_{mn}} = L \not\to 0 \tag{10}$$

Define $x = (x_{mn})$ by

$$((m+n)!x_{mn})^{1/m+n} = t_0 \sum_{v=0}^{m,n-\eta} (-1)^\eta \times$$

$$\binom{\gamma + (m,n) - v - 1}{(m,n) - v}$$

for $m, n = 1, 2, \cdots$. Thus $x \notin \chi^2_M[\hat{c}, \Delta^m, p]$ by (10), but $x \in \Lambda^2[\hat{c}, \Delta^m, p]$. Hence (9) must hold.

Conversely, Suppose that (9) hold and $x \in \Lambda^2[\hat{c}, \Delta^m, p]$. Then for every m, n and s

$$|\Delta^m x_{m+s,n+s}|^{1/m_s+n_s} \leq N < \infty$$

Therefore

$$\left[M_{mn}\left(\frac{((m_s+n_s)!\,|\Delta^m x_{m+s,n+s}|)^{1/m_s+n_s}}{\rho}\right)\right]^{p_{mn}} \leq \left[M_{mn}\left(\frac{N}{\rho}\right)\right]^{p_{mn}}$$

and

$$\frac{1}{\mu\gamma} \sum_{m,n=1}^{\mu\gamma} \left[M_{mn}\left(\frac{((m_s+n_s)!\,|\Delta^m x_{m+s,n+s}|)^{1/m_s+n_s}}{\rho}\right)\right]^{p_{mn}} \leq$$

$$\frac{1}{\mu\gamma} \sum_{m,n=1}^{\mu\gamma} \left[M_{mn}\left(\frac{N}{\rho}\right)\right]^{p_{mn}} \to 0 \text{ as } \mu, \gamma \to \infty$$

by (9). Hence $x \in \chi^2_M[\hat{c}, \Delta^m, p]$. This completes the proof.

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