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On integral operators of meromorphic functions [1]

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Abstract

Let $p \in \mathbb{N}^*, \Phi, \varphi \in H[1,p]$, $\Phi(z)\varphi(z) \neq 0$, $z \in U$, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$, and let Σ_p denote the class of meromorphic functions of the form $g(z) = \dfrac{a_{-p}}{z^p} + a_0 + a_1 z + \cdots$, $z \in \dot{U}$, $a_{-p} \neq 0$.
We consider the integral operator $J^{\Phi,\varphi}_{p,\alpha,\beta,\gamma,\delta} : K \subset \Sigma_p \to \Sigma_p$ defined by

$$J^{\Phi,\varphi}_{p,\alpha,\beta,\gamma,\delta}(g)(z) = \left[\frac{\gamma - p\beta}{z^\gamma \Phi(z)} \int_0^z g^\alpha(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}}, \; g \in K, \; z \in \dot{U}.$$

The first result of this paper gives us the conditions for which $J^{\Phi,\varphi}_{p,\alpha,\beta,\gamma,\delta}$ has some important properties. Furthermore, we study the image of the set $\Sigma^*_p(\alpha,\delta)$ through the operator $J_{p,\beta,\gamma} = J^{1,1}_{p,\beta,\beta,\gamma,\gamma}$ and the image of the sets $\Sigma K_p(\alpha,\delta)$, $\Sigma\mathcal{C}_{p,0}(\alpha,\delta;\varphi)$ through the operator $J_{p,\gamma} = J_{p,1,\gamma}$.

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1 Introduction and preliminaries

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane, $\dot{U} = U \setminus \{0\}$, $H(U) = \{f : U \to \mathbb{C} : f$ is holomorphic in $U\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

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For $p \in \mathbb{N}^*$, let Σ_p denote the class of meromorphic functions of the form

$$g(z) = \frac{a_{-p}}{z^p} + a_0 + a_1 z + \cdots,\ z \in \dot{U},\ a_{-p} \neq 0.$$

We will also use the following notations:
$\Sigma_{p,0} = \{g \in \Sigma_p : a_{-p} = 1\}$, $\Sigma_0 = \{g \in \Sigma_{p,0} : g(z) \neq 0,\ z \in \dot{U}\}$,
$\Sigma_p^*(\alpha) = \left\{g \in \Sigma_p : \mathrm{Re}\left[-\frac{zg'(z)}{g(z)}\right] > \alpha,\ z \in U\right\}$, where $\alpha < p$,
$\Sigma_p^*(\alpha, \delta) = \left\{g \in \Sigma_p : \alpha < \mathrm{Re}\left[-\frac{zg'(z)}{g(z)}\right] < \delta,\ z \in U\right\}$, where $\alpha < p < \delta$,
$\Sigma K_p(\alpha) = \left\{g \in \Sigma_p : \mathrm{Re}\left[1 + \frac{zg''(z)}{g'(z)}\right] < -\alpha,\ z \in U\right\}$, where $\alpha < p$,
$\Sigma K_{p,0}(\alpha) = \Sigma K_p(\alpha) \cap \Sigma_{p,0}$,
$\Sigma K_p(\alpha, \delta) = \left\{g \in \Sigma_p : \alpha < \mathrm{Re}\left[-1 - \frac{zg''(z)}{g'(z)}\right] < \delta,\ z \in U\right\}$, where $\alpha < p < \delta$,
$\Sigma K_{p,0}(\alpha, \delta) = \Sigma K_p(\alpha, \delta) \cap \Sigma_{p,0}$,
$\Sigma \mathcal{C}_{p,0}(\alpha, \delta; \varphi) = \left\{g \in \Sigma_{p,0} : \alpha < \mathrm{Re}\left[\frac{g'(z)}{\varphi'(z)}\right] < \delta,\ z \in U\right\}$, where $\alpha < 1 \leq p < \delta$,
$\varphi \in \Sigma K_{p,0}(\alpha, \delta)$.

We remark that $\Sigma_1^*(\alpha)$, $0 \leq \alpha < 1$, is the classes of meromorphic starlike functions of order α and $\Sigma K_{1,0}(\alpha) \cap \Sigma_0$ is the classes of meromorphic convex functions of order α. These classes are classes of univalent functions.

$H[a, n] = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots\}$ for $a \in \mathbb{C}$, $n \in \mathbb{N}^*$,
$A_n = \{f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots\}$, $n \in \mathbb{N}^*$, and for $n = 1$ we denote A_1 by A and this set is called *the class of analytic functions normalized at the origin.*

Definition 1. *[3, p.4], [4, p.45] Let $f, g \in H(U)$. We say that the function f is subordinate to the function g, and we denote this by $f(z) \prec g(z)$, if there is a function $w \in H(U)$, with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that*

$$f(z) = g[w(z)],\ z \in U.$$

Remark 1. *If $f(z) \prec g(z)$, then $f(0) = g(0)$ and $f(U) \subseteq g(U)$.*

Theorem 1. *[3, p.4], [4, p.46] Let $f, g \in H(U)$ and let g be a univalent function in U. Then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.*

Theorem 2. *[3, Theorem 2.4f.], [4, p.212] Let* $p \in H[a,n]$ *with* $\operatorname{Re} a > 0$ *and let* $P : U \to \mathbb{C}$ *be a function with* $\operatorname{Re} P(z) > 0$, $z \in U$. *If*

$$\operatorname{Re}\left[p(z) + P(z)zp'(z)\right] > 0,\ z \in U,$$

then $\operatorname{Re} p(z) > 0$, $z \in U$.

Definition 2. *[3, p.46], [4, p.228] Let* $c \in \mathbb{C}$ *with* $\operatorname{Re} c > 0$ *and* $n \in \mathbb{N}^*$. *We consider*

$$C_n = C_n(c) = \frac{n}{\operatorname{Re} c}\left[|c|\sqrt{1 + \frac{2\operatorname{Re} c}{n}} + \operatorname{Im} c\right].$$

If the univalent function $R : U \to \mathbb{C}$ *is given by* $R(z) = \dfrac{2C_n z}{1 - z^2}$, *then we will denote by* $R_{c,n}$ *the "Open Door" function, defined as*

$$R_{c,n}(z) = R\left(\frac{z+b}{1+\bar{b}z}\right) = 2C_n \frac{(z+b)(1+\bar{b}z)}{(1+\bar{b}z)^2 - (z+b)^2}\ ,$$

where $b = R^{-1}(c)$.

Lemma 1. *[3, p.35], [4, pg. 209] Let* $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ *be a function that satisfies the condition*

$$\operatorname{Re} \psi(\rho i, \sigma; z) \le 0\,,$$

when $\rho, \sigma \in \mathbb{R}, \sigma \le -\dfrac{n}{2}(1+\rho^2)$, $z \in U, n \ge 1$.

If $p \in H[1,n]$ *and*

$$\operatorname{Re} \psi(p(z), zp'(z); z) > 0, \quad z \in U,$$

then

$$\operatorname{Re} p(z) > 0, \quad z \in U.$$

Theorem 3. *[3, Theorem 2.5c.] Let* $\Phi, \varphi \in H[1,n]$ *with* $\Phi(z) \ne 0, \varphi(z) \ne 0$, *for* $z \in U$. *Let* $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *with* $\beta \ne 0$, $\alpha + \delta = \beta + \gamma$ *and* $\operatorname{Re}(\alpha + \delta) > 0$. *Let the function* $f(z) = z + a_{n+1}z^{n+1} + \cdots \in A_n$ *and suppose that*

$$\alpha \frac{zf'(z)}{f(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec R_{\alpha+\delta,n}(z).$$

If $F = I^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}(f)$ *is defined by*

$$F(z) = I^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}(f)(z) = \left[\frac{\beta+\gamma}{z^\gamma \Phi(z)} \int_0^z f^\alpha(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}}, \tag{1}$$

then $F \in A_n$ *with* $\frac{F(z)}{z} \neq 0$, $z \in U$, *and*

$$\mathrm{Re}\left[\beta\frac{zF'(z)}{F(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0,\ z \in U.$$

All powers in (1) are principal ones.

Theorem 4. *[3, Lemma 1.2c.] Let* $n \geq 0$ *be an integer and let* $\gamma \in \mathbb{C}$, *with* $\mathrm{Re}\,\gamma > -n$. *If* $f(z) = \sum_{m\geq n} a_m z^m$ *is analytic in* U *and* F *is defined by*

$$F(z) = I[f](z) = \frac{1}{z^\gamma}\int_0^z f(\zeta)\zeta^{\gamma-1}d\zeta = \int_0^1 f(tz)t^{\gamma-1}dt,$$

then $F(z) = \sum_{m\geq n} \frac{a_m z^m}{m+\gamma}$ *is analytic in* U.

2 Main results

Theorem 5. *Let* $p \in \mathbb{N}^*$, $\Phi, \varphi \in H[1,p]$ *with* $\Phi(z)\varphi(z) \neq 0$, $z \in U$. *Let* $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *with* $\beta \neq 0$, $\delta + p\beta = \gamma + p\alpha$ *and* $\mathrm{Re}\,(\gamma - p\beta) > 0$. *Let* $g \in \Sigma_p$ *and suppose that*

$$\alpha\frac{zg'(z)}{g(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec R_{\delta-p\alpha,p}(z),\ z \in U.$$

If $G = J^{\Phi,\varphi}_{p,\alpha,\beta,\gamma,\delta}(g)$ *is defined by*

$$\text{(2)} \qquad G(z) = J^{\Phi,\varphi}_{p,\alpha,\beta,\gamma,\delta}(g)(z) = \left[\frac{\gamma - p\beta}{z^\gamma\Phi(z)}\int_0^z g^\alpha(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}},$$

then $G \in \Sigma_p$ *with* $z^pG(z) \neq 0$, $z \in U$, *and*

$$\mathrm{Re}\left[\beta\frac{zG'(z)}{G(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0,\ z \in U.$$

All powers in (2) are principal ones.

Proof. Let $g \in \Sigma_p$ be of the form $g(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} a_k z^k$, $z \in \dot{U}$, $a_{-p} \neq 0$. It's easy to see that the function $f(z) = \frac{z^{p+1}g(z)}{a_{-p}}$ belongs to the class A_p.

After a simple computation we have

$$\alpha\frac{zf'(z)}{f(z)}+\frac{z\varphi'(z)}{\varphi(z)}=\alpha\frac{zg'(z)}{g(z)}+\frac{z\varphi'(z)}{\varphi(z)}+\alpha(p+1),$$

hence

$$\alpha\frac{zf'(z)}{f(z)}+\frac{z\varphi'(z)}{\varphi(z)}+\delta-\alpha(p+1)\prec R_{\delta-p\alpha,p}(z).$$

By denoting $\delta-\alpha(p+1)=\delta_1$ and $\gamma-\beta(p+1)=\gamma_1$, after using the fact that $\delta+p\beta=\gamma+p\alpha$ and $\operatorname{Re}(\gamma-p\beta)>0$, we obtain that $\alpha+\delta_1=\beta+\gamma_1$ and $\operatorname{Re}(\beta+\gamma_1)>0$.

Now we remark that the conditions of Theorem 3 are satisfied for the functions f,Φ,φ and the numbers $\alpha,\beta,\gamma_1,\delta_1$, so, we obtain that

$$F(z)=I^{\Phi,\varphi}_{\alpha,\beta,\gamma_1,\delta_1}(f)(z)=\left[\frac{\beta+\gamma_1}{z^{\gamma_1}\Phi(z)}\int_0^z f^{\alpha}(t)\varphi(t)t^{\delta_1-1}dt\right]^{\frac{1}{\beta}}\in A_p,$$

with $\dfrac{F(z)}{z}\neq 0$, $z\in U$, and

$$\operatorname{Re}\left[\beta\frac{zF'(z)}{F(z)}+\frac{z\Phi'(z)}{\Phi(z)}+\gamma_1\right]>0,\ z\in U. \tag{3}$$

It's not difficult to see that

$$F^{\beta}(z)(a_{-p})^{\alpha}=G^{\beta}(z)z^{\beta(p+1)}, \tag{4}$$

where

$$G(z)=J^{\Phi,\varphi}_{p,\alpha,\beta,\gamma,\delta}(g)(z)=\left[\frac{\gamma-p\beta}{z^{\gamma}\Phi(z)}\int_0^z g^{\alpha}(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}}.$$

Since $\dfrac{F(z)}{z}\neq 0$, $z\in U$, we have from (4), $z^pG(z)\neq 0$, $z\in U$.
Using the logharitmic differential and the multiplying with z for (4), we obtain

$$\beta\frac{zF'(z)}{F(z)}=\beta\frac{zG'(z)}{G(z)}+\beta(p+1),\ z\in U.$$

From this last equality and (3), we get

$$\operatorname{Re}\left[\beta\frac{zG'(z)}{G(z)}+\frac{z\Phi'(z)}{\Phi(z)}+\gamma\right]>0,\ z\in U.$$

Taking $\alpha=\beta$ and $\gamma=\delta$ in the above theorem and using the notation $J^{\Phi,\varphi}_{p,\beta,\gamma}$ instead of $J^{\Phi,\varphi}_{p,\beta,\beta,\gamma,\gamma}$, we obtain the next corollary:

Corollary 1. *Let $p \in \mathbb{N}^*$, $\Phi, \varphi \in H[1,p]$ with $\Phi(z)\varphi(z) \neq 0$, $z \in U$. Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\operatorname{Re}(\gamma - p\beta) > 0$. If $g \in \Sigma_p$ and*

$$\beta \frac{zg'(z)}{g(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \gamma \prec R_{\gamma - p\beta, p}(z),$$

then

$$G(z) = J^{\Phi,\varphi}_{p,\beta,\gamma}(g)(z) = \left[\frac{\gamma - p\beta}{z^\gamma \Phi(z)} \int_0^z g^\beta(t)\varphi(t)t^{\gamma-1}dt\right]^{\frac{1}{\beta}} \in \Sigma_p,$$

with $z^p G(z) \neq 0$, $z \in U$, and

$$\operatorname{Re}\left[\beta \frac{zG'(z)}{G(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0,\ z \in U.$$

Considering $\Phi = \varphi \equiv 1$ in Corollary 1, and using the notation $J_{p,\beta,\gamma}$ instead of $J^{1,1}_{p,\beta,\beta,\gamma,\gamma}$, we obtain:

Corollary 2. *Let $p \in \mathbb{N}^*$, $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\operatorname{Re}(\gamma - p\beta) > 0$. If $g \in \Sigma_p$ and*

$$\beta \frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p\beta, p}(z),$$

then

$$G(z) = J_{p,\beta,\gamma}(g)(z) = \left[\frac{\gamma - p\beta}{z^\gamma} \int_0^z g^\beta(t)t^{\gamma-1}dt\right]^{\frac{1}{\beta}} \in \Sigma_p,$$

with $z^p G(z) \neq 0$, $z \in U$, and

$$\operatorname{Re}\left[\beta \frac{zG'(z)}{G(z)} + \gamma\right] > 0,\ z \in U.$$

Let $p \in \mathbb{N}^*$, $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, $g \in \Sigma_p$, $G = J_{p,\beta,\gamma}(g)$ and let us denote $P(z) = -\dfrac{zG'(z)}{G(z)}$, $z \in U$. If we suppose that $P \in H(U)$, we obtain from

$$G(z) = \left[\frac{\gamma - p\beta}{z^\gamma} \int_0^z t^{\gamma-1} g^\beta(t)dt\right]^{\frac{1}{\beta}},\ z \in \dot{U},$$

that

$$P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} = -\frac{zg'(z)}{g(z)},\ z \in U. \tag{5}$$

Theorem 6. *Let $p \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$. If $g \in \Sigma_p$, then $J_{p,\lambda}(g) \in \Sigma_p$, where $J_{p,\lambda}(g)(z) = J_{p,1,\lambda}(g)(z) = \dfrac{\lambda - 1}{z^\lambda} \displaystyle\int_0^z g(t)t^{\lambda-1}dt$.*

Proof. Let g be of the form $g(z) = \dfrac{a_{-p}}{z^p} + a_0 + a_1 z + \cdots$, $z \in \dot{U}$, $a_{-p} \neq 0$. Since $g \in \Sigma_p$ we have $z^p g \in H[a_{-p}, p]$. Let us denote $f(z) = z^p g(z)$, $z \in U$, and $\gamma = \lambda - p$.

We know that $\operatorname{Re}\lambda > p$, so, $\operatorname{Re}\gamma > 0$, and using Theorem 4 for f and γ we get that

$$F(z) = \frac{1}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt$$

is analytic in U, so $F \in H\left[\dfrac{a_{-p}}{\gamma}, p\right]$. It's easy to see that

$$F(z) = \frac{1}{z^{\lambda-p}} \int_0^z g(t) t^{\lambda-1} dt = z^p \frac{1}{\lambda-1} J_{p,\lambda}(g)(z),$$

therefore $J_{p,\lambda}(g) \in \Sigma_p$.

Remark 2. *Let $p \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$. From the above theorem, it's easy to see that we have $J_{p,\lambda}(g) \in \Sigma_{p,0}$, when $g \in \Sigma_{p,0}$.*

For the next results we need the following lemmas:

Lemma 2. *Let $n \in \mathbb{N}^*$, $\alpha, \beta \in \mathbb{R}$, $\gamma \in \mathbb{C}$ with $\operatorname{Re}[\gamma - \alpha\beta] \geq 0$. If $P \in H[P(0), n]$ with $P(0) \in \mathbb{R}$ and $P(0) > \alpha$, then we have*

$$\operatorname{Re}\left[P(z) + \frac{zP'(z)}{\gamma - \beta P(z)}\right] > \alpha \Rightarrow \operatorname{Re} P(z) > \alpha,\ z \in U.$$

Proof. If we take $R(z) = \dfrac{P(z) - \alpha}{P(0) - \alpha}$, we have $R(z) \in H[1, 1]$ and from

$$\operatorname{Re}\left[P(z) + \frac{zP'(z)}{\gamma - \beta P(z)}\right] > \alpha,\ z \in U,$$

since $P(0) - \alpha > 0$, we obtain

$$\operatorname{Re}\left[R(z) + \frac{zR'(z)}{\gamma - \beta\alpha - \beta(P(0) - \alpha)R(z)}\right] > 0,\ z \in U.$$

Now let us put

$$\psi(R(z), zR'(z); z) = R(z) + \frac{zR'(z)}{\gamma - \beta\alpha - \beta(P(0) - \alpha)R(z)}.$$

We have $\operatorname{Re}\psi(R(z), zR'(z); z) > 0$, $z \in U$.

To apply Lemma 1 we need to show that $\operatorname{Re}\psi(\rho i, \sigma; z) \leq 0$, when $\rho \in \mathbb{R}$, $\sigma \leq -\dfrac{1+\rho^2}{2}$, $z \in U$. We have

$$\operatorname{Re}\psi(\rho i, \sigma; z) = \operatorname{Re}\frac{\sigma}{\gamma-\beta\alpha-\beta(P(0)-\alpha)\rho i} = \operatorname{Re}\frac{\sigma}{\gamma_1+i\gamma_2-\beta\alpha-\beta(P(0)-\alpha)\rho i} =$$

$$\frac{\sigma(\gamma_1-\beta\alpha)}{(\gamma_1-\beta\alpha)^2+(\gamma_2-\beta(P(0)-\alpha)\rho)^2} \leq 0,\ z\in U,\ \rho\in\mathbb{R},\ \sigma\leq -\frac{1+\rho^2}{2},\ \gamma_1=\operatorname{Re}\gamma\geq\alpha\beta.$$

Applying now Lemma 1 we obtain $\operatorname{Re}R(z) > 0$, $z \in U$, hence $\operatorname{Re}P(z) > \alpha$.

Lemma 3. *Let* $n \in \mathbb{N}^*$, $\delta, \beta \in \mathbb{R}$, $\gamma \in \mathbb{C}$ *with* $\operatorname{Re}[\gamma-\delta\beta] \geq 0$. *If* $P \in H[P(0), n]$ *with* $P(0) \in \mathbb{R}$ *and* $P(0) < \delta$, *then we have*

$$\operatorname{Re}\left[P(z) + \frac{zP'(z)}{\gamma-\beta P(z)}\right] < \delta \Rightarrow \operatorname{Re}P(z) < \delta,\ z \in U.$$

Proof. Let us denote $R(z) = -P(z)$, $\alpha = -\delta$, $\beta_1 = -\beta$. It is easy to see that the conditions from Lemma 2 holds for the function R and the numbers α, β_1, γ, so we obtain $\operatorname{Re}R(z) > \alpha$, $z \in U$, which is equivalent to $\operatorname{Re}P(z) < \delta$, $z \in U$.

Next we will study the properties of the image of a function $g \in \Sigma_p^*(\alpha, \delta)$ through the integral operator $J_{p,\beta,\gamma}$ defined by

$$J_{p,\beta,\gamma}(g)(z) = \left[\frac{\gamma-p\beta}{z^\gamma}\int_0^z g^\beta(t)t^{\gamma-1}dt\right]^{\frac{1}{\beta}}. \tag{6}$$

Theorem 7. *Let* $p \in \mathbb{N}^*$, $\beta > 0, \gamma \in \mathbb{C}$ *and* $\alpha < p < \delta \leq \dfrac{\operatorname{Re}\gamma}{\beta}$.
If $g \in \Sigma_p^*(\alpha, \delta)$, *then* $G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha, \delta)$.

Proof. We know that $g \in \Sigma_p^*(\alpha, \delta)$ is equivalent to

$$\alpha < \operatorname{Re}\left[-\frac{zg'(z)}{g(z)}\right] < \delta,\ z \in U, \tag{7}$$

so,

$$\operatorname{Re}\gamma - \beta\delta < \operatorname{Re}\left[\gamma + \beta\frac{zg'(z)}{g(z)}\right] < \operatorname{Re}\gamma - \beta\alpha,\ z \in U, \quad \text{when} \quad \beta > 0.$$

Because $\delta \leq \dfrac{\operatorname{Re}\gamma}{\beta}$, we have $\operatorname{Re}\left[\gamma+\beta\dfrac{zg'(z)}{g(z)}\right]>0,\ z\in U$, and using Corollary 2, we obtain that $G=J_{p,\beta,\gamma}(g)\in\Sigma_p,\ z^pG(z)\neq 0,\ z\in U$, and $\operatorname{Re}\left[\gamma+\beta\dfrac{zG'(z)}{G(z)}\right]$ $>0,\ z\in U$.

From (5) we know that

$$P(z)+\frac{zP'(z)}{\gamma-\beta P(z)}=-\frac{zg'(z)}{g(z)},\quad \text{where}\quad P(z)=-\frac{zG'(z)}{G(z)} \text{ is analytic in } U.$$

Using (7) we get

$$\alpha<\operatorname{Re}\left[P(z)+\frac{zP'(z)}{\gamma-\beta P(z)}\right]<\delta,\ z\in U. \tag{8}$$

Since $\alpha<P(0)=p<\delta$ and $0\leq\operatorname{Re}\gamma-\beta\delta<\operatorname{Re}\gamma-\beta\alpha$, we obtain from (8), after applying Lemma 2 and Lemma 3, that

$$\alpha<\operatorname{Re}P(z)<\delta,\ z\in U,$$

which is equivalent to

$$\alpha<\operatorname{Re}\left[-\frac{zG'(z)}{G(z)}\right]<\delta,\ z\in U. \tag{9}$$

Since $G\in\Sigma_p$ we get from (9) that $G\in\Sigma_p^*(\alpha,\delta)$.

We remark that for $p=1$ all members of the class $\Sigma_1^*(\alpha,\delta)$ are univalent functions, when $0\leq\alpha<1<\delta$, so $G=J_{1,\beta,\gamma}(g)$ is an univalent function when $g\in\Sigma_1^*(\alpha,\delta)$ and $0\leq\alpha<1<\delta\leq\dfrac{\operatorname{Re}\gamma}{\beta},\ \beta>0$.

Taking $\beta=1$ in the above theorem and using the notation $J_{p,\gamma}$ instead of $J_{p,1,\gamma}$, we obtain:

Corollary 3. *Let* $p\in\mathbb{N}^*$, $\gamma\in\mathbb{C}$ *and* $\alpha<p<\delta\leq\operatorname{Re}\gamma$. *If* $g\in\Sigma_p^*(\alpha,\delta)$, *then*

$$G=J_{p,\gamma}(g)=\frac{\gamma-p}{z^\gamma}\int_0^z t^{\gamma-1}g(t)dt\in\Sigma_p^*(\alpha,\delta).$$

The properties of the integral operator $J_{1,\gamma}$, were studied by many authors in different papers, from which we remember [1], [2], [5], [6], [7].

Theorem 8. *Let* $p \in \mathbb{N}^*$, $\beta > 0, \gamma \in \mathbb{C}$ *and* $\alpha < p < \dfrac{\operatorname{Re}\gamma}{\beta} \leq \delta$.
If $g \in \Sigma_p^*(\alpha, \delta)$, *with*

$$\beta \frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p\beta, p}(z),\ z \in U,$$

then $G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha, \delta)$.

Proof. Because $\beta \dfrac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p\beta, p}(z)$, $z \in U$, we obtain from Corollary 2 that $G \in \Sigma_p$, with $z^p G(z) \neq 0$, $z \in U$, and

$$\operatorname{Re}\left[\beta \frac{zG'(z)}{G(z)} + \gamma\right] > 0,\ z \in U,\ \text{where } G = J_{p,\beta,\gamma}(g). \tag{10}$$

Since $\dfrac{\operatorname{Re}\gamma}{\beta} \leq \delta$, we get from (10),

$$\operatorname{Re}\frac{zG'(z)}{G(z)} + \delta > 0,\ z \in U. \tag{11}$$

From (5) we know that

$$P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} = -\frac{zg'(z)}{g(z)}, \quad \text{where} \quad P(z) = -\frac{zG'(z)}{G(z)}. \tag{12}$$

Since $g \in \Sigma_p^*(\alpha, \delta)$, we obtain from (12) that

$$\alpha < \operatorname{Re}\left[P(z) + \frac{zP'(z)}{\gamma - \beta P(z)}\right] < \delta,\ z \in U. \tag{13}$$

Because we know from (11) that $\operatorname{Re} P(z) < \delta$, $z \in U$, we have only to verify that $\operatorname{Re} P(z) > \alpha$. To show this we will use Lemma 2.

We know that P is analytic in U with $P(0) = p > \alpha$. We also have $\operatorname{Re}\gamma - \alpha\beta > 0$. Since the conditions from Lemma 2 are met, we obtain $\operatorname{Re} P(z) > \alpha$, which is equivalent to

$$-\operatorname{Re}\frac{zG'(z)}{G(z)} > \alpha. \tag{14}$$

Since $G \in \Sigma_p$, from (11) and (14) we have $G \in \Sigma_p^*(\alpha, \delta)$.

If we consider $\delta \to \infty$, in the above theorem, we obtain the next corollary:

Corollary 4. *Let* $p \in \mathbb{N}^*$, $\beta > 0, \gamma \in \mathbb{C}$ *and* $\alpha < p < \frac{\operatorname{Re}\gamma}{\beta}$.
If $g \in \Sigma_p^*(\alpha)$, *with*

$$\beta \frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p\beta, p}(z),\ z \in U,$$

then $G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha)$.

We make the remark that we can obtain a similar result, without the condition $\beta \frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p\beta, p}(z), z \in U$, as it follows:

Theorem 9. *Let* $p \in \mathbb{N}^*$, $\beta > 0, \gamma \in \mathbb{C}$, $\alpha < p < \frac{\operatorname{Re}\gamma}{\beta}$ *and* $g \in \Sigma_p^*(\alpha)$. *Let* $G = J_{p,\beta,\gamma}(g)$. *If* $G \in \Sigma_p$ *and* $z^p G(z) \neq 0$, $z \in U$, *then* $G \in \Sigma_p^*(\alpha)$.

Proof. Let us denote $P(z) = -\frac{zG'(z)}{G(z)}$, $z \in U$. Because $G \in \Sigma_p$ and $z^p G(z) \neq 0$, $z \in U$, we have that P analytic in U, hence from $G = J_{p,\beta,\gamma}(g)$ and (5) we have that

$$P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} = -\frac{zg'(z)}{g(z)},\ z \in U. \tag{15}$$

Since $g \in \Sigma_p^*(\alpha)$, we obtain from (15) that

$$\operatorname{Re}\left[P(z) + \frac{zP'(z)}{\gamma - \beta P(z)}\right] > \alpha,\ z \in U. \tag{16}$$

We have to verify that $\operatorname{Re} P(z) > \alpha$. To show this we will use Lemma 2.

We have P analytic in U with $P(0) = p > \alpha$ and $\operatorname{Re}\gamma - \alpha\beta > 0$. Since the conditions from Lemma 2 are met, we obtain $\operatorname{Re} P(z) > \alpha$, which is equivalent to

$$-\operatorname{Re}\frac{zG'(z)}{G(z)} > \alpha,\ z \in U. \tag{17}$$

Because $G \in \Sigma_p$, from (17), we get $G \in \Sigma_p^*(\alpha)$.

Since we know from Theorem 6 that for $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$, we have $J_{p,\gamma}(g) \in \Sigma_p$ when $g \in \Sigma_p$, we obtain for the above theorem, taking $\beta = 1$, the next corollary:

Corollary 5. *Let* $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ *and* $\alpha < p < \operatorname{Re}\gamma$.
If $g \in \Sigma_p^*(\alpha)$ *with* $z^p J_{p,\gamma}(g)(z) \neq 0$, $z \in U$, *then* $G = J_{p,\gamma}(g) \in \Sigma_p^*(\alpha)$.

Taking $\beta = 1$ in Theorem 8, we get:

Corollary 6. *Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ and $\alpha < p < \operatorname{Re}\gamma \le \delta$.*
If $g \in \Sigma_p^*(\alpha, \delta)$, with

$$\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma-p,p}(z),\ z \in U,$$

then $G = J_{p,\gamma}(g) \in \Sigma_p^*(\alpha, \delta)$.

Theorem 10. *Let $p \in \mathbb{N}^*$, $\beta < 0$, $\gamma \in \mathbb{C}$ and $\dfrac{\operatorname{Re}\gamma}{\beta} \le \alpha < p < \delta$.*
If $g \in \Sigma_p^*(\alpha, \delta)$, then $G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha, \delta)$.

Proof. We know that $g \in \Sigma_p^*(\alpha, \delta)$ is equivalent to

$$\alpha < \operatorname{Re}\left[-\frac{zg'(z)}{g(z)}\right] < \delta,\ z \in U, \tag{18}$$

so,

$$\operatorname{Re}\gamma - \beta\alpha < \operatorname{Re}\left[\gamma + \beta\frac{zg'(z)}{g(z)}\right] < \operatorname{Re}\gamma - \beta\delta,\ z \in U, \quad \text{when} \quad \beta < 0.$$

Because $\alpha \ge \dfrac{\operatorname{Re}\gamma}{\beta}$, we have $\operatorname{Re}\left[\gamma + \beta\dfrac{zg'(z)}{g(z)}\right] > 0$, $z \in U$, and using Corollary 2, we obtain that $G = J_{p,\beta,\gamma}(g) \in \Sigma_p$, $z^pG(z) \ne 0$, $z \in U$ and $\operatorname{Re}\left[\gamma + \beta\dfrac{zG'(z)}{G(z)}\right] > 0$, $z \in U$.

From (5) we know that

$$P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} = -\frac{zg'(z)}{g(z)}, \quad \text{where} \quad P(z) = -\frac{zG'(z)}{G(z)} \text{ is analytic in } U.$$

We will use the same idea as at the proof of Theorem 7.

Using (18) we get

$$\alpha < \operatorname{Re}\left[P(z) + \frac{zP'(z)}{\gamma - \beta P(z)}\right] < \delta,\ z \in U. \tag{19}$$

Since $\alpha < P(0) = p < \delta$ and $\operatorname{Re}\gamma - \beta\delta > \operatorname{Re}\gamma - \beta\alpha \ge 0$, we obtain from (19), after applying Lemma 2 and Lemma 3, that

$$\alpha < \operatorname{Re} P(z) < \delta,\ z \in U,$$

which is equivalent to

$$\alpha < \operatorname{Re}\left[-\frac{zG'(z)}{G(z)}\right] < \delta,\ z \in U. \tag{20}$$

Since $G \in \Sigma_p$ we have from (20) that $G \in \Sigma_p^*(\alpha, \delta)$.

If we consider $\delta \to \infty$, in the above theorem, we obtain the next corollary:

Corollary 7. *Let $p \in \mathbb{N}^*$, $\beta < 0$, $\gamma \in \mathbb{C}$ and $\frac{\operatorname{Re}\gamma}{\beta} \le \alpha < p$. Then we have*

$$g \in \Sigma_p^*(\alpha) \Rightarrow G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha).$$

Theorem 11. *Let $p \in \mathbb{N}^*$, $\beta < 0, \gamma \in \mathbb{C}$ and $\alpha \le \frac{\operatorname{Re}\gamma}{\beta} < p < \delta$.
If $g \in \Sigma_p^*(\alpha, \delta)$, with*

$$\beta\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p\beta, p}(z),\ z \in U,$$

then $G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha, \delta)$.

Proof. Because $\beta\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p\beta, p}(z)$, $z \in U$, we obtain from Corollary 2 that $G \in \Sigma_p$ with $z^pG(z) \neq 0$, $z \in U$, and

$$\operatorname{Re}\left[\beta\frac{zG'(z)}{G(z)} + \gamma\right] > 0,\ z \in U,\ \text{where } G = J_{p,\beta,\gamma}(g). \tag{21}$$

Since $\alpha \le \frac{\operatorname{Re}\gamma}{\beta}$, and $\beta < 0$, we get from (21) that

$$\operatorname{Re}\frac{zG'(z)}{G(z)} + \alpha < 0,\ z \in U. \tag{22}$$

From (5) we know that

$$P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} = -\frac{zg'(z)}{g(z)}, \quad \text{where} \quad P(z) = -\frac{zG'(z)}{G(z)}. \tag{23}$$

Since $g \in \Sigma_p^*(\alpha, \delta)$, we obtain from (23) that

$$\alpha < \operatorname{Re}\left[P(z) + \frac{zP'(z)}{\gamma - \beta P(z)}\right] < \delta. \tag{24}$$

Because we know from (22) that $\operatorname{Re} P(z) > \alpha$, $z \in U$, we have only to verify that $\operatorname{Re} P(z) < \delta$.

To show this we will use Lemma 3.

We know that P is analytic in U with $P(0) = p < \delta$. Also we have $\operatorname{Re}\gamma - \delta\beta > 0$. Since the conditions from Lemma 3 are met, we obtain $\operatorname{Re} P(z) < \delta$, which is equivalent to

$$-\operatorname{Re}\frac{zG'(z)}{G(z)} < \delta. \tag{25}$$

From (22) and (25), since $G \in \Sigma_p$, we have $G \in \Sigma_p^*(\alpha, \delta)$.

If we consider $\delta \to \infty$, in the above theorem, we obtain the next corollary:

Corollary 8. *Let $p \in \mathbb{N}^*$, $\beta < 0, \gamma \in \mathbb{C}$ and $\alpha \leq \dfrac{\operatorname{Re}\gamma}{\beta} < p$. If $g \in \Sigma_p^*(\alpha)$, with*

$$\beta\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p\beta, p}(z),\ z \in U,$$

then $G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha)$.

We make the remark that we can obtain a similar result, without the condition $\beta\dfrac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p\beta, p}(z), z \in U$, as it follows:

Theorem 12. *Let $p \in \mathbb{N}^*$, $\beta < 0, \gamma \in \mathbb{C}$, $\alpha \leq \dfrac{\operatorname{Re}\gamma}{\beta} < p$ and $g \in \Sigma_p^*(\alpha)$. Let $G = J_{p,\beta,\gamma}(g)$. If $G \in \Sigma_p$ and $z^pG(z) \neq 0$, $z \in U$, then $G \in \Sigma_p^*(\alpha)$.*

We omit the proof because it is similar to that of Theorem 9.

The next results concern the sets $\Sigma K_p(\alpha, \delta)$, $\Sigma\mathcal{C}_{p,0}(\alpha, \delta; \varphi)$ and the operator $J_{p,\gamma} = J_{p,1,\gamma}$.

Theorem 13. *Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$ and let $\alpha < p < \delta \leq \operatorname{Re}\gamma$. If $g \in \Sigma K_p(\alpha, \delta)$ and $z^{p+1}J'_{p,\gamma}(g)(z) \neq 0$, $z \in U$, then*

$$J_{p,\gamma}(g) \in \Sigma K_p(\alpha, \delta).$$

Proof. Let us denote $G = J_{p,\gamma}(g)$. We know from Theorem 6 that $G \in \Sigma_p$. Let $P(z) = -1 - \dfrac{zG''(z)}{G'(z)}$, $z \in U$. Since $G \in \Sigma_p$ and $z^{p+1}G'(z) \neq 0$, $z \in U$, we have $P \in H(U)$.

Using the definition of the operator $J_{p,\gamma}$ and the logharitmic differential, two times, we obtain

$$P(z)+\frac{zP'(z)}{\gamma-P(z)}=-1-\frac{zg''(z)}{g'(z)},\ z\in U. \tag{26}$$

From $g\in\Sigma K_p(\alpha,\delta)$, we have

$$\alpha<\operatorname{Re}\left[-1-\frac{zg''(z)}{g'(z)}\right]<\delta,\ z\in U,$$

so, using (26), we obtain

$$\alpha<\operatorname{Re}\left[P(z)+\frac{zP'(z)}{\gamma-P(z)}\right]<\delta,\ z\in U. \tag{27}$$

Since $\alpha<P(0)=p<\delta$ and $0\le\operatorname{Re}\gamma-\delta<\operatorname{Re}\gamma-\alpha$, we obtain from (27), after applying Lemma 2 and Lemma 3 (in the case $\beta=1$), that

$$\alpha<\operatorname{Re}P(z)<\delta,\ z\in U,$$

which is equivalent to

$$\alpha<\operatorname{Re}\left[-1-\frac{zG''(z)}{G'(z)}\right]<\delta,\ z\in U. \tag{28}$$

Since $G=J_{p,\gamma}(g)\in\Sigma_p$, we have from (28), that $J_{p,\gamma}(g)\in\Sigma K_p(\alpha,\delta)$.

From the proof of the above theorem we remark that we also have the next result.

Theorem 14. *Let $p\in\mathbb{N}^*$, $\alpha\in\mathbb{R}$, $\gamma\in\mathbb{C}$ with $\alpha<p<\operatorname{Re}\gamma$. If $g\in\Sigma K_p(\alpha)$ and $z^{p+1}J'_{p,\gamma}(g)(z)\neq 0$, $z\in U$, then*

$$J_{p,\gamma}(g)\in\Sigma K_p(\alpha).$$

Theorem 15. *Let $p\in\mathbb{N}^*$, $\gamma\in\mathbb{C}$ with $\operatorname{Re}\gamma>p$, and $\alpha<1\le p<\delta\le\operatorname{Re}\gamma$. Let φ be a function in $\Sigma K_{p,0}(\alpha,\delta)$ and $g\in\Sigma\mathcal{C}_{p,0}(\alpha,\delta;\varphi)$ such that $z^{p+1}J'_{p,\gamma}(\varphi)\neq 0$, $z\in U$, then*

$$J_{p,\gamma}(g)\in\Sigma\mathcal{C}_{p,0}(\alpha,\delta;\Phi),$$

where $\Phi=J_{p,\gamma}(\varphi)$.

Proof. From $g \in \Sigma\mathcal{C}_{p,0}(\alpha, \delta; \varphi)$, we have

$$\alpha < \mathrm{Re}\, \frac{g'(z)}{\varphi'(z)} < \delta,\ z \in U. \tag{29}$$

Let $G = J_{p,\gamma}(g)$. We know from Remark 2 that $G, \Phi \in \Sigma_{p,0}$.

From $G = J_{p,\gamma}(g)$ and $\Phi = J_{p,\gamma}(\varphi)$, we get

$$\gamma G(z) + zG'(z) = (\gamma - p)g(z) \text{ and } \gamma\Phi(z) + z\Phi'(z) = (\gamma - p)\varphi(z),\ z \in \dot{U},$$

hence

$$(\gamma+1)G'(z)+zG''(z) = (\gamma-p)g'(z) \text{ and } (\gamma+1)\Phi'(z)+z\Phi''(z) = (\gamma-p)\varphi'(z).$$

Let us denote

$$p(z) = \frac{G'(z)}{\Phi'(z)},\ z \in U.$$

Since $G, \Phi \in \Sigma_{p,0}$ and $z^{p+1}\Phi'(z) \neq 0$, $z \in U$, we have $p \in H(U)$. Of course, $p(0) = 1$.

From $p(z)\Phi'(z) = G'(z)$, we get $G''(z) = p'(z)\Phi'(z)+p(z)\Phi''(z)$, so, the equality

$$(\gamma + 1)G'(z) + zG''(z) = (\gamma - p)g'(z),\ z \in U,$$

can be rewritten as

$$(\gamma + 1)p(z)\Phi'(z) + z[p'(z)\Phi'(z) + p(z)\Phi''(z)] = (\gamma - p)g'(z). \tag{30}$$

Using the equality $(\gamma + 1)\Phi'(z) + z\Phi''(z) = (\gamma - p)\varphi'(z)$, we obtain from (30) that

$$p(z) + \frac{zp'(z)}{\gamma + 1 + \dfrac{z\Phi''(z)}{\Phi'(z)}} = \frac{g'(z)}{\varphi'(z)},\ z \in U,$$

which is equivalent to

$$p(z) + \frac{zp'(z)}{P(z)} = \frac{g'(z)}{\varphi'(z)},\ \text{where } P(z) = \gamma + 1 + \frac{z\Phi''(z)}{\Phi'(z)}.$$

Since $\alpha < \mathrm{Re}\, \dfrac{g'(z)}{\varphi'(z)} < \delta$, $z \in U$, we obtain

$$\alpha < \mathrm{Re}\left[p(z) + \frac{zp'(z)}{P(z)}\right] < \delta,\ z \in U. \tag{31}$$

Let us denote $p_1(z) = p(z) - \alpha$ and $p_2(z) = \delta - p(z)$. Using now (31), we have

$$\operatorname{Re}\left[p_k(z) + \frac{zp_k'(z)}{P(z)}\right] > 0,\ z \in U,\ k = 1, 2. \tag{32}$$

It is easy to see that $p_k(0) > 0$, so, to apply Theorem 2 we need only to verify that $\operatorname{Re} P(z) > 0$, $z \in U$, where $P(z) = \gamma + 1 + \dfrac{z\Phi''(z)}{\Phi'(z)}$.

As we know that $\varphi \in \Sigma K_{p,0}(\alpha, \delta)$ with $z^{p+1}J_{p,\gamma}'(\varphi)(z) \neq 0$, $z \in U$, we obtain from Theorem 13 that

$$\Phi = J_{p,\gamma}(\varphi) \in \Sigma K_{p,0}(\alpha, \delta),$$

which is equivalent to

$$\alpha < \operatorname{Re}\left[-1 - \frac{z\Phi''(z)}{\Phi'(z)}\right] < \delta,\ z \in U,$$

hence

$$\operatorname{Re}\gamma - \delta < \operatorname{Re} P(z) < \operatorname{Re}\gamma - \alpha,\ z \in U.$$

Since $\operatorname{Re}\gamma \geq \delta$, we get $\operatorname{Re} P(z) > 0$, $z \in U$, and we can now apply Theorem 2 to obtain $\operatorname{Re} p_k(z) > 0$, $z \in U$, $k = 1, 2$. Therefore, we have

$$\alpha < \operatorname{Re}\frac{G'(z)}{\Phi'(z)} < \delta,\ z \in U. \tag{33}$$

Since we know that $G \in \Sigma_{p,0}$ and $\Phi \in \Sigma K_{p,0}(\alpha, \delta)$, we have from (33) that $G = J_{p,\gamma}(g) \in \Sigma\mathcal{C}_{p,0}(\alpha, \delta; \Phi)$.

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