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On sufficient condition for starlikeness [1]

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Abstract

In this paper,we give a condition for starlikeness of the integral operator of the form $F(z) = \int_0^z \prod_{\imath=1}^{k} \left(\frac{f_\imath(s)}{s}\right)^{\frac{1}{\alpha}} ds.$

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1 Introduction

Let A be the class of all analytic functions $f(z)$ defined in the open unit disk $U = \{z \in C :| z |< 1\}$ and S the subclass of A consisting of univalent functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

$$S^* = \{f \in S : Re(\frac{zf'(z)}{f(z)}) > 0, z \in U\},$$

$$M_\alpha = \{f \in S : \frac{f(z)f'(z)}{z} \neq 0, ReJ(\alpha, f; z) > 0, z \in U\}$$

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where $J(\alpha, f; z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha(1 + \frac{zf''(z)}{f'(z)})$ be the class of starlike and $\alpha - convex$ functions respectively.
Let $p(z)$ be the class of functions that are regular in U and of the form :

(2) $$p(z) = 1 + \sum_{k=1}^{\infty} b_k z^k$$

Furthermore, let $h(z) = \frac{1+z}{1-z}$.
Let T be the univalent [5] subclass of A consisting of functions $f(z)$ satisfying $|\frac{z^2 f'(z)}{f(z)^2} - 1| < 1, (z \in U)$
Let T_n be the subclass of T for which $f^k(0) = 0$ $(k = 2, 3, ..., n)$.
Let $T_{n,\mu}$ be the subclass of T_n consisting of functions of the form $\int_0^z \prod_{\imath=1}^k (\frac{f_\imath(s)}{s})^{\frac{1}{\alpha}} ds$ satisfying: $|\frac{z^2 f'(z)}{f(z)^2} - 1| < \mu, (z \in U)$ for some $\mu (0 < \mu \leq 1)$.

2 Preliminaries

Theorem 1 *[1] Let M and N be analytic in U with $M(0) = N(0) = 0$. If $N(z)$ maps onto a many sheeted region which is starlike with respect to the origin and $Re\{\frac{M'(z)}{N'(z)}\} > 0$ in U, then $Re\{\frac{M(z)}{N(z)}\} > 0$ in U.*

Theorem 2 *[6] Let $f_\imath \in T_{n,\mu_\imath} (\imath = 1, 2, ..., k; k \in N^*)$ be defined by*

(3) $$f_i(z) = z + \sum_{n=2}^{\infty} a_n^i z^n$$

for all $i = 1, 2, ..., k; \alpha, \beta \in \mathcal{C}; R\{\beta\} \geq \gamma$ and $\gamma = \sum_{i=1}^{k} \frac{1 + (1 + \mu_i)M}{|\alpha|} (M \geq 1, 0 < \mu_i < 1, k \in N^*)$. If $|f_i(z)| \leq M (z \in U), i = 1, 2, ..., k$ then, the integral operator

(4) $$F_{\alpha,\beta}(z) = \{\beta \int_0^z t^{\beta-1} \prod_{i=1}^{k} (\frac{f_i(t)}{t})^{\frac{1}{\alpha}} dt\}^{\frac{1}{\beta}}$$

is univalent.

Theorem 3 *[2] Let h be convex in U and $Re\{\beta h(z) + \gamma\} > 0, z \in U$.If $p \in H(U)$ where $H(U)$ is the class of functions which are analytic in the unit disk, with $p(0) = h(0)$ and p satisfies the Briot-Bouquet differential subordinations: $p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), z \in U$. Then, $p(z) \prec h(z), z \in U$.*

3 Main Results

We now give the proof of the following results:

Theorem 4 *Let $F_\alpha(z)$ be the function in U defined by*

$$F_\alpha(z) = \int_0^z \prod_{i=1}^{k} (\frac{f_i(s)}{s})^{\frac{1}{\alpha}} ds, \alpha \in C. \tag{5}$$

If $f_i \in S^*$ then, $F(z) \in S^*$ where f_i is as in equation (3) above.

Proof. By differentiating (5), we obtain: $F'(z) = \prod_{i=1}^{k}(\frac{f_i(z)}{z})^{\frac{1}{\alpha}}$. Thus, $\dfrac{zF'(z)}{F(z)} = \dfrac{\prod_{i=1}^{k}(\frac{f_i(z)}{z})^{\frac{1}{\alpha}}}{\int_0^z \prod_{i=1}^{k}(\frac{f_i(s)}{s})^{\frac{1}{\alpha}} ds}$.

Let

$$M = zF'(z), N(z) = F(z) \tag{6}$$

From (5) and (6) we have:

$$\frac{M'(z)}{N'(z)} = 1 + \frac{zF''(z)}{F'(z)}, \ \frac{M'(z)}{N'(z)} = 1 + \frac{\sum_{i=1}^{k} \frac{1}{\alpha}(\frac{zf_i'(z)}{f(z)} - 1)}{\prod_{i=1}^{k}(\frac{f_i(z)}{z})^{\frac{1}{\alpha}}}$$

$$|\frac{M'(z)}{N'(z)} - 1| = \frac{|\sum_{i=1}^{k} \frac{1}{\alpha}(\frac{zf_i'(z)}{f(z)} - 1)|}{|\prod_{i=1}^{k}(\frac{f_i(z)}{z})^{\frac{1}{\alpha}}|} \leq \frac{\sum_{i=1}^{k} |\frac{1}{\alpha}||\frac{zf_i'(z)}{f(z)} - 1|}{|\prod_{i=1}^{k}(\frac{f_i(z)}{z})^{\frac{1}{\alpha}}|}.$$

By hypothesis $f_i \in S^*$. This means that $|\frac{zf_i'(z)}{f(z)} - 1| < 1$, which implies that $|\frac{M'(z)}{N'(z)} - 1| < 1$. Thus $Re\{\frac{M'(z)}{N'(z)}\} > 0$ and by Theorem 1, $Re\{\frac{M(z)}{N(z)}\} > 0$. This implies that $Re\{\frac{zF'(z)}{F(z)}\} > 0$. Hence $F \in S^*$.

Remark 1 *The integral in (5) is equivalent to that in (4) of section 2 with $\beta = 1$.*

Let $S = \{f : U \to C\} \cap S$. Let $F(z) \in U$ be defined by

$$F(z) = \int_0^z \prod_{i=1}^{k} (\frac{f_i(s)}{s})^{\frac{1}{\alpha}} ds. \tag{7}$$

Theorem 5 *Let* $z \in U, \alpha \in C, Re\alpha > 0$ *and* $m_\alpha = M_\alpha \cap s$. *If* $F \in m_\alpha$, *then* $F \in S^*$ *that is* $m_\alpha \subset S^*$.

Proof. From (6) above, we have $\frac{F(z)F'(z)}{z} \neq 0$ and for $F \in m_\alpha$, we have

$$ReJ(\alpha, f; z) = Re\{(1-\alpha)\frac{zF'(z)}{F(z)} + \alpha(1 + \frac{zF'(z)}{F(z)})\} \tag{8}$$

for $p(z) = \frac{zF'(z)}{F(z)}$, $\frac{zp'(z)}{p(z)} = 1 + \frac{zF"(z)}{F'(z)} - p(z)$. This implies that

$$1 + \frac{zF''(z)}{F'(z)} = \frac{zp'(z)}{p(z)} + p(z) \tag{9}$$

using (7) and (9) in (8), we obtain

$$ReJ(\alpha, f; z) = Re\{(1-\alpha)p(z) + \alpha(\frac{zp'(z)}{p(z)} + p(z))\}. \tag{10}$$

Simplifying (10), we obtain $ReJ(\alpha, f; z) = Re\{p(z) + \alpha(\frac{zp'(z)}{p(z)})\}$
$p(0) + \frac{\alpha zp'(0)}{p(0)} = 1$ and $p(0) = h(0) = 1$. Thus, using Theorem 3 with $\beta = 1$ and $\gamma = 0$,we have $p(z) + \frac{\alpha zp'(z)}{p(z)} < h(z) = \frac{1+z}{1-z}$. This implies that $p(z) \prec h(z)$. That is $Re\{p(z)\} > 0$. Thus, $Re\{\frac{zF'(z)}{F(z)}\} > 0$. Hence, $F \in S^*$.

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