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A new characterization of the golden number [1]

Cristinel Mortici, Andrei Vernescu

Abstract

The so-called golden number $\varphi = (1+\sqrt{5})/2$ admits two well-known representations: one as a limit of an standard sum of iterated square roots and other as a continuous fraction, both containing only the number 1. In a previous paper [8] of one of the authors, it is proved that this property characterizes the golden number, i.e., if

$$\sqrt{a+\sqrt{a+\sqrt{a+\ldots}}} = a + \cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{\ddots}}},$$

then it results that $a = 1$ and the common value of the two parts of the equality is the golden number φ. In this paper we revisit this result and we compare the two expressions from above in general case. We obtain an interesting result, i.e., in the case $a < 1$ the inequality holds in certain sense, while in the case $a > 1$, the reversed inequality appears.

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1 Introduction

First, if consider only the structure as real number of the so-called *golden number* $\varphi = (1+\sqrt{5})/2$, then it is a quadratic algebraic number, being the greatest root of the equation $\varphi^2 - \varphi - 1 = 0$, while the other root of this equation will be denoted by $\overline{\varphi}$.

However, the golden number consists the object of a very special attention, because of its interesting and useful properties, some of them given below:

(i) the golden number appears in the division of a segment of length a in two parts of lengths u and v ($u > v$, $u + v = a$) such that

$$\frac{u+v}{u} = \frac{u}{v},$$

which gives $u/v = \varphi$ (from here it results also for φ the name of *golden ratio*).

(ii) the expressions of the general term of the Fibonacci's sequence $(F_n)_{n\geq 0}$ and respective Lucas' sequence $(L_n)_{n\geq 0}$ defined by

$$F_{n+2} = F_{n+1} + F_n \quad ; \quad F_0 = 0\ , \quad F_1 = 1$$

$$L_{n+2} = L_{n+1} + L_n \quad ; \quad L_0 = 2\ , \quad L_1 = 1$$

involve φ and $\overline{\varphi}$, namely $F_n = (\varphi^n - \overline{\varphi}^n)/\sqrt{5}$, respective $L_n = \varphi^n + \overline{\varphi}^n$.

(iii) Fibonacci considered the following (naive) model of the growth of a rabbit population. If it is assumed that the population starts having bunnies once a month after they are two months old; they always give birth to twins (one male and one female bunny), they never die and they never stop propagating. Then the number of rabbits pairs after n months is equal to F_n. The Fibonacci sequence is considered so important such that a special journal in pure mathematics is entitled "The Fibonacci Quarterly". It was founded in 1963 and appears as "The official journal of the Fibonacci Association, devoted to the study of integers with special properties".

(iv) The golden number is closely related to the logarithmic spiral (see [2]).

(v) The golden number is involved in a lot of results concerning the summations of some series, as

$$\left(1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}\right)+\left(\frac{1}{11}-\frac{1}{12}-\frac{1}{13}+\frac{1}{14}\right)+\ldots=\frac{2\sqrt{5}}{5}\ln\varphi$$

(see [1]), a beautiful illustration of a celebrated theorem of Riemann, or

$$\frac{1}{F_1}+\frac{1}{F_2}+\frac{1}{F_4}+\frac{1}{F_8}+\frac{1}{F_{16}}+\ldots=4-\varphi$$

(see [5]). For proofs and other properties see for example [2] - [6], the journal [7] and the reference therein.

All these mathematical properties as well as some properties in biology (we do not consider here also mystic, esoteric or aesthetic properties) explain the wide attention given to φ and these can be completed by the following two equalities:

$$\varphi=\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}$$

and

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}$$

(the first being in the rigourous sense that $\varphi = \lim_{n\to\infty} x_n$, where

$$x_n = \left.\sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots + \sqrt{1}}}}\right\} n \text{ radicals},$$

while the second containing an usual continued fraction). Hence φ admits the most simple representation in terms of the numbers

$$x(a) = \sqrt{a + \sqrt{a + \sqrt{a + \ldots}}} \tag{1}$$

and

$$y(b) = b + \cfrac{1}{b + \cfrac{1}{b + \cfrac{1}{\ddots}}} \tag{2}$$

with $a = 1$ and $b = 1$. In [8], the converse implication is proved, that is if a real number A admits the both representations (1) and (2) with the same parameter $a = b$, then $a = b = 1$ and $A = \varphi$. In other words, if

$$A = \sqrt{a + \sqrt{a + \sqrt{a + \ldots}}} = a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{\ddots}}}, \tag{3}$$

then $a = 1$ and consequently, $A = \varphi$. In the present paper we will give a new simple proof of the above result and in the last section we will compare the iterated radicals and the continuous fraction from (3) in general case $a \neq 1$.

2 The second proof

In [8] it is remembered the known convergence of the sequences

$$x_n(a) = \underbrace{\sqrt{a + \sqrt{a + \sqrt{a + ... + \sqrt{a}}}}}_{n \text{ radicals}}$$

and

$$y_n(b) = b + \cfrac{1}{b + \cfrac{1}{b + \ddots + \cfrac{1}{b}}} \qquad (n-1 \text{ fractions})$$

using the Weierstrass theorem of bounded, monotone sequences. For sake of completeness, we will compute here the general term of the sequence $(y_n(b))_{n\geq 1}$, using a nice idea of an adequate substitution and we will find also connections of that sequence with the Fibonacci's sequence. We have the recurrence relation

$$y_{n+1}(b) = b + \frac{1}{y_n(b)}$$

or $y_{n+1}(b)y_n(b) = by_n(b)+1$. With the notation $y_n(b) = u_{n+1}/u_n$ and $u_1 = 1$, we obtain

$$\frac{u_{n+2}}{u_{n+1}} \cdot \frac{u_{n+1}}{u_n} = b \cdot \frac{u_{n+1}}{u_n} + 1,$$

thus $u_{n+2} = bu_{n+1} + u_n$ (note that in case $b = 1$, this is the recurrence from the Fibonacci's sequence or from Lucas's sequence). Further, if $r_{12} = (b \pm \sqrt{b^2 + 4})/2$ are the roots of the associated quadratic equation $r^2 - br - 1 = 0$, then we have $u_n = \lambda r_1^n + \mu r_2^n$, where λ, μ are given by the system

$$\begin{cases} \lambda r_1 + \mu r_2 = 1 \\ \lambda r_1^2 + \mu r_2^2 = b \end{cases}.$$

In consequence, after some easy computations, we deduce that

$$y_n(b) = \frac{(b+\sqrt{b^2+4})^{n+1} - (b-\sqrt{b^2+4})^{n+1}}{(b+\sqrt{b^2+4})^n - (b-\sqrt{b^2+4})^n}$$

and we see that the sequence $(y_n(b))_{n\geq 1}$ converges to $(b+\sqrt{b^2+4})/2$. We can write that

$$b + \cfrac{1}{b + \cfrac{1}{b + \cfrac{1}{\ddots}}} = \frac{b+\sqrt{b^2+4}}{2}.$$

We have also from [8] that

$$\sqrt{a+\sqrt{a+\sqrt{a+...}}} = \frac{1+\sqrt{4a+1}}{2}.$$

Remember that in [8] there are given two methods to prove that if

$$\sqrt{a+\sqrt{a+\sqrt{a+...}}} = a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{\ddots}}},$$

then $a = 1$. Now we are in position to give a simpler proof of this result, by using a nice idea. To do this, assume that (3) is true. Then from

$$A = \sqrt{a+\sqrt{a+\sqrt{a+...}}}$$

it results that $A = \sqrt{a+A}$ and from

$$a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{\ddots}}} = A$$

it results that $a + \dfrac{1}{A} = A$. In consequence,

$$A = \sqrt{a + A} = a + \frac{1}{A}$$

and from here, the common value of a is

$$a = A^2 - A = A - \frac{1}{A}. \tag{4}$$

In terms of A, we have $A^3 - 2A^2 + 1 = 0$ or $(A - 1)(A^2 - A - 1) = 0$. The case $A = 1$ is not acceptable because it should attract $a = 0$ in (4). As $A > 0$, we derive $A = \varphi$, so we are done.

3 The main result

We discuss here the more general case $a \neq 1$ in order to compare $x(a)$ and $y(a)$ given by (1)-(2). The answer to this problem, which also solve the result from [8], is given by the next

Theorem 1 *Let there be given $a > 0$. Then*

$$\sqrt{a + \sqrt{a + \sqrt{a + \ldots}}} > a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{\ddots}}} \quad , \qquad \text{for every } a \in (0, 1)$$

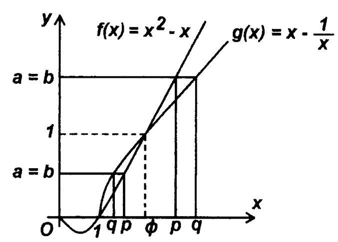
and $\sqrt{a + \sqrt{a + \sqrt{a + \ldots}}} < a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{\ddots}}}$, *for every* $a \in (1, \infty)$.

In consequence, $\sqrt{a + \sqrt{a + \sqrt{a + \ldots}}} = a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{\ddots}}}$ *if and only if* $a = 1$.

Proof. Here we denote by p and q the involved expressions:

$$p=\sqrt{a+\sqrt{a+\sqrt{a+\ldots}}} \quad \text{and} \quad q=a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{\ddots}}}, \tag{5}$$

from which it results $p=\sqrt{a+p}$, respective $q=a+\dfrac{1}{q}$. Further, $a=f(p)$ and $a=g(q)$, where $f,\ g:(0,\infty)\to\mathbb{R}$ are $f(x)=x^2-x$, $g(x)=x-\dfrac{1}{x}$, which can be viewed in the next figure.



The graphs of the functions f and g meet at the point $(1,0)$ and $(\varphi,1)$. As we can see from the above figure or as we can be easily prove, we have $g(x)>f(x)$, for every $x\in(1,\varphi)$ and $g(x)<f(x)$, for every $x\in(\varphi,\infty)$. In our first hypotesis, we have $a\in(0,1)$ (on the Oy axis) then there exists an unique point $p\in(1,\varphi)$ such that $a=f(p)$ and there exists an unique point $q\in(1,\varphi)$ such that $a=g(q)$. More precisely, p and q are given by (5) and we have to prove that $p>q$. If we assume by contrary that $p\leq q$, then, using also the monotony of the function f, we get the following contradiction:

$$a=f(p)\leq f(q)<g(q)=a.$$

As we explained, $p>q$, thus from (5), the first assertion of the theorem is proved.

In our second hypotesis, we have $a \in (1, \infty)$ (on the Oy axis) then there exists an unique point $p \in (\varphi, \infty)$ such that $a = f(p)$ and there exists an unique point $q \in (\varphi, \infty)$ such that $a = g(q)$. More precisely, p and q are given by (5) and we have to prove that $p < q$. If assume by contrary that $p \geq q$, then using also the monotony of the function f, we get the following contradiction:

$$a = f(p) \geq f(q) > g(q) = a.$$

As we explained, $p < q$, thus the proof is completed with (5).

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Cristinel Mortici
Valahia University of Târgovişte
Department of Mathematics,
Bd. Unirii 18, 130082 Târgovişte, Romania
e-mail: cmortici@valahia.ro

Andrei Vernescu
Valahia University of Târgovişte
Department of Mathematics,
Bd. Unirii 18, 130082 Târgovişte, Romania
e-mail: avernescu@clicknet.ro