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Strong converse results for Bernstein-Durrmeyer operators and their quasi-interpolants [1]

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Abstract

The purpose of this paper is the investigation of strong converse results for Bernstein-Durrmeyer operators and their quasi-interpolants. For lower dimensions we improve known results by Knoop and Zhou [16] and Chen, Ditzian and Ivanov [5] in regard to the constants in their estimates and prove a strong converse theorem of type B for the quasi-interpolants introduced by Berdysheva, Jetter and Stöckler (see e. g. [2]) with respect to a modified K-functional in the univariate case.

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1 Introduction

The so-called Bernstein-Durrmeyer operators were first defined by Durrmeyer [11] and independently developed by Lupaş [15].

The research on these operators was initiated by Derriennic [6] in 1981. Ditzian and Ivanov [9] gave a fresh impetus to this topic by stating the commutativity of the operators and using the Ditzian-Totik modulus of smoothness and appropriate equivalent K-functionals (see [10]) for their direct and converse results.

The multivariate Bernstein-Durrmeyer operators given in the following definition were also introduced by Derriennic [7].

Definition 1 *Let S^d denote the simplex* $S^d = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_i \geq 0, \sum_{i=1}^d x_i \leq 1\}$. *For $f \in L_p(S^d)$, $1 \leq p \leq \infty$, $n \in \mathbb{N}$, the multivariate Bernstein-Durrmeyer operators are given by*

$$\begin{aligned}(M_n f)(x) &= \sum_{k/n \in S} p_{nk}(x) \frac{(n+d)!}{n!} \int_S f(t) p_{nk}(t) dt \\ &= \sum_{k/n \in S} p_{nk}(x) \frac{(n+d)!}{n!} < p_{nk}, f >,\end{aligned}$$

where

$$p_{n,k}(x) = \frac{n!}{k!(n-|k|)!} x^k (1-|x|)^{n-|k|},\ x \in S$$

and

$$k! = k_1! \cdot \dots \cdot k_d!,\ |k| = \sum_{i=1}^{d} k_i,\ |x| = \sum_{i=1}^{d} x_i,\ x^k = x_1^{k_1} \cdot \dots \cdot x_d^{k_d}.$$

To study the approxiamtion behaviour of the operators M_n we use a modified K-functional given in [5] by

$$K^2(f,t)_p = \inf\left\{ \|f-g\|_p + t\|\widetilde{D}^2 g\|_p :\ g \in C^2(S^d) \right\}, \tag{1}$$

where $\widetilde{D}^2$ denotes the partial differential operator

$$\sum_{i=1}^{d} \frac{\partial}{\partial x_i} x_i(1-|x|) \frac{\partial}{\partial x_i} + \sum_{i<j} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) x_i x_j \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right)$$

of second order.

For $f \in L_p(S^d)$, $1 \le p \le \infty$, the direct estimate

$$\|M_n f - f\|_p \le 4K^2\left(f, \frac{1}{n}\right)_p$$

is proved in [4, Theorem 3.1]. Considering [3, Theorem 19] for the special case $r = 0$, $\mu = 0$, with $c_{0,0} = 1$ and $\alpha_n(d) = \dfrac{C_{n,0,d}}{\binom{n}{r+1}} = \displaystyle\sum_{l=n+1}^{\infty} \frac{1}{l(l+d)}$ $= \dfrac{1}{d}\left[\dfrac{1}{n+1} + \dots + \dfrac{1}{n+d}\right]$ in the notation of [3], we have

$$\|f - M_n f\|_p \le 2\|f-g\|_p + \alpha_n(d)\|\tilde{D}^2 g\|_p$$

for each $g \in C^2(S^d)$, i. e.

$$\|f - M_n f\|_p \le 2K^2\left(f, \frac{\alpha_n(d)}{2}\right)_p. \tag{2}$$

Concerning the strong converse result of type A in the terminology of [9], it was proved in [5, Theorem 6.1] that for $f \in L_p(S^d)$, $1 \le p \le \infty$, $d \le 3$

$$K^2\left(f, \frac{1}{n}\right)_p \le A\, \|M_n f - f\|_p\,. \tag{3}$$

Knoop and Zhou could prove (3) for all $f \in L_p(S^d)$, $1 \le p \le \infty$, without the restriction on the dimension d and stated, that $A = A(d) \le 1 + d^3 + 5d^7$, i. e. $A(1) \le 7$, $A(2) \le 649$ and $A(3) \le 10963$.
In order to derive significantly better constants in case $d \le 3$, we improve the strong Voronovskaja-type result [5, Theorem 4.1] in section 2 .

To increase the order of approximation Jetter and Stöckler constructed quasi-interpolants of the multivariate Bernstein-Durrmeyer operators in [13]. In this paper we restrict ourselfes to the case $d = 1$.

Definition 2 *The quasi-interpolants $M_{n,r}$ of type (r, n), $0 \le r \le n$, are given by*

$$M_{n,r}f = \sum_{l=0}^{r} (-1)^l \frac{(n-l)!}{n!l!} \widetilde{D}^{2l}(M_n f),$$

where $\widetilde{D}^{2l}$ denotes the differential operator $\frac{d^l}{dx^l}\left(x^l(1-x)^l \frac{d^l}{dx^l}\right)$ of order $2l$.

The quasi-interpolants of type (r, n) reproduce polynomials of order r and contain the Bernstein-Durrmeyer operators as a special case for $r = 0$. To study the approximation behaviour of the quasi-interpolants we use the K-functional

$$K^{2(r+1)}(f,t)_p = \inf\left\{||f-g||_p + t||\widetilde{D}^{2(r+1)}g||_p : \; g \in C^{2(r+1)}[0,1]\right\},$$

introduced in [13]. For the quasi-interpolants the direct result

$$\|M_{n,r}f - f\|_p \le 2^{r+1} K^{2(r+1)}\left(f, \beta_n(r)\right)_p, \; \beta_n(r) = \frac{(n-r)!}{(n+1)!(r+1)!},$$

was proved in [2, Theorem 4].

2 Better constants in strong converse results of type A

In this section we determine good constants in the strong converse results of type A for the operators M_n. To do so, we first have to improve the strong Voronovskaja-type result by Chen, Ditzian and Ivanov. In [5, Theorem 4.1] they showed, that for $f \in C^4(S^d)$, $n \in \mathbb{N}$, $n > 1$,

$$\left\|M_n f - f - \frac{1}{2}\alpha_n(d)\widetilde{D}^2\left[f + M_n f\right]\right\|_p \leq \left\|(\widetilde{D}^2)^2 f\right\|_p \cdot \left\{\frac{\alpha_n(d)^2}{4} + \frac{1}{2(n+1)^3}\right\},$$

with $\alpha_n(d) = \sum_{l=n+1}^{\infty} \frac{1}{l(l+d)} = \frac{1}{d}\sum_{i=1}^{d} \frac{1}{n+i}$ given as above. For $d \leq 3$ this estimate will be improved in the following lemma.

Lemma 1 *Let $f \in C^4(S^d)$, $n \in \mathbb{N}$, $d \leq 3$. Then*

$$\left\|M_n f - f - \frac{1}{2}\alpha_n(d)\widetilde{D}^2\left[f + M_n f\right]\right\|_p \leq \left\|(\widetilde{D}^2)^2 f\right\|_p \cdot \psi(n),$$

where

$$\psi(n) = \begin{cases} \frac{1}{4(n+1)^2} + \frac{1}{6(n+1)^4}, & d = 1 \\ \frac{1}{4(n+1)^2}, & d = 2, 3 \end{cases}.$$

Proof

From [5, Proof of Theorem 4.1] we have

$$\begin{aligned} &\left\|M_n f - f - \frac{1}{2}\alpha_n(d)\widetilde{D}^2\left[f + M_n f\right]\right\|_p \\ &\leq \frac{1}{2}\left[S_1(d) + S_2(d) - S_3(d)\right]\|(\widetilde{D}^2)^2 f\|_p, \end{aligned} \tag{4}$$

where

$$S_1(d) = \begin{cases} 2\alpha_n(1)\alpha_{2n+2}(1) - \alpha_{2n+2}^2(1) - \alpha_{2n+1}^2(1), & d = 1 \\ 2\alpha_{2n+2}(d)\left[\alpha_n(d) - \alpha_{2n+2}(d)\right], & d = 2,3 \end{cases},$$

$$S_2(d) = \sum_{j=n+1}^{j_0} \frac{1}{j^2(j+d)^2} \quad \text{and} \quad S_3(d) = \sum_{j=2n+3}^{\infty} \frac{1}{j^2(j+d)^2}.$$

Here and in what follows, we denote by j_0 the index

$$j_0 = \max\{j \,:\, 2\alpha_{j-1}(d) - \alpha_n(d) > 0\}.$$

For the dimensions d discussed in this paper we have

$$j_0 = \begin{cases} 2n+1, & d = 1 \\ 2n+2, & d = 2,3 \end{cases}.$$

Note that $2 \cdot \alpha_{2n+2}(1) - \alpha_n(1) = 0$.

$S_1(d)$ can be calculated directly with the definition of the $\alpha_n(k)$ to

$$(5) \quad S_1(d) = \begin{cases} \dfrac{1}{4} \cdot \dfrac{8n^2+20n+11}{(n+1)^2(2n+3)^2}, & d = 1 \\ \dfrac{1}{8} \cdot \dfrac{(4n+7)(4n^2+13n+11)}{(n+1)(n+2)^2(2n+3)^2}, & d = 2 \\ \dfrac{1}{18} \cdot \dfrac{(12n^2+48n+47)(12n^4+96n^3+287n^2+380n+189)}{(n+1)(n+2)^2(n+3)(2n+3)^2(2n+5)^2}, & d = 3 \end{cases}.$$

Taking into account that $j(j+d) \geq (j-1)(j+d+1)$ we get the estimate

$$
\begin{aligned}
(6)\, S_2(d) & \\
\leq\ & \sum_{j=n+1}^{j_0} \frac{1}{(j-1)j(j+d)(j+d+1)} \\
=\ & \frac{1}{d+1}\left\{\frac{1}{d}\sum_{j=n+1}^{j_0}\left[\frac{1}{j+d}-\frac{1}{j}\right]+\frac{1}{d+2}\sum_{j=n+1}^{j_0}\left[\frac{1}{j-1}-\frac{1}{j+d+1}\right]\right\} \\
=\ & \frac{1}{d+1}\left\{\frac{1}{d}\sum_{i=1}^{d}\left[\frac{1}{i+j_0}-\frac{1}{i+n}\right]+\frac{1}{d+2}\sum_{i=0}^{d+1}\left[\frac{1}{i+n}-\frac{1}{i+j_0}\right]\right\} \\
=\ & \begin{cases} \frac{1}{6}\cdot\frac{7n^2+14n+6}{n(n+1)(n+2)(2n+1)(2n+3)}, & d=1 \\ \frac{1}{24}\cdot\frac{28n^3+157n^2+291n+180}{n(n+1)(n+2)(n+3)(2n+3)(2n+5)}, & d=2 \\ \frac{1}{60}\cdot\frac{70n^4+560n^3+1669n^2+2196n+1080}{n(n+1)(n+2)(n+3)(n+4)(2n+3)(2n+5)}, & d=3 \end{cases}.
\end{aligned}
$$

For the estimate of $S_3(d)$ we use that $j(j+1)^2 \leq (j-1)(j+2)(j+3)$ in case $d=1$ and $j(j+d)^2 \leq (j+1)(j+2)(j+3)$ in case $d=2,3$ respectively. Thus we derive

$$
\begin{aligned}
(7)\qquad & S_3(d) \\
\geq\ & \begin{cases} \sum_{j=2n+3}^{\infty}\left[\frac{1}{6}\left(\frac{1}{j+2}-\frac{1}{j}\right)+\frac{1}{12}\left(\frac{1}{j-1}-\frac{1}{j+3}\right)\right], & d=1 \\ \sum_{j=2n+3}^{\infty}\left[\frac{1}{6}\left(\frac{1}{j}-\frac{1}{j+3}\right)+\frac{1}{2}\left(\frac{1}{j+2}-\frac{1}{j+1}\right)\right], & d=2,3 \end{cases} \\
=\ & \begin{cases} \frac{1}{24}\cdot\frac{4n+7}{(n+1)(n+2)(2n+3)(2n+5)} & d=1 \\ \frac{1}{6}\cdot\frac{1}{(n+2)(2n+3)(2n+5)} & d=2,3 \end{cases}.
\end{aligned}
$$

For the different dimensions $d = 1, 2, 3$ in question we derive from (5), (6) and (7)

$$\begin{aligned} & S_1(1) + S_2(1) - S_3(1) \\ \leq \; & \frac{1}{8} \cdot \frac{64n^6 + 512n^5 + 1588n^4 + 2412n^3 + 1887n^2 + 741n + 120}{n(n+1)^2(n+2)(2n+1)(2n+3)^2(2n+5)} \\ \leq \; & \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^4}, \end{aligned}$$

$$\begin{aligned} & S_1(2) + S_2(2) - S_3(2) \\ \leq \; & \frac{1}{8} \cdot \frac{32n^6 + 352n^5 + 1540n^4 + 3402n^3 + 3925n^2 + 2133n + 360}{n(n+1)(n+2)^2(n+3)(2n+3)^2(2n+5)} \\ \leq \; & \frac{1}{2(n+1)^2}, \end{aligned}$$

$$\begin{aligned} & S_1(3) + S_2(3) - S_3(3) \\ \leq \; & \frac{1}{180} \cdot \big\{1440n^8 + 23760n^7 + 165360n^6 + 633888n^5 + 1458370n^4 \\ & \qquad + 2043371n^3 + 1669696n^2 + 694440n + 97200\big\} \, / \\ & \big\{n(n+1)(n+2)^2(n+3)(n+4)(2n+3)^2(2n+5)^2\big\} \\ \leq \; & \frac{1}{2(n+1)^2}. \end{aligned}$$

Together with (4) we have proved our lemma.

We are now in the position to prove the main result of this section.

Theorem 1 *Let $f \in L_p(S^d)$, $1 \leq p \leq \infty$, $d \leq 3$. Then*

$$K^2(f, \alpha_n(d))_p \leq A(d) \, \|M_n f - f\|_p$$

for every $n \in \mathbb{N}$, where $A(1) \leq \frac{19}{4}$, $A(2) \leq 8$ and $A(3) \leq \dfrac{85}{4}$.

Proof.

Following the lines of [5, Proof of Theorem 6.3] we choose $g = \frac{1}{2}\left[M_n^4 f + M_n^3 f\right]$. As the operators M_n are contractions we first get

$$\|f - g\|_p \leq \frac{7}{2}\|M_n f - f\|_p \,. \tag{8}$$

Application of the strong Voronovskaja-type result in lemma 1 to the function $M_n^3 f$ leads us to

$$\alpha_n(d)\|\widetilde{D}^2 g\|_p \leq ||M_n f - f||_p + \psi(n)||(\widetilde{D}^2)^2 M_n^3 f||_p. \tag{9}$$

In [5, Lemma 3.1, Lemma 3.2] it was proved that $\|\widetilde{D}^2(M_n f)\|_p \leq 2dn\|f\|_p$ and $\|\widetilde{D}^2(M_n^2 f)\|_p \leq dn\|f\|_p$. So, as the operators M_n and the differential operators $\widetilde{D}^2$ commute (see [12, Lemma 3.1] for $d = 1$ and [4, Lemma 2.5] for the general case) and the operators M_n are contractions, we derive

$$\begin{aligned}
\|(\widetilde{D}^2)^2(M_n^3 f)\|_p &= \|\widetilde{D}^2(M_n^2 \widetilde{D}^2 M_n f)\|_p \\
&\leq dn\|\widetilde{D}^2(M_n f)\|_p \\
&\leq dn\Big\{\|\widetilde{D}^2 g\|_p + \|\widetilde{D}^2(M_n(f - M_n f)\|_p \\
&\quad + \|\widetilde{D}^2(M_n^2(f - M_n f))\|_p + \frac{1}{2}\|\widetilde{D}^2(M_n^2(M_n f - M_n^2 f))\|_p\Big\} \\
&\leq dn\left\{\|\widetilde{D}^2 g\|_p + \frac{7}{2}dn\|M_n f - f\|_p\right\}.
\end{aligned}$$

Together with (9) we get

$$\alpha_n(d)\|\widetilde{D}^2 g\|_p \leq \|M_n f - f\|_p + \psi(n)dn\|\widetilde{D}^2 g\|_p + \frac{7}{2}\psi(n)d^2n^2||M_n f - f||_p.$$

This yields

$$\frac{\alpha_n(d)}{2}\|\widetilde{D}^2 g\|_p \leq B_n(d)\,\|M_n f - f\|_p \,, \tag{10}$$

where

$$B_n(d) = \alpha_n(d)\frac{1+\frac{7}{2}d^2n^2\psi(n)}{2[\alpha_n(d)-dn\psi(n)]}.$$

With the definition of $\psi(n)$ in lemma 1 we get for each $n \in \mathbb{N}$

$$\begin{aligned} B_n(1) &= \frac{1}{4}\cdot\frac{45n^4+138n^3+179n^2+96n+24}{(n+1)(9n^3+30n^2+31n+12)} \leq \frac{5}{4}, \\ B_n(2) &= \frac{1}{4}\cdot\frac{(2n+3)(9n^2+4n+2)}{(n+1)(n^2+3n+3)} \leq \frac{9}{2}, \\ B_n(3) &= \frac{1}{4}\cdot\frac{(3n^2+12n+11)(71n^2+16n+8)}{(n+1)(3n^3+15n^2+38n+44)} \leq \frac{71}{4}. \end{aligned}$$

Together with (8), (10) and the definition of the K-functional we derive our proposition.

3 Strong converse result of type B for quasi-interpolants

In this section we prove a strong converse inequality of type B in the terminology of [8] for the quasi-interpolants of the Bernstein-Durrmeyer operators in the univariate case.

First we need some auxiliary results, given in the following lemmas, which were proved by Berdysheva, Jetter and Stöckler in [2] and [3] or follow from the Bernstein-type inequality

$$(11) \qquad \|\widetilde{D}^{2l}(M_n f)\|_p \leq 2^l \frac{n!l!}{(n-l)!}\|f\|_p,$$

proved in [2, Theorem 3.1].

First we need, that the norms of the quasi-interpolants are uniformly bounded, (see [2, 4.1]).

Lemma 2 *Let* $f \in L_p[0,1]$, $1 \leq p \leq \infty$. *Then*

$$\|M_{n,r}f\|_p \leq (2^{r+1}-1)\|f\|_p.$$

The next two estimates follow from the Bernstein-type inequality.

Lemma 3 *Let* $f \in L_p[0,1]$, $1 \leq p \leq \infty$. *Then*

$$\left\|\widetilde{D}^2(M_{n,r}f)\right\|_p \leq \gamma_n(r)\|f\|_p,$$

where

$$\gamma_n(r) = (n-r)2^{r+1}(r+1) + (n+2)[2^{r+1}(r-1)+2].$$

Proof.

It follows from [3, (11)], that

$$\begin{aligned}
&\widetilde{D}^2(M_{n,r}f) \\
&= \sum_{l=0}^{r}(-1)^l \frac{(n-l)!}{l!n!}\widetilde{D}^{2(l+1)}(M_nf) - \sum_{l=1}^{r}(-1)^l \frac{(n-l)!}{l!n!} l(l+1)\widetilde{D}^{2l}(M_nf) \\
&= (-1)^r \frac{(n-r)!}{r!n!}\widetilde{D}^{2(r+1)}(M_nf) + (n+2)\sum_{l=0}^{r-1}(-1)^l \frac{(n-l-1)!}{l!n!}\widetilde{D}^{2(l+1)}(M_nf).
\end{aligned}$$

Together with the Bernstein-type inequality (11) we get

$$||\widetilde{D}^2(M_{n,r}f)||_p \leq \left\{2^{r+1}(n-r)(r+1) + (n+2)\sum_{l=0}^{r-1}(l+1)2^{l+1}\right\}||f||_p.$$

As $\sum_{l=0}^{r-1}(l+1)2^{l+1} = 2^{r+1}(r-1)+2$, which can easily proved by induction, it follows that

$$||\widetilde{D}^2(M_{n,r}f)||_p \leq \left\{2^{r+1}(n-r)(r+1) + (n+2)\left[2^{r+1}(r-1)+2\right]\right\}||f||_p.$$

Lemma 4 *Let $f \in L_p[0,1]$, $1 \le p \le \infty$. Then*

$$\left\|\widetilde{D}^{2(r+1)}(M_{n,r}f)\right\|_p \le c_n(r)\|f\|_p,$$

where

$$c_n(r) = \frac{n!}{(n-r-1)!}\frac{(2r+1)!}{r!}2^r(3^{r+1}-1).$$

Proof.

In [1, Lemma 2.3] it was proved, that for $g \in C^{2(r+1+l)}[0,1]$

$$\begin{aligned}&[x(1-x)]^{r+1}D^{r+1}(\widetilde{D}^{2l}g)\\ &= \sum_{k=0}^{l}\binom{l}{k}\frac{(r+l+1)!}{(r+k+1)!}(-1)^{l-k}D^k\left\{[x(1-x)]^{r+1+k}D^{r+1+k}g\right\},\end{aligned}$$

where D^k denotes the usual differential operator.

Differentiating both sides $(r+1)$-times, we derive for $g \in C^{2(r+1+l)}[0,1]$

$$\widetilde{D}^{2(r+1)}\widetilde{D}^{2l}g = \sum_{k=0}^{l}\binom{l}{k}\frac{(r+l+1)!}{(r+k+1)!}(-1)^{l-k}\widetilde{D}^{2(r+1+k)}g.$$

Applying the definition of $M_{n,r}$ and substituting $g = M_n f$ in the last equatity leads to the estimate

$$\begin{aligned}||\widetilde{D}^{2(r+1)}M_{n,r}f||_p &\le \sum_{l=0}^{r}\frac{(n-l)!}{l!n!}||\widetilde{D}^{2(r+1)}\widetilde{D}^{2l}M_nf||_p\\ &\le \sum_{l=0}^{r}\frac{(n-l)!}{l!n!}\sum_{k=0}^{l}\binom{l}{k}\frac{(r+l+1)!}{(r+k+1)!}||\widetilde{D}^{2(r+1+k)}M_nf||_p.\end{aligned}$$

With (11) and as $\dfrac{1}{(n-r-1-k)!} \le \dfrac{1}{(n-r-1-l)!}$ we get

$$\begin{aligned}||\widetilde{D}^{2(r+1)}M_{n,r}f||_p &\le ||f||_p\sum_{l=0}^{r}\frac{(n-l)!}{(n-r-1-l)!}\frac{(r+1+l)!}{l!}\sum_{k=0}^{l}\binom{l}{k}2^{r+1+k}\\ &= 2^{r+1}||f||_p\sum_{l=0}^{r}\frac{(n-l)!}{(n-r-1-l)!}\frac{(r+1+l)!}{l!}3^l.\end{aligned}$$

As $\frac{(n-l)!}{(n-r-1-l)!} \leq \frac{n!}{(n-r-1)!}$ and $\frac{(r+1+l)!}{l!} \leq \frac{(2r+1)!}{r!}$ we derive

$$||\widetilde{D}^{2(r+1)}M_{n,r}f||_p \leq 2^{r+1}\frac{n!}{(n-r-1)!} \cdot \frac{(2r+1)!}{r!} \cdot \frac{1}{2}(3^{r+1}-1)||f||_p.$$

The following strong Voronovskaja-type theorem was shown in [2] in a more general setting.

Lemma 5 *Let $f \in C^{2(r+2)}[0,1]$. Then*

$$\left\|M_{n,r}f - f - (-1)^r\beta_n(r)\widetilde{D}^{2(r+1)}f\right\|_p \leq \frac{1}{n+2}\beta_n(r)\|\widetilde{D}^{2(r+1)}(\widetilde{D}^2 f)\|_p,$$

where

$$\beta_n(r) = \frac{(n-r)!}{(n+1)!(r+1)!}.$$

Now we are able to prove the main result of this section.

Theorem 2 *Let $f \in L_p[0,1]$, $1 \leq p \leq \infty$, $n, r \in \mathbb{N}$, $n \geq r$, $k \in \mathbb{N}$ with $k \geq \frac{\gamma(n)}{\rho} - 2$, $\gamma(n)$ as defined in lemma 3 and a constant $\rho < 1$ independent of k and n. Then we have*

$$K^{2(r+1)}(f, \beta_n(r))_p \leq C(r,\rho)\frac{\beta_n(r)}{\beta_k(r)}\left\{\|M_{n,r}f - f\|_p + \|M_{k,r}f - f\|_p\right\},$$

with $\beta_n(r)$ as defined in lemma 5 and a constant $C(r,\rho)$ independent of n and k.

Proof.

We choose $g = M_{n,r}^2 f$ and prove

(12) $$\|f-g\|_p \leq 2^{r+1}\|M_{n,r}f - f\|_p$$

(13) $$\beta_n(r)\left\|\widetilde{D}^{2(r+1)}g\right\|_p \leq C(r)\frac{\beta_n(r)}{\beta_k(r)}\{||M_{n,r}f - f||_p + ||M_{k,r}f - f||_p\}$$

From lemma 2 we get

$$\begin{aligned}\|f-g\|_p &\le \|M_{n,r}(f-M_{n,r}f)\|_p+\|f-M_{n,r}f\|_p\\ &\le 2^{r+1}\|M_{n,r}f-f\|_p.\end{aligned}$$

Thus (12) holds true.

Now we apply the strong Voronovskaja-type result of lemma 5 to the function g and use $M_{k,r}$ instead of $M_{n,r}$. Together with the Bernstein-type inequalities in lemma 3 and lemma 4 this leads to

$$\begin{aligned}&\left\|M_{k,r}(M_{n,r}^2f)-M_{n,r}^2f-(-1)^r\beta_k(r)\widetilde{D}^{2(r+1)}(M_{n,r}^2f)\right\|_p\\ \le\;& \frac{1}{k+2}\beta_k(r)\left\|\widetilde{D}^{2(r+1)}(\widetilde{D}^2M_{n,r}^2f)\right\|_p\\ \le\;& \frac{\beta_k(r)}{k+2}\gamma_n(r)\left\|\widetilde{D}^{2(r+1)}(M_{n,r}f)\right\|_p\\ \le\;& \frac{\beta_k(r)}{k+2}\gamma_n(r)\left\{\left\|\widetilde{D}^{2(r+1)}(M_{n,r}^2f)\right\|_p+\left\|\widetilde{D}^{2(r+1)}(M_{n,r}(M_{n,r}f-f))\right\|_p\right\}\\ \le\;& \frac{\beta_k(r)}{k+2}\gamma_n(r)\left\{\left\|\widetilde{D}^{2(r+1)}(M_{n,r}^2f)\right\|_p+c_n(r)\|M_{n,r}f-f\|_p\right\}.\end{aligned}$$

From this inequality we derive by using the inverse triangle inequality, lemma 2 and the commutativity of the operators (see [3, Lemma 6])

$$\begin{aligned}&(14)\quad \beta_n(r)\left\{1-\frac{1}{k+2}\gamma_n(r)\right\}\left\|\widetilde{D}^{2(r+1)}g\right\|_p\\ &\le\frac{\beta_n(r)}{\beta_k(r)}\left\{\left\|M_{n,r}^2(M_{k,r}f-f)\right\|_p+\frac{1}{k+2}\beta_k(r)\gamma_n(r)c_n(r)\|M_{n,r}f-f\|_p\right\}\\ &\le\frac{\beta_n(r)}{\beta_k(r)}\left\{\underbrace{(2^{r+1}-1)^2}_{=C_1(r)}\|M_{k,r}f-f\|_p+\frac{1}{k+2}\beta_k(r)\gamma_n(r)c_n(r)\|M_{n,r}f-f\|_p\right\}.\end{aligned}$$

By our assumption on k we have

$$\frac{1}{k+2}\gamma_n(r)\le\rho<1,$$

with a constant ρ independent of k and n.

Thus

$$\left\{1-\frac{1}{k+2}\gamma_n(r)\right\}^{-1}\le\frac{1}{1-\rho}.$$

Furthermore with $C_2(r)=\dfrac{(2r+1)!}{(r+1)!r!}\cdot 2^r(3^{r+1}-1)$ we get

$$\begin{aligned}\frac{1}{k+2}\beta_k(r)\gamma_n(r)c_n(r) &\le \rho\cdot\frac{(k-r)!}{(k+1)!}\cdot\frac{n!}{(n-r-1)!}\cdot C_2(r)\\ &\le \rho\cdot C_2(r).\end{aligned}$$

Together with (14) we have proved that

$$\beta_n(r)\left\|\widetilde{D}^{2(r+1)}g\right\|_p\le\frac{\beta_n(r)}{\beta_k(r)}\frac{1}{1-\rho}\left\{C_1(r)\|M_{k,r}f-f\|_p+\rho\cdot C_2(r)\|M_{n,r}f-f\|_p\right\},$$

which implies our theorem with $C(r,\rho)=\max\left\{\dfrac{C_1(r)}{1-\rho},\rho\dfrac{C_2(r)}{1-\rho}\right\}$.

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