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Asymptotic behaviour of differentiated Bernstein polynomials revisited [1]

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Abstract

We give a refined version of a non-quantitative theorem by Floater dealing with the asymptotic behaviour of differentiated Bernstein polynomials. Orderwise we thus improve a previous result by Gonska and Raşa dealing with the same question. The assertion which we present here generalizes the classical Voronovskaya theorem and, in particular, a hardly known quantitative version of this theorem which can be traced to Sikkema and van der Meer [7] and which is also due to Videnskiĭ [8].

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1 Introduction

In 2005 Floater [1] proved a Voronovskaya-type result for simultaneous approximation by the classical Bernstein operators B_n.

The latter are given for a function $f : [0,1] \to \mathbb{R}$ and $x \in [0,1]$ by

$$B_n f(x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) p_{n,i}(x),$$

where

$$p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \;\; i = 0, \ldots, n.$$

In doing so, Floater was aware of the fact that his Voronovskaya formula for derivatives had been established earlier and using a completely different approach by López-Moreno et al. in 2002 (see [6]).

Recently this statement was brought into quantitative form by two of the present authors (see [4]). Details concerning these theorems are provided in the text below. In the present note we continue to consider the quantitative aspect of the matter. While in [4] the least concave majorant of the first order modulus of the $(k+2)$nd derivative played a central role in the estimate, here we use a combination of the first and second order moduli of the same derivative in the upper bound. This combination explains the convergence in the "simultaneous" Voronovskaya theorem even better, as can be seen from the concluding remark.

For completeness we mention that in a forthcoming article by R. Păltănea and one of the present authors (see [2]) the problem of simultaneous approximation by certain operators including those of Bernstein was also investigated, but from a somewhat different point of view.

2 Notation and auxiliary results

For a given integer $k \geq 0$ consider the operator $I_k : C[0,1] \longrightarrow C[0,1]$ given by $I_k f = f$, if $k = 0$, and

$$(I_k f)(x) = \int_0^x \frac{(x-t)^{k-1}}{(k-1)!} f(t)\, dt\,, \text{ if } k \geq 1.$$

Let $D^k := \frac{d^k}{dx^k}$, and $Q_n^k := D^k B_n I_k$, where B_n, $n \geq 1$, are the classical Bernstein operators on $C[0,1]$. Then Q_n^k is a positive linear operator on $C[0,1]$; more details can be found, e. g., in [4].

For each $i \geq 0$ let $e_i(x) := x^i$, $x \in [0,1]$. Consider also the moment of order i of the operator Q_n^k, i. e., the function

$$M_{n,i}^k(x) := Q_n^k((e_1 - x)^i; x),\; x \in [0,1].$$

Theorem 1 *1. For each $x \in [0,1]$ we have*

$$\frac{\left|M_{n,3}^k(x)\right|}{M_{n,2}^k(x)} \leq \frac{3k+2}{2} \cdot \frac{1}{n}. \tag{1}$$

2. There exists a constant $A = A(k)$ such that

$$\frac{M_{n,4}^k(x)}{M_{n,2}^k(x)} \leq A(k) \cdot \frac{1}{n},\; x \in [0,1]. \tag{2}$$

Assertion (2) was already used in [4], where a sketch of proof was presented. Here we mention only the following <u>exact</u> representations, with $X := x(1-x)$.

$$M_{n,2}^k(x) = \frac{n!}{n^{k+2}(n-k)!} \cdot \frac{1}{12}\left[k(3k+1) + 12(n - k(k+1))X\right], \tag{3}$$

$$(4) \quad M_{n,3}^k(x) = \frac{n!}{n^{k+3}(n-k)!} \cdot \frac{X'}{8} \{4\left[n(3k+2) - k(k+1)(k+2)\right] X + k^2(k+1)\},$$

$$(5) \quad M_{n,4}^k(x) = \frac{n!}{n^{k+4}(n-k)!} \cdot \frac{1}{240} \{240\left[n(3n-6-6k^2-14k)+k(k+1)(k+2)(k+3)\right] X^2 +120\left[n(k+1)(3k+2) - k(k+1)^2(k+2)\right] X +15k^2(k+1)^2 - 2k(5k+1)\}.$$

Detailed proofs of (1)-(5) will appear elsewhere.

Theorem 1 will be used in conjunction with the following slight extension of Theorem 3 of [5].

Theorem 2 *Suppose* $L : C[0,1] \longrightarrow C[0,1]$ *is a positive linear operator. If* $f \in C^2[0,1]$*, then for any* $0 < h \leq \frac{1}{2}$ *the following inequality holds:*

$$(6) \quad \left| L(f;x) - L(e_0;x)f(x) - L(e_1 - x;x)f'(x) - \frac{1}{2}L((e_1-x)^2;x)f''(x) \right| \leq L\left((e_1-x)^2;x\right) \left\{ \frac{|L((e_1-x)^3;x)|}{L\left((e_1-x)^2;x\right)} \cdot \frac{5}{6h}\omega_1(f'';h) + \left(\frac{3}{4} + \frac{L\left((e_1-x)^4;x\right)}{L\left((e_1-x)^2;x\right)} \cdot \frac{1}{16h^2} \right) \omega_2(f'';h) \right\},$$

where ω_1 *and* ω_2 *are the first and second moduli of smoothness, respectively.*

3 Main result

M. S. Floater proved the following theorem dealing with the asymptotic behaviour of differentiated Bernstein polynomials.

Theorem 3 *([1]) If $f \in C^{k+2}[0,1]$ for some $k \geq 0$, then*

$$\lim_{n\to\infty} n\left\{(B_n f)^{(k)}(x) - f^{(k)}(x)\right\} = \frac{1}{2}\frac{d^k}{dx^k}\left\{x(1-x)f''(x)\right\},$$

uniformly for $x \in [0,1]$.

A quantitative version of Floater's convergence result was obtained in [4, Theorem 4]:

Theorem 4 *If $f \in C^{k+2}[0,1]$ for some $k \geq 0$, then*

$$\left|n\left[(B_n f)^{(k)}(x) - f^{(k)}(x)\right] - \frac{1}{2}\frac{d^k}{dx^k}\left\{x(1-x)f''(x)\right\}\right| \tag{7}$$

$$\leq \mathcal{O}\left(\frac{1}{n}\right)\max_{k\leq i\leq k+2}\left\{\left|f^{(i)}(x)\right|\right\} + \mathcal{O}\left(1\right)\widetilde{\omega}\left(f^{(k+2)};\frac{1}{\sqrt{n}}\right).$$

Here $\mathcal{O}\left(\frac{1}{n}\right)$ and $\mathcal{O}\left(1\right)$ represent sequences of order $\mathcal{O}\left(\frac{1}{n}\right)$ and $\mathcal{O}\left(1\right)$, respectively, which depend on the fixed k, and $\widetilde{\omega}$ is the least concave majorant of ω_1, satisfying

$$\omega_1(f;\varepsilon) \leq \widetilde{\omega}(f;\varepsilon) \leq 2\omega_1(f;\varepsilon),\ \varepsilon \geq 0.$$

In this article we shall give another quantitative version of Floater's result, involving ω_1 and ω_2 instead of $\widetilde{\omega}$. From this new version we shall get a better order of convergence for functions $f \in C^{k+4}[0,1]$, for example.

In fact, our main result is:

Theorem 5 *For $n \geq 4$ and $f \in C^{k+2}[0,1]$, $k \geq 0$ fixed, we have*

$$\left|n[(B_n f)^{(k)}(x) - f^{(k)}(x)] - \frac{1}{2}\frac{d^k}{dx^k}\left\{x(1-x)f''(x)\right\}\right| \tag{8}$$

$$\leq \mathcal{O}\left(\frac{1}{n}\right)\max_{k\leq i\leq k+2}\left\{\left|f^{(i)}(x)\right|\right\}$$

$$+\mathcal{O}\left(1\right)\left\{\frac{1}{\sqrt{n}}\omega_1\left(f^{(k+2)};\frac{1}{\sqrt{n}}\right) + \omega_2\left(f^{(k+2)};\frac{1}{\sqrt{n}}\right)\right\}.$$

Proof. Let $f \in C^{k+2}[0,1]$. Denoting Q_n^k by L, we may apply Theorem 2. As a consequence, we get for any $0 < h \le \frac{1}{2}$ the inequality

$$\begin{aligned}
&\Bigg| L(f^{(k)};x) - f^{(k)}(x) - \frac{1}{2n}\frac{d^k}{dx^k}\{x(1-x)f''(x)\} \\
&-\{(L(e_0;x)-1)f^{(k)}(x) + L(e_1-x;x)f^{(k+1)}(x) \\
&+\frac{1}{2}L((e_1-x)^2;x)f^{(k+2)}(x) - \frac{1}{2n}\frac{d^k}{dx^k}\{x(1-x)f''(x)\}\Bigg| \\
&\le L\left((e_1-x)^2;x\right)\left\{\frac{\left|L((e_1-x)^3;x)\right|}{L\left((e_1-x)^2;x\right)} \cdot \frac{5}{6h}\omega_1\left(f^{(k+2)};h\right)\right. \\
&\left.+\left(\frac{3}{4} + \frac{L\left((e_1-x)^4;x\right)}{L\left((e_1-x)^2;x\right)} \cdot \frac{1}{16h^2}\right)\omega_2\left(f^{(k+2)};h\right)\right\}.
\end{aligned}$$

Multiplying both sides by n and using the triangular inequality yields

$$\left| n\{L(f^{(k)};x) - f^{(k)}(x)\} - \frac{1}{2}\frac{d^k}{dx^k}\{x(1-x)f''(x)\}\right| \le A + B, \tag{9}$$

where

$$\begin{aligned}
A \ := \ & n\Big|(L(e_0;x)-1)f^{(k)}(x) + L(e_1-x;x)f^{(k+1)}(x) \\
&+\frac{1}{2}L((e_1-x)^2;x)f^{(k+2)}(x) - \frac{1}{2n}\frac{d^k}{dx^k}\{x(1-x)f''(x)\}\Big|
\end{aligned}$$

and

$$\begin{aligned}
B \ := \ & nL\left((e_1-x)^2;x\right)\left\{\frac{\left|L\left((e_1-x)^3;x\right)\right|}{L\left((e_1-x)^2;x\right)} \cdot \frac{5}{6h}\omega_1\left(f^{(k+2)};h\right)\right. \\
&\left.+\left(\frac{3}{4} + \frac{L\left((e_1-x)^4;x\right)}{L\left((e_1-x)^2;x\right)} \cdot \frac{1}{16h^2}\right)\omega_2\left(f^{(k+2)};h\right)\right\}.
\end{aligned}$$

It was shown in [4, pp.56-57] that

$$A \le A_n^k\left|f^{(k)}(x)\right| + B_n^k\left|f^{(k+1)}(x)\right| + C_n^k\left|f^{(k+2)}(x)\right|$$

with A_n^k, B_n^k, $C_n^k = \mathcal{O}\left(\frac{1}{n}\right)$, and $A_n^k = B_n^k = 0$ for $k \in \{0,1\}$ and $C_n^k = 0$ for $k = 0$.

Hence

$$A = \mathcal{O}\left(\frac{1}{n}\right) \max\{|f^{(k)}(x)|, |f^{(k+1)}(x)|, |f^{(k+2)}(x)|\}. \tag{10}$$

Moreover, it was proved in [4, p.57] that

$$nL\left((e_1 - x)^2; x\right) = nQ_n^k\left((e_1 - x)^2; x\right) = \mathcal{O}(1). \tag{11}$$

Let $n \geq 4$ and $h = \frac{1}{\sqrt{n}}$. From (11), (1) and (2) we get

$$B = \mathcal{O}(1)\left\{\frac{1}{\sqrt{n}}\omega_1\left(f^{(k+2)}; \frac{1}{\sqrt{n}}\right) + \omega_2\left(f^{(k+2)}; \frac{1}{\sqrt{n}}\right)\right\}. \tag{12}$$

It remains to remark that

$$L\left(f^{(k)}; x\right) = Q_n^k\left(f^{(k)}; x\right) = (B_n f)^{(k)}(x) \tag{13}$$

(see also [4, p.55]).

Now (8) is a consequence of (9), (13), (10) and (12).

For $k = 0$, the $\mathcal{O}\left(\frac{1}{n}\right)$ in (8) equals 0. So in this case (8) becomes

$$\begin{aligned}
&\left|n[(B_n f)(x) - f(x)] - \frac{1}{2}x(1-x)f''(x)\right| \\
\leq\ & \mathcal{O}\left(\left\{\frac{1}{\sqrt{n}}\omega_1\left(f''; \frac{1}{\sqrt{n}}\right) + \omega_2\left(f''; \frac{1}{\sqrt{n}}\right)\right\}\right) \\
=\ & \begin{cases} \mathcal{O}\left(\frac{1}{\sqrt{n}}\right), & \text{if } f \in C^3[0,1] \\ \mathcal{O}\left(\frac{1}{n}\right), & \text{if } f \in C^4[0,1] \end{cases}.
\end{aligned}$$

In fact, for the special case $k = 0$ an even more precise inequality was given in [5, Theorem 4].

Remark 1 *The quantity* $\widetilde{\omega}\left(f^{(k+2)}; \frac{1}{\sqrt{n}}\right)$ *in (7) is replaced in (8) by* $\frac{1}{\sqrt{n}}\omega_1\left(f^{(k+2)}; \frac{1}{\sqrt{n}}\right) + \omega_2\left(f^{(k+2)}; \frac{1}{\sqrt{n}}\right)$. *Here* $f \in C^{k+2}[0,1]$. *Let us remark that*

$$\widetilde{\omega}\left(f^{(k+2)}; \frac{1}{\sqrt{n}}\right) = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right),\ f \in C^{k+3}[0,1],$$

and

$$\frac{1}{\sqrt{n}}\omega_1\left(f^{(k+2)}; \frac{1}{\sqrt{n}}\right) + \omega_2\left(f^{(k+2)}; \frac{1}{\sqrt{n}}\right) = \begin{cases} \mathcal{O}\left(\frac{1}{\sqrt{n}}\right), & \text{if } f \in C^{k+3}[0,1] \\ \mathcal{O}\left(\frac{1}{n}\right), & \text{if } f \in C^{k+4}[0,1] \end{cases}.$$

This better order of approximation for $f \in C^{k+4}[0,1]$ *cannot be read off the inequality in terms of* $\widetilde{\omega}\left(f^{(k+2)}; \frac{1}{\sqrt{n}}\right)$; *this follows from the saturation property of the first order modulus of continuity.*

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