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A generalization of some classical quadrature formulas[1]

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Abstract

In this paper is obtained a quadrature formula with higher degree of exactness. This formula is a generalization of some classical quadrature rules.

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1 Introduction

Let $[\alpha, \beta]$ be an interval on the real axis.

For a positive integer n, a function $F \in C^n[\alpha, \beta]$ and a point $\delta \in [\alpha, \beta]$ let denote by $\Theta_n(F; \delta)(x)$ the corresponding Taylor polynomial, i.e. $\Theta_n(F; \delta)(x) = \sum\limits_{i=0}^{n} \frac{F^{(i)}(\delta)}{i!}(x - \delta)^i$.

Lemma 1. *(see [1]) If $F \in C^{n+1}[\alpha, \beta]$ and $\delta \in [\alpha, \beta]$ then*

$$F(x) - \Theta_n(F; \delta)(x) = \frac{F^{(n+1)}(\xi)}{(n+1)!}(x - \delta)^{n+1},\ x \in [\alpha, \beta],$$

where ξ is a point between x and δ.

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In the sequel we consider a parameter $\gamma \in (0,1]$ and we define the points $\delta_{1,\gamma} = \frac{\alpha+\beta}{2} - \gamma\frac{\beta-\alpha}{2}$, $\delta_2 = \frac{\alpha+\beta}{2}$, $\delta_{3,\gamma} = \frac{\alpha+\beta}{2} + \gamma\frac{\beta-\alpha}{2}$.

Further we assume that p is a given even integer. Using a remark from the paper [2] and an idea suggested by Al. Lupaş in the paper [3] we consider the following quadrature formula which depend on parameters $\gamma \in (0,1]$, $\rho \in \mathbb{R}$:

$$\int\limits_{\alpha}^{\beta} F(x)dx = \Lambda_p(F;\gamma,\rho) + \Omega_p(F;\gamma,\rho),\ F \in C^{p+1}[\alpha,\beta], \tag{1}$$

where

$$\Lambda_p(F;\gamma,\rho) = \tag{2}$$

$$= \rho\int\limits_{\alpha}^{\beta} \Theta_{p+1}(F;\delta_2)(x)dx + \frac{1}{2}(1-\rho)\int\limits_{\alpha}^{\beta}(\Theta_p(F;\delta_{1,\gamma})(x) + \Theta_p(F;\delta_{3,\gamma})(x))dx$$

and $\Omega_p(F;\gamma,\rho)$ being the remainder term.

2 Main result

Theorem 1. *The remainder term in the quadrature formula (1) has the following representation*

$$\Omega_p(F;\gamma,\rho) = \tag{3}$$

$$= \rho\int\limits_{\alpha}^{\beta}(F(x) - \Theta_{p+1}(F;\delta_2)(x))dx + \frac{1}{2}(1-\rho)\int\limits_{\alpha}^{\beta}(F(x) - \Theta_p(F;\delta_{1,\gamma})(x))dx +$$

$$+ \frac{1}{2}(1-\rho)\int\limits_{\alpha}^{\beta}(F(x) - \Theta_p(F;\delta_{3,\gamma})(x))dx,\ F \in C^{p+1}[\alpha,\beta].$$

The proof is obtained directly from the relations (1), (2).

Further we consider the functions $\varepsilon_k(x) = x^k$, $k \in \mathbb{N}$.

Lemma 2. $\Omega_p(\varepsilon_k;\gamma,\rho) = 0$ *for any* $k \in \{0,...,p+1\}$, $\gamma \in (0,1]$ *and* $\rho \in \mathbb{R}$.

Proof. For $k \in \{0,...,p\}$ using Lemma 1 we have $\varepsilon_k(x) - \Theta_{p+1}(\varepsilon_k;\delta_2)(x) = 0$, $\varepsilon_k(x) - \Theta_p(\varepsilon_k;\delta_{1,\gamma})(x) = 0$, $\varepsilon_k(x) - \Theta_p(\varepsilon_k;\delta_{3,\gamma})(x) = 0$, where $x \in [\alpha,\beta]$. Therefore from (3) it follows that $\Omega_p(\varepsilon_k;\gamma,\rho) = 0$ for any $k \in \{0,...,p\}$, $\gamma \in (0,1]$ and $\rho \in \mathbb{R}$.

If $k = p+1$ using Lemma 1 we deduce $\varepsilon_{p+1}(x) - \Theta_{p+1}(\varepsilon_{p+1};\delta_2)(x) = 0$, $\varepsilon_{p+1}(x) - \Theta_p(\varepsilon_{p+1};\delta_{1,\gamma})(x) = (x-\delta_{1,\gamma})^{p+1}$, $\varepsilon_{p+1}(x) - \Theta_p(\varepsilon_{p+1};\delta_{3,\gamma})(x) = (x-\delta_{3,\gamma})^{p+1}$, where $x \in [\alpha,\beta]$, and substituting in (3) we obtain $\Omega_p(\varepsilon_{p+1};\gamma,\rho) = 0$ for any $\gamma \in (0,1]$ and $\rho \in \mathbb{R}$.

Lemma 3. $\Omega_p(\varepsilon_k;\gamma,\overline{\rho}_p(\gamma)) = 0$ *for any* $k \in \{p+2,p+3\}$ *and* $\gamma \in (0,1]$, *where*

$$\overline{\rho}_p(\gamma) =$$
$$= \frac{\gamma(p+3)((1+\gamma)^{p+2} - (1-\gamma)^{p+2}) - (1+\gamma)^{p+3} - (1-\gamma)^{p+3}}{\gamma(p+3)((1+\gamma)^{p+2} - (1-\gamma)^{p+2}) - (1+\gamma)^{p+3} - (1-\gamma)^{p+3} + 2}.$$

The proof is similar with Lemma 2, taking into account Lemma 1 and Theorem 1.

Theorem 2. *The degree of exactness in the quadrature formula (1) for* $\rho := \overline{\rho}_p(\gamma)$ *is at least* $p+3$ *for any* $\gamma \in (0,1]$, *i.e.* $\Omega_p(Q;\gamma,\overline{\rho}_p(\gamma)) = 0$, *for all* $Q : [\alpha,\beta] \to \mathbb{R}$ *polynomials of degree at least* $p+3$.

The proof follows directly from Lemma 2 and Lemma 3.

Theorem 3. *The quadrature formula (1) for* $\rho := \overline{\rho}_p(\gamma)$ *has the following representation*

$$\int_\alpha^\beta F(x)dx = \sum_{i=0}^{p} \sigma_{p,i}(\gamma,\overline{\rho}_p(\gamma))F^{(i)}(\delta_{1,\gamma}) + \sum_{i=0}^{p/2} \tau_{p,2i}(\gamma,\overline{\rho}_p(\gamma))F^{(2i)}(\delta_2) +$$
$$+ \sum_{i=0}^{p} \varphi_{p,i}(\gamma,\overline{\rho}_p(\gamma))F^{(i)}(\delta_{3,\gamma}) + \Omega_p(F;\gamma,\overline{\rho}_p(\gamma)),\ F \in C^{p+1}[\alpha,\beta],$$

where the coefficients $(\sigma_{p,i}(\gamma,\overline{\rho}_p(\gamma)))_{i\in\{0,...,p\}}$, $(\tau_{p,2i}(\gamma,\overline{\rho}_p(\gamma)))_{i\in\{0,...,p/2\}}$, $(\varphi_{p,i}(\gamma,\overline{\rho}_p(\gamma)))_{i\in\{0,...,p\}} \in \mathbb{R}$ *are given by*

$$\sigma_{p,i}(\gamma,\overline{\rho}_p(\gamma)) = \frac{1}{2(i+1)!}(1-\overline{\rho}_p(\gamma))((1+\gamma)^{i+1}+$$

$$+(-1)^i(1-\gamma)^{i+1})\left(\frac{\beta-\alpha}{2}\right)^{i+1},\ i\in\{0,...,p\},$$

$$\tau_{p,2i}(\gamma,\overline{\rho}_p(\gamma)) = \frac{2}{(2i+1)!}\overline{\rho}_p(\gamma)\left(\frac{\beta-\alpha}{2}\right)^{2i+1},\ i\in\{0,...,p/2\},$$

$$\varphi_{p,i}(\gamma,\overline{\rho}_p(\gamma)) = \frac{(-1)^i}{2(i+1)!}(1-\overline{\rho}_p(\gamma))((1+\gamma)^{i+1}+$$

$$+(-1)^i(1-\gamma)^{i+1})\left(\frac{\beta-\alpha}{2}\right)^{i+1} = (-1)^i\sigma_{p,i}(\gamma,\overline{\rho}_p(\gamma)),\ i\in\{0,...,p\}.$$

The proof is obtained from the relations (1), (2) considering $\rho := \overline{\rho}_p(\gamma)$.

In the sequel we consider the particular case $p=0$ and for some values of the parameter $\gamma\in(0,1]$ we obtain some classical quadrature rules.

For $\gamma=1$ one obtains the Simpson quadrature formula

$$\int_\alpha^\beta F(x)dx = \frac{\beta-\alpha}{6}\left(F(\alpha)+4F(\frac{\alpha+\beta}{2})+F(\beta)\right)-$$

$$-\frac{(\beta-\alpha)^5}{2880}F^{(4)}(\xi),\ \xi\in[\alpha,\beta].$$

For $\gamma=\frac{2}{3}$ one obtains the Maclaurin quadrature formula

$$\int_\alpha^\beta F(x)dx = \frac{3(\beta-\alpha)}{8}\left(F(\frac{5\alpha+\beta}{6})+\frac{2}{3}F(\frac{\alpha+\beta}{2})+F(\frac{\alpha+5\beta}{6})\right)+$$

$$+\frac{7(\beta-\alpha)^5}{51840}F^{(4)}(\xi),\ \xi\in[\alpha,\beta].$$

For $\gamma=\frac{1}{2}$ one obtains the "open" Newton-Cotes quadrature formula

$$\int_{\alpha}^{\beta}F(x)dx=\frac{2(\beta-\alpha)}{3}\left(F(\frac{3\alpha+\beta}{4})-\frac{1}{2}F(\frac{\alpha+\beta}{2})+F(\frac{\alpha+3\beta}{4})\right)+$$

$$+\frac{7(\beta-\alpha)^5}{23040}F^{(4)}(\xi),\ \xi\in[\alpha,\beta].$$

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