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Some Results On Janowski Starlike Log-harmonic Mappings Of Complex Order b

Melike Aydoğan

Abstract

Let $H(\mathbb{D})$ be a linear space of all analytic functions defined on the open unit disc $\mathbb{D} = \{z \mid |z| < 1\}$. A sense-preserving log-harmonic function is the solution of the non-linear elliptic partial differential equation

$$\overline{f_{\overline{z}}} = w\frac{\overline{f}}{f}f_z,$$

where $w(z)$ is analytic, satisfies the condition $|w(z)| < 1$ for every $z \in \mathbb{D}$ and is called the second dilatation of f. It has been shown that if f is a non-vanishing log-harmonic mapping then f can be represented by

$$f(z) = h(z)\overline{g(z)},$$

where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ with $h(0) \neq 0$, $g(0) = 1([1])$. If f vanishes at $z = 0$ but it is not identically zero, then f admits the representation

$$f(z) = z\,|z|^{2\beta}\,h(z)\overline{g(z)},$$

where $Re\beta > -\frac{1}{2}$, $h(z)$ and $g(z)$ are analytic in D with $g(0) = 1$ and $h(0) \neq 0$. The class of sense-preserving log-harmonic mappins is denoted by $\mathcal{S}_{LH}$. We say that f is a Janowski starlike log-harmonic mapping.If

$$1 + \frac{1}{b}\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f} - 1\right) = \frac{1 + A\phi(z)}{1 + B\phi(z)}$$

where $\phi(z)$ is Schwarz function. The class of Janowski starlike log-harmonic mappings is denoted by $\mathcal{S}^*_{LH}(A, B, b)$. We also note that, if $(zh(z))$ is a starlike function, then the Janowski starlike log-harmonic mappings will be called a perturbated Janowski starlike log-harmonic mappings. And the family of such mappings will be denoted by $\mathcal{S}^*_{PLH}(A, B, b)$. The aim of this paper is to give some distortion theorems of the class $\mathcal{S}^*_{LH}(A, B, b)$.

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1 Introduction

Let Ω be the family of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.
Next, denote by $\mathcal{P}(A, B)$ the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + ...$$

regular in $\mathbb{D}$, such that $p(z)$ is in $\mathcal{P}(A, B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)}, \quad -1 \leq B < A \leq 1 \tag{1}$$

for some function $\phi(z) \in \Omega$ and for every $z \in \mathbb{D}$. Therefore we have $p(0) = 1$, $Rep(z) > \frac{1-A}{1-B} > 0$ whenever $p(z) \in \mathcal{P}(A, B)$. Moreover, let $\mathcal{S}^*(A, B)$ denote the family of functions

$$s(z) = z + a_2 z^2 + ...$$

regular in $\mathbb{D}$, and such that $s(z)$ is in $\mathcal{S}^*$ if and only if

$$Re\left(z\frac{s'(z)}{s(z)}\right) = p(z) = \frac{1+\phi(z)}{1-\phi(z)}, p(z) \in \mathcal{P}(1, -1) \tag{2}$$

Let $S_1(z)$ and $S_2(z)$ be analytic functions in $\mathbb{D}$ with $S_1(0) = S_2(0)$. We say that $S_1(z)$ subordinated to $S_2(z)$ and denote by $S_1(z) \prec S_2(z)$, if $S_1(z) = S_2(\phi(z))$ for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$. If $S_1(z) \prec S_2(z)$, then $S_1(\mathbb{D}) \subset S_2(\mathbb{D})$([5]).

The radius of starlikeness of the class of sense-preserving log-harmonic mapping is

$$r_s = sup\left\{r | Re\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) > 0, 0 < r < 1\right\}.$$

Finally, let $H(D)$ be the linear space of all analytic functions defined on the open unit disc $\mathbb{D}$. A sense-preserving log-harmonic mapping is the solution of the non-linear elliptic partial differential equation

$$\frac{\overline{f_{\overline{z}}}}{\overline{f}} = w(z)\frac{f_z}{f}, \tag{3}$$

where $w(z) \in H(\mathbb{D})$ is the second dilatation of f such that $|w(z)| < 1$ for every $z \in \mathbb{D}$. It has been shown that if f is a non-vanishing log-harmonic mapping, then f can be expressed as

$$f = h(z)\overline{g(z)} \tag{4}$$

where $h(z)$ and $g(z)$ are analytic functions in $\mathbb{D}$.
On the other hand, if f vanishes at $z = 0$ and at no other point, then f admits the representation,

$$f = z\,|z|^{2\beta}\,h(z)\overline{g(z)}, \tag{5}$$

where $Re\beta > -1/2$, $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ with $g(0) = 1$ and $h(0) \neq 0$. We note that the class of log-harmonic mappings is denoted by $\mathcal{S}_{LH}$.
Let $f = zh(z)\overline{g(z)}$ be an element of $\mathcal{S}_{LH}$. We say that f is a Janowski starlike log-harmonic mapping if

$$1 + \frac{1}{b}\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f} - 1\right) = p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)}, p(z) \in \mathcal{P}(A, B) \tag{6}$$

where $-1 \leq B < A \leq 1$, $b \neq 0$ and complex and denote by $\mathcal{S}^*_{LH}(A, B, b)$ the set of all starlike log-harmonic mappings. Also we denote $\mathcal{S}^*_{PLH}(A, B, b)$ the class of all functions in $\mathcal{S}^*_{LH}(A, B, b)$ for which $(zh(z)) \in \mathcal{S}^*(A, B)$ for all $z \in \mathbb{D}$.
We note that if we give special values to b, then we obtain important subclasses of Janowski starlike log-harmonic mappings

i. For $b = 0$, we obtain the class of starlike log-harmonic mappings.

ii. For $b = 1 - \alpha$, $0 \leq \alpha < 1$, we obtain the class of starlike log-harmonic mappings of order α.

iii. For $b = e^{-i\lambda}cos\lambda$, $|\lambda| < \frac{\pi}{2}$, we obtain the class of $\lambda-$ spirallike log-harmonic mappings.

iv. For $b=(1-\alpha)e^{-i\lambda}cos\lambda$, $0\le\alpha<1$, $|\lambda|<\frac{\pi}{2}$, we obtain the class of $\lambda-$ spirallike log-harmonic mappings of order α.

2 Main results

Theorem 1 *Let $f=zh(z)\overline{g(z)}$ be an element of $\mathcal{S}^*_{PLH}(A,B,b)$. Then*

(7)

$$f=zh(z)\overline{g(z)}\in\mathcal{S}^*_{LH}(A,B,b)\ \ iff\ \begin{cases} z\frac{h'(z)}{h(z)}-\overline{z}\frac{\overline{g'(z)}}{\overline{g(z)}}\prec\frac{b(A-B)z}{1+Bz}; & B\neq 0,\\ z\frac{h'(z)}{h(z)}-\overline{z}\frac{\overline{g'(z)}}{\overline{g(z)}}\prec bAz; & B=0.\end{cases}$$

Proof. Let $f\in\mathcal{S}^*_{LH}(A,B,b)$. Using the principle of subordination then we have

$$1+\frac{1}{b}\left(\frac{zfz-\overline{z}f_{\overline{z}}}{f}-1\right)=1+\frac{1}{b}\left(z\frac{h'(z)}{h(z)}-\overline{z}\frac{\overline{g'(z)}}{\overline{g(z)}}\right)=\begin{cases}\frac{1+A\phi(z)}{1+B\phi(z)}; & B\neq 0,\\ 1+A\phi(z); & B=0,\end{cases}$$

$$iff\ \ z\frac{h'(z)}{h(z)}-\overline{z}\frac{\overline{g'(z)}}{\overline{g(z)}}=\begin{cases}\frac{b(A-B)\phi(z)}{1+B\phi(z)}; & B\neq 0,\\ bA\phi(z); & B=0,\end{cases}$$

$$iff\ \ z\frac{h'(z)}{h(z)}-\overline{z}\frac{\overline{g'(z)}}{\overline{g(z)}}\prec\begin{cases}\frac{b(A-B)z}{1+Bz}; & B\neq 0,\\ bAz; & B=0.\end{cases}$$

Theorem 2 *Let $f=zh(z)\overline{g(z)}$ be an element of $\mathcal{S}^*_{PLH}(A,B,b)$.Then*

$$\begin{cases}G(A,B,-r)\le\left|\frac{h(z)}{g(z)}\right|\le G(A,B,r); & B\neq 0,\\ G_1(A,-r)\le\left|\frac{h(z)}{g(z)}\right|\le G_1(A,r); & B=0,\end{cases}$$

where

(8)
$$\begin{cases} G(A,B,r) = \frac{(1+Br)^{\frac{(A-B)(|b|-Reb)}{2B}}}{(1-Br)^{\frac{(A-B)(|b|+Reb)}{2B}}}; & B \neq 0, \\ G_1(A,r) = \frac{(1+r)^{\frac{A|b|}{2}}}{(1-r)^{\frac{A|b|}{2}}}; & B = 0. \end{cases}$$

Proof. The function $\left(\frac{1+Az}{1+Bz}\right)$ maps $|z| = r$ on to crcle with the centre $C(r) = (\frac{1-ABr^2}{1-B^2r^2}, 0)$ and the raidus $p(r) = \frac{(A-B)r}{1-B^2r^2}$ .Therefore using the definition of subordination and Theorem 1,we get

(9)
$$\begin{cases} \left|\left(1 + \frac{1}{b}\left(z\frac{h'(z)}{h(z)} - \overline{z}\overline{\frac{g'(z)}{g(z)}}\right) - \frac{1-ABr^2}{1-B^2r^2}\right)\right| \leq \frac{(A-B)r}{1-B^2r^2}; B \neq 0, \\ \left|\left(1 + \frac{1}{b}\left(z\frac{h'(z)}{h(z)} - \overline{z}\frac{g'(z)}{g(z)}\right) - 0\right)\right| \leq Ar; B = 0. \end{cases}$$

The inequality (9) takes in the form,

(10)
$$\begin{cases} \frac{(B(A-B)Reb)r^2 - |b|(A-B)r}{1-B^2r^2} \leq Re\left(z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}\right) \leq \frac{b(A-B)r}{1-B^2r^2} \\ \qquad\qquad + \frac{(B(A-B)Reb)r^2}{1-B^2r^2};\ B \neq 0, \\ -\frac{A|b|r}{1-r^2} \leq Re(z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}) \leq \frac{A|b|r}{1-r^2};\ B = 0, \end{cases}$$

on the other hand we have,

$$Re\left(z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}\right) = r\frac{\partial}{\partial r}\left(\log|h(z)| - \log|g(z)|\right)$$

Thus the inequality (10) can be written in the form,

(11)
$$\begin{cases} \frac{[B(A-B)Reb.r - |b|(A-B)]}{(1-Br)(1+Br)} \leq \frac{\partial}{\partial r}\log|h(z) - g(z)| \leq \frac{[B(A-B)Reb.r + |b|(A-B)]}{(1-Br)(1+Br)}; B \neq 0, \\ -\frac{A|b|}{(1-r)(1+r)} \leq \frac{\partial}{\partial r}\log|h(z) - g(z)| \leq \frac{A|b|}{(1-r)(1+r)}; B = 0, \end{cases}$$

integrating both sides of (11) from 0 to r we get (8).

Corollary 1 *The radius of starlikeness of the class $\mathcal{S}^*_{PLH}$ is*

$$r_s = \begin{cases} \frac{2}{(A-B)|b|+\sqrt{(A-B)^2|b|^2+4[B^2+(AB-B^2)Reb]}}; & B \neq 0, \\ \frac{1}{|b|A}; & B = 0. \end{cases} \tag{12}$$

Proof. The inequality (9) can be written in the form

$$\begin{cases} \left| \frac{zf_{;z}-\overline{z}f_{\overline{z}}}{f} - \left[\frac{1-\left(B^2+(AB-B^2Reb)r^2-i((AB-B^2)Imb)r^2\right)}{1-B^2r^2} \right] \right| \leq \frac{|b|(A-B)r}{1-B^2r^2}; & B \neq 0, \\ \left| \frac{zf_z-\overline{z}f_{\overline{z}}}{f} - 1 \right| \leq |b|\, Ar; & B = 0. \end{cases}$$

Therefore we have

$$Re\left(\frac{zf_z-\overline{z}f_{\overline{z}}}{f}\right) \geq \begin{cases} \frac{1-(A-B)|b|r-(B^2+(AB-B^2)Reb)r^2}{1-B^2r^2}; & B \neq 0, \\ 1-|b|\, Ar; & B = 0, \end{cases}$$

which gives (12).

Lemma 1 *Let $f = z|z|^{2\beta} h(z)\overline{g(z)} \in \mathcal{S}_{LH}$ and let $w(z)$ be the second dilatation of f. Then*

$$\frac{||\beta| - |\beta+1|\, r|}{||\beta+1| - |\beta|\, r|} \leq |w(z)| \leq \frac{||\beta| + |\beta+1|\, r|}{||\beta+1| + |\beta|\, r|}. \tag{13}$$

This inequality is sharp because the extremal function is

$$w(z) = e^{i\theta} \frac{e^{i\ell}z - \left|\frac{\overline{\beta}}{\beta+1}\right|}{1-\left|\frac{\overline{\beta}}{\beta+1}\right| e^{i\ell}z}, \quad z \in \mathbb{D}, \ell \in \mathbb{R}.$$

Proof. $f = z|z|^{2\beta} h(z)\overline{g(z)} \in \mathcal{S}_{LH}$ and let $1^{2\beta} = 1$. Then f is the solution of the nonlinear elliptic partial differential equation

$$w(z) = \frac{\overline{f_{\overline{z}}}}{\overline{f}}.\frac{f}{f_z}$$

$$f_z = \left(\frac{1}{z} + \frac{\beta}{z} + \frac{h'(z)}{h(z)}\right) f,$$

$$f_{\overline{z}} = \left(\frac{\overline{\beta}}{z} + \frac{g'(z)}{g(z)}\right) \overline{f}$$

$$w(z) = \frac{\overline{f_{\overline{z}}}}{\overline{f}} . \frac{f}{f_z} = \frac{\overline{\beta} + z\frac{g'(z)}{g(z)}}{(\beta+1) + z\frac{h'(z)}{h(z)}}, \quad w(0) = \frac{\overline{\beta}}{\beta+1}, \quad |w(0)| < 1.$$

On the other hand for $Re\beta > -\frac{1}{2}$, we have $\left|\frac{\overline{\beta}}{\beta+1}\right| < 1$. Therefore we can take $w(0) = c_0 = \left|\frac{\overline{\beta}}{\beta+1}\right| e^{i\theta}, \quad \theta \in \mathbb{R}$.

Now consider the function

$$\phi(z) = \frac{e^{-i\theta} w(z) - \left|\frac{\overline{\beta}}{\beta+1}\right|}{1 - \left|\frac{\overline{\beta}}{\beta+1}\right| e^{i\theta} w(z)}, \quad z \in \mathbb{D},$$

which satisfies the conditions Schwarz lemma and use the estimate $|\phi(z)| \leq |z| < r$, to get

$$\left| e^{-i\theta} w(z) - \left|\frac{\overline{\beta}}{\beta+1}\right| \right| \leq r \left| \left|\frac{\overline{\beta}}{\beta+1}\right| e^{-i\theta} w(z) - 1 \right|.$$

This is equivalent to

$$\left| w(z) - \frac{\left|\frac{\overline{\beta}}{\beta+1}\right| (1-r^2)}{1 - \left|\frac{\overline{\beta}}{\beta+1}\right|^2 r^2} \right| \leq \frac{r\left(1 - \left|\frac{\overline{\beta}}{\beta+1}\right|^2\right)}{1 - \left|\frac{\overline{\beta}}{\beta+1}\right|^2 r^2} \tag{14}$$

and the equality holds only for a function of the form

$$w(z) = e^{i\theta}\frac{e^{i\ell}z - \left|\frac{\overline{\beta}}{\beta+1}\right|}{1-\left|\frac{\overline{\beta}}{\beta+1}\right|e^{i\ell}z}, \quad z \in \mathbb{D}, \ell \in \mathbb{R}.$$

From the inequality (14)) we have then

$$|w(z)| = \left|e^{-i\theta}w(z)\right| \geq \left|\frac{\left|\frac{\overline{\beta}}{\beta+1}\right|(1-r^2)}{1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2 r^2} - \frac{r\left(1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2\right)}{1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2 r^2}\right| = \frac{\left|\left|\frac{\overline{\beta}}{\beta+1}\right| - r\right|}{1-\left|\frac{\overline{\beta}}{\beta+1}\right| r}$$

$$|w(z)| = \left|e^{-i\theta}w(z)\right| \leq \frac{\left|\frac{\overline{\beta}}{\beta+1}\right|(1-r^2)}{1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2 r^2} - \frac{r\left(1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2\right)}{1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2 r^2} = \frac{\left|\frac{\overline{\beta}}{\beta+1}\right| + r}{1+\left|\frac{\overline{\beta}}{\beta+1}\right| r}.$$

Lemma 2 $\phi(z) \in \mathcal{S}^*(A,B)$ *ise*

$$(15) \qquad \begin{cases} \frac{1-Ar}{r(1-Br)} \leq \left|\frac{\phi'(z)}{\phi(z)}\right| \leq \frac{1+Ar}{r(1+Br)}, & B \neq 0; \\ \\ \frac{1-Ar}{r} \leq \left|\frac{\phi'(z)}{\phi(z)}\right| \leq \frac{1+Ar}{r}, & B = 0; \end{cases}$$

we get the result.

Proof. $\phi(z) = z.h(z) \in \mathcal{S}^*(A,B)$;

$$(16) \qquad \begin{cases} \left|z\frac{\phi'(z)}{\phi(z)} - \frac{1-ABr^2}{1-B^2r^2}\right| \leq \frac{(A-B)r}{1-B^2r^2}, & B \neq 0; \\ \left|z\frac{\phi'(z)}{\phi(z)} - 1\right| \leq Ar, & B = 0; \end{cases}$$

these inequalities can be written.Then we have,

$$(17) \qquad \begin{cases} \frac{1-Ar}{1-Br} \leq \left|z\frac{\phi'(z)}{\phi(z)}\right| \leq \frac{1+Ar}{1+Br}, & B \neq 0; \\ 1-Ar \leq \left|z\frac{\phi'(z)}{\phi(z)}\right| \leq 1+Ar, & B = 0; \end{cases}$$

.And if we divide by $|z| = r$ each of these ;we obtain the result easily.

Lemma 3 *$f(z) = zh(z)\overline{g(z)} = \phi(z)\overline{g(z)} \in \mathcal{S}^*_{lh}(A,B)$ ve $\phi(z) = zh(z) \in \mathcal{S}^*(A,B)$*

$$(18) \qquad \begin{cases} -\frac{1-Ar}{1-Br} < \left|\frac{g'(z)}{g(z)}\right| < \frac{1+Ar}{1+Br}, & B \neq 0; \\ \\ -(1-Ar) < \left|\frac{g'(z)}{g(z)}\right| < 1+Ar, & B = 0; \end{cases}$$

we get the result.

Proof. $f(z) = \phi(z)\overline{g(z)}$ and we use the second dilatation function's elliptic differentional solution,

$$w(z) = \frac{\frac{g'(z)}{g(z)}}{\frac{\phi'(z)}{\phi(z)}}$$

we hold this result. Therefore , $w(z)$ function is analytic at $\mathbb{D}$ disc; $|w(z)| < 1$ (sense-preserving) and $w(0) = 0$;because of Schwarz Lemma,

$$-r < |w(z)| < r$$

We can write these inequalities.Then we have,

$$-r < \left|\frac{\frac{g'(z)}{g(z)}}{\frac{\phi'(z)}{\phi(z)}}\right| < r$$

And if we use Lemma (11),We can take the result easily.

Theorem 3 *Let $f = zh(z)\overline{g(z)}$ be an element of $\mathcal{S}^*_{PLH}(A,B,b)$ then*

$$F(A,B,-r) \leq |g(z)| \leq F(A,B,r),$$

$$F_1(A,-r) \leq |g(z)| \leq F_1(A,r),$$

where

$$F(A,B,r) = \frac{1}{(1+Br)^{\frac{B-A}{B}}}.\frac{(1+Br)^{\frac{(A-B)(|b|+Reb)}{2B}}}{(1-Br)^{\frac{(A-B)(|b|-Reb)}{2B}}}$$

$$F_1(A,r) = e^{Ar}.\frac{(1-r)^{\frac{|-b|A}{2}}}{(1+r)^{\frac{|-b|A}{2}}}.$$

Proof. $f \in \mathcal{S}^*_{lh}(A,B)$ then we $h(z)$ function satisfies, $h(0) = 1$ and has a Taylor formula $h(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ We know that,$\phi(z) = zh(z) \in \mathcal{S}^*(A,B)$ from starlikeness radius formula,

(19)

$$Re\left(z\frac{\phi'(z)}{\phi(z)}\right) = Re\left(z\frac{(zh(z))'}{zh(z)}\right) = Re\left(\frac{(zh(z))'}{h(z)}\right) = Re(1+z\frac{h'(z)}{h(z)}) > 0$$

satisfied. And also for Janowski Starlike logharmonic mappings,

$$\begin{cases} \left|\frac{(zh(z))'}{h(z)} - \frac{1-ABr^2}{1-B^2r^2}\right| \leq \frac{(A-B)r}{1-B^2r^2}, & B \neq 0; \\ \\ \left|\frac{(zh(z))'}{h(z)} - 1\right| \leq Ar, & B = 0; \end{cases} \tag{20}$$

Then we have,

$$\begin{cases} \left|1+\frac{zh'(z)}{h(z)} - \frac{1-ABr^2}{1-B^2r^2}\right| \leq \frac{(A-B)r}{1-B^2r^2}, & B \neq 0; \\ \\ \left|1+\frac{zh'(z)}{h(z)} - 1\right| \leq Ar, & B = 0; \end{cases} \tag{21}$$

we can write these inequalities.

$$-|z| \leq Rez \leq |z|$$

if we use this inequality ;

$$(22)\qquad \begin{cases} \frac{1}{(1-Br)^{\frac{B-A}{B}}} \leq |h(z)| \leq \frac{1}{(1+Br)^{\frac{B-A}{B}}}, & B \neq 0; \\ \\ e^{-Ar} \leq |h(z)| \leq e^{Ar}, & B = 0; \end{cases}$$

Then we have the result.

Corollary 2 *Let* $f = zh(z)\overline{g(z)} \in \mathcal{S}^*_{LH}(A,B,b)$ *and let* $(zh(z)) \in \mathcal{S}^*(A,B)$. *Then*

$$(23)\qquad \begin{cases} -(\frac{1-Ar}{1-Br})F(A,B,-r) < |g'(z)| < (\frac{1+Ar}{1+Br})F(A,B,r),\ B \neq 0; \\ -(1-Ar)F_1(A,-r) < |g'(z)| < (1+Ar)F_1(A,r),\ B = 0; \end{cases}$$

Proof. Follows immediately from Lemma (12) and Theorem3

Corollary 3 *Let* $f = zh(z)\overline{g(z)} \in \mathcal{S}^*_{LH}(A,B,b)$. *Then*

$$(24)\qquad \begin{cases} \frac{1}{(1-Br)^{\frac{B-A}{B}}} \left[\frac{1-(A-B)r-ABr^2}{1-B^2r^2}\right] \leq |h(z)+zh'(z)| \\ \qquad\qquad \leq \frac{1}{(1+Br)^{\frac{B-A}{B}}} \left[\frac{1+(A-B)r-ABr^2}{1-B^2r^2}\right],\ B \neq 0; \\ e^{-Ar}(1-Ar) \leq |h(z)+zh'(z)| \leq e^{Ar}(1+Ar), \qquad B = 0; \end{cases}$$

Proof. This result is a simple consequence of Lemma (12).

Corollary 4 *Let* $f = zh(z)\overline{g(z)} \in \mathcal{S}^*_{LH}(A,B,b)$.*Then*

$$(25)\qquad \begin{cases} \frac{r}{(1-Br)^{\frac{B-A}{B}}}.F(A,B,-r) \leq |f| \leq \frac{r}{(1+Br)^{\frac{B-A}{B}}}F(A,B,r),\ B \neq 0; \\ r.e^{-Ar}.F_1(A,-r) \leq |f| \leq r.e^{Ar}.F_1(A,r),\ B = 0; \end{cases}$$

Proof. This result is a simple consequence of Theorem 3.

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Melike Aydoğan

İstanbul Aydn University

Department of Mathematics and Computer Science

İstanbul Aydn University, İstanbul, Turkey

E-mail: melikeaydogan@aydin.edu.tr