General Mathematics Vol. 17, No. 2 (2009), 113–125

q-Extensions for the Apostol-Genocchi Polynomials[1]

Qiu-Ming Luo

Abstract

In this paper, we define the Apostol-Genocchi polynomials and q-Apostol-Genocchi polynomials. We give the generating function and some basic properties of q-Apostol-Genocchi polynomials. Several interesting relationships are also obtained.

2000 Mathematics Subject Classification: Primary 05A30; Secondary 11B83, 11M35, 33E20.

Key Words and Phrases: Genocchi polynomials , q-Genocchi polynomials; Apostol-Genocchi polynomials , q-Apostol-Genocchi polynomials; Hurwitz-Lerch Zeta function; q-Hurwitz-Lerch Zeta function; Goyal-Laddha-Hurwitz-Lerch Zeta function; q-Goyal-Laddha-Hurwitz-Lerch Zeta function.

1 Introduction, definitions and motivation

Throughout this paper, we always make use of the following notation: $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0^- = \{0, -1, -2, -3, \ldots\}$ denotes the set of

[1] *Received 20 September, 2008*

Accepted for publication (in revised form) 27 November, 2008

nonpositive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers.

The *falling factorial* is $\{n\}_0 = 1, \{n\}_k = n(n-1)\cdots(n-k+1)$ $(n \in \mathbb{N})$; The *rising factorial* is $(n)_0 = 1, (n)_k = n(n+1)\cdots(n+k-1)$; The *q-shifted factorial* is $(a;q)_0 = 1; (a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}), k = 1, 2, \ldots$; $(a;q)_\infty = (1-a)(1-aq)\cdots(1-aq^k)\cdots = \prod_{k=0}^{\infty}(1-aq^k), (|q| < 1;\ a, q \in \mathbb{C})$. Clearly, $(a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}$.

The *q-number* or *q-basic number* is defined by $[a]_q = \dfrac{1-q^a}{1-q}, q \neq 1,\ (|q| < 1;\ a, q \in \mathbb{C})$; The *q-numbers factorial* is defined by $[n]_q! = [1]_q[2]_q\cdots[n]_q,\ (n \in \mathbb{N})$. The *q-numbers shifted factorial* is defined by $([a]_q)_n = [a]_{q;n} = [a]_q[a+1]_q\cdots[a+n-1]_q$ $(n \in \mathbb{N},\ a \in \mathbb{C})$. Clearly, $\lim_{q\to 1}[a]_q = a,\ \lim_{q\to 1}[n]_q! = n!,\ \lim_{q\to 1}([a]_q)_n = (a)_n$.

The usual binomial theorem

$$\frac{1}{(1-z)^\alpha} = \sum_{n=0}^{\infty}\binom{-\alpha}{n}(-z)^n := \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}z^n, \qquad (z, \alpha \in \mathbb{C};\ |z| < 1). \tag{1.1}$$

The q-binomial theorem

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \qquad (z, q \in \mathbb{C};\ |z| < 1, |q| < 1). \tag{1.2}$$

A special case of (1.2), for $a = q^\alpha(\alpha \in \mathbb{C})$, can be written as follows:

$$\frac{1}{(z;q)_\alpha} = \frac{(q^\alpha z;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty}\frac{(q^\alpha;q)_n}{(q;q)_n}z^n := \sum_{n=0}^{\infty}\frac{([\alpha]_q)_n}{[n]_q!}z^n, \tag{1.3}$$
$$(z, q, \alpha \in \mathbb{C};\ |z| < 1, |q| < 1).$$

The above q-standard notation can be found in [2].

The Genocchi numbers G_n and polynomials $G_n(x)$ together with their generalizations $G_n^{(\alpha)}$ and $G_n^{(\alpha)}(x)$ (α is real or complex), are usually defined by means of the following generating functions (see [5, p. 532-533]):

$$\left(\frac{2z}{e^z+1}\right)^\alpha = \sum_{n=0}^{\infty}G_n^{(\alpha)}\frac{z^n}{n!} \qquad (|z| < \pi), \tag{1.4}$$

$$\left(\frac{2z}{e^z+1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x)\frac{z^n}{n!} \qquad (|z|<\pi). \tag{1.5}$$

Obviously, for $\alpha = 1$, Genocchi polynomials $G_n(x)$ and numbers G_n are

$$G_n(x) := G_n^{(1)}(x) \quad \text{and} \quad G_n := G_n(0) \quad (n \in \mathbb{N}_0), \tag{1.6}$$

respectively.

We now intrduce the following extensions of Genocchi polynomials of higher order based on the idea of Apostol (see, for details, [1]).

Definition 1.1. *The Apostol-Genocchi numbers and polynomials of order α are respectively defined by means of the generating functions:*

$$\left(\frac{2z}{\lambda e^z+1}\right)^{\alpha} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(\lambda)\frac{z^n}{n!} \qquad (|z|<|\log(-\lambda)|), \tag{1.7}$$

$$\left(\frac{2z}{\lambda e^z+1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} \qquad (|z|<|\log(-\lambda)|). \tag{1.8}$$

Clearly, we have

$$\begin{aligned} G_n^{(\alpha)}(x) = \mathcal{G}_n^{(\alpha)}(x;1), \qquad \mathcal{G}_n^{(\alpha)}(\lambda) := \mathcal{G}_n^{(\alpha)}(0;\lambda), \\ \mathcal{G}_n(x;\lambda) := \mathcal{G}_n^{(1)}(x;\lambda) \qquad \text{and} \qquad \mathcal{G}_n(\lambda) := \mathcal{G}_n^{(1)}(\lambda), \end{aligned} \tag{1.9}$$

where $\mathcal{G}_n(\lambda)$, $\mathcal{G}_n^{(\alpha)}(\lambda)$ and $\mathcal{G}_n(x;\lambda)$ denote the so-called Apostol-Genocchi numbers, Apostol-Genocchi numbers of order α and Apostol-Genocchi polynomials respectively.

It follows that we give the following q-extensions for Apostol-Genocchi polynomials of order α.

Definition 1.2. *The q-Apostol-Genocchi numbers and polynomials of order α are respectively defined by means of the generating functions:*

(1.10)

$$W_{\lambda;q}^{(\alpha)}(t) = (2t)^{\alpha}\sum_{n=0}^{\infty}\frac{([\alpha]_q)_n}{[n]_q!}(-\lambda)^n q^n e^{[n]_q t} = \sum_{n=0}^{\infty}\mathcal{G}_{n;q}^{(\alpha)}(\lambda)\frac{t^n}{n!}, \quad (q,\alpha,\lambda \in \mathbb{C};\ |q|<1).$$

$$W^{(\alpha)}_{x;\lambda;q}(t) = (2t)^\alpha \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!}(-\lambda)^n q^{n+x} e^{[n+x]_q t} = \sum_{n=0}^{\infty} \mathcal{G}^{(\alpha)}_{n;q}(x;\lambda)\frac{t^n}{n!}, \quad (q,\alpha,\lambda \in \mathbb{C};\ |q|<1). \tag{1.11}$$

Obviously,

$$\lim_{q\to 1} \mathcal{G}^{(\alpha)}_{n;q}(x;\lambda) = \mathcal{G}^{(\alpha)}_{n}(x;\lambda), \qquad \lim_{q\to 1} \mathcal{G}^{(\alpha)}_{n;q}(\lambda) = \mathcal{G}^{(\alpha)}_{n}(\lambda)$$

and

$$\lim_{q\to 1} G^{(\alpha)}_{n;q}(x) = G^{(\alpha)}_{n}(x), \qquad \lim_{q\to 1} G^{(\alpha)}_{n;q} = G^{(\alpha)}_{n}.$$

We recall that a family of the Hurwitz-Lerch Zeta function $\Phi^{(\rho,\sigma)}_{\mu,\nu}(z,s,a)$ [4, p. 727, Eq. (8)] is defined by

$$\Phi^{(\rho,\sigma)}_{\mu,\nu}(z,s,a) := \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \frac{z^n}{(n+a)^s}, \tag{1.12}$$

$$(\mu \in \mathbb{C};\ a,\nu \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ \rho,\sigma \in \mathbb{R}^+;\ \rho<\sigma \quad \text{when} \quad s,z\in\mathbb{C}; \\ \rho=\sigma \quad \text{and} \quad s\in\mathbb{C} \quad \text{when} \quad |z|<1;\ \rho=\sigma \quad \text{and} \\ \Re(s-\mu+\nu)>1 \quad \text{when} \quad |z|=1),$$

contains, as its *special* cases, not only the Hurwitz-Lerch Zeta function

$$\Phi^{(\sigma,\sigma)}_{\nu,\nu}(z,s,a) = \Phi^{(0,0)}_{\mu,\nu}(z,s,a) = \Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \tag{1.13}$$

but also the following generalized Hurwitz-Zeta function introduced and studied earlier by Goyal and Laddha [3, p. 100, Eq. (1.5)]

$$\Phi^{(1,1)}_{\mu,1}(z,s,a) = \Phi_\mu(z,s,a) := \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n+a)^s}, \tag{1.14}$$

which, for convenience, are called the *Goyal-Laddha-Hurwitz-Lerch Zeta function*.

It follows that we introduce the following definitions.

Definition 1.3. *The q-Goyal-Laddha-Hurwitz-Lerch Zeta function is defined by*

(1.15)

$$\Phi_{\mu;q}(z,s,a) := \sum_{n=0}^{\infty} \frac{([\mu]_q)_n}{[n]_q!} \frac{z^n q^{n+a}}{[n+a]_q^s}, \qquad (\mu, s \in \mathbb{C};\ \Re(a) > 0;\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

Setting $\mu = 1$ in (1.15), we have

Definition 1.4. *The q-Hurwitz-Lerch Zeta function is defined by*

$$\Phi_q(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n q^{n+a}}{[n+a]_q^s}, \qquad (s \in \mathbb{C};\ \Re(a) > 0;\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-). \tag{1.16}$$

The aim of this paper is to give another generating function of q-Apostol-Genocchi polynomials. Some basic properties are also studied. We obtain several interesting relationships between these polynomials and the generalized Zeta functions.

2 Generating functions of the q-Apostol-Genocchi polynomials of higher order

By (1.3) and (1.11), yields

$$\begin{aligned} W_{x;\lambda;q}^{(\alpha)}(t) &= (2t)^{\alpha} \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!} (-\lambda)^n q^{n+x} e^{[n+x]_q t} \\ &= (2t)^{\alpha} e^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!} (-\lambda)^n q^{n+x} e^{-\frac{q^{n+x}}{1-q} t} \\ &= (2t)^{\alpha} e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)x}}{(1-q)^k} \frac{t^k}{k!} \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!} (-\lambda q^{k+1})^n \\ &= (2t)^{\alpha} e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)x}}{(-\lambda q^{k+1}; q)_{\alpha}} \left(\frac{1}{1-q}\right)^k \frac{t^k}{k!}. \end{aligned} \tag{2.1}$$

Therefor, we obtain the generating function of $\mathcal{G}_{n;q}^{(\alpha)}(x;\lambda)$ as follows:

(2.2)
$$W_{x;\lambda;q}^{(\alpha)}(t) = (2t)^{\alpha} e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)x}}{(-\lambda q^{k+1};q)_{\alpha}} \left(\frac{1}{1-q}\right)^k \frac{t^k}{k!} = \sum_{n=0}^{\infty} \mathcal{G}_{n;q}^{(\alpha)}(x;\lambda) \frac{t^n}{n!}.$$

Clearly,

$$W_{\lambda;q}^{(\alpha)}(t) = (2t)^{\alpha} e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(-\lambda q^{k+1};q)_{\alpha}} \left(\frac{1}{1-q}\right)^k \frac{t^k}{k!} = \sum_{n=0}^{\infty} \mathcal{G}_{n;q}^{(\alpha)}(\lambda) \frac{t^n}{n!}. \tag{2.3}$$

Setting $\lambda = 1$ in (2.2) and (2.3) respectively, we deduce the generating functions of $G_{n;q}^{(\alpha)}(x)$ and $G_{n;q}^{(\alpha)}$ as follows:

(2.4)
$$W_{x;q}^{(\alpha)}(t) = (2t)^{\alpha} e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)x}}{(-q^{k+1};q)_{\alpha}} \left(\frac{1}{1-q}\right)^k \frac{t^k}{k!} = \sum_{n=0}^{\infty} G_{n;q}^{(\alpha)}(x;\lambda) \frac{t^n}{n!}$$

and

$$W_{q}^{(\alpha)}(t) = (2t)^{\alpha} e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(-q^{k+1};q)_{\alpha}} \left(\frac{1}{1-q}\right)^k \frac{t^k}{k!} = \sum_{n=0}^{\infty} G_{n;q}^{(\alpha)}(\lambda) \frac{t^n}{n!}. \tag{2.5}$$

It follows that we derive readily the following formulas by (2.2) and (2.3) for $\alpha = \ell \in \mathbb{N}$.

$$\mathcal{G}_{n;q}^{(\ell)}(\lambda) = \frac{2^{\ell}}{(1-q)^{n-\ell}} \sum_{k=\ell}^{n} \binom{n}{k} \frac{(-1)^{k-\ell}\{k\}_{\ell}}{(-\lambda q^{k-\ell+1};q)_{\ell}} \tag{2.6}$$

and

$$\mathcal{G}_{n;q}^{(\ell)}(x;\lambda) = \frac{2^{\ell}}{(1-q)^{n-\ell}} \sum_{k=\ell}^{n} \binom{n}{k} \frac{(-1)^{k-\ell}\{k\}_{\ell} q^{(k-\ell+1)x}}{(-\lambda q^{k-\ell+1};q)_{\ell}}. \tag{2.7}$$

Setting $\lambda = 1$ in (2.6) and (2.7) respectively, we deduce the explicit formulas as follows:

$$G_{n;q}^{(\ell)} = \frac{2^{\ell}}{(1-q)^{n-\ell}} \sum_{k=\ell}^{n} \binom{n}{k} \frac{(-1)^{k-\ell}\{k\}_{\ell}}{(-q^{k-\ell+1};q)_{\ell}} \tag{2.8}$$

and

$$G_{n;q}^{(\ell)}(x) = \frac{2^\ell}{(1-q)^{n-\ell}} \sum_{k=\ell}^{n} \binom{n}{k} \frac{(-1)^{k-\ell}\{k\}_\ell q^{(k-\ell+1)x}}{(-q^{k-\ell+1};q)_\ell}. \tag{2.9}$$

3 Some properties of the q-Apostol-Genocchi polynomials of higher order

In this Section, we shall derive some basic properties of the q-Apostol-Genocchi polynomials.

Proposition 3.1. *The special values for q-Apostol-Genocchi polynomials and numbers of higher order* $(n, \ell \in \mathbb{N};\ \alpha, \lambda \in \mathbb{C})$

$$\begin{aligned} \mathcal{G}_{n;q}^{(\alpha)}(\lambda) = \mathcal{G}_{n;q}^{(\alpha)}(0;\lambda), \qquad \mathcal{G}_{n;q}^{(0)}(x;\lambda) = q^x[x]_q^n, \\ \mathcal{G}_{0;q}^{(\alpha)}(x;\lambda) = \mathcal{G}_{0;q}^{(\alpha)}(\lambda) = \delta_{\alpha,0}, \qquad \mathcal{G}_{n;q}^{(\ell)}(x;\lambda) = 0 \quad (0 \leqq n \leqq \ell - 1). \end{aligned} \tag{3.1}$$

$\delta_{n,k}$ *being the Kronecker symbol.*

Proposition 3.2. *The formula of q-Apostol-Genocchi polynomials of higher order in terms of q-Apostol-Genocchi numbers of higher order*

$$\mathcal{G}_{n;q}^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{G}_{k;q}^{(\alpha)}(\lambda) q^{(k-\alpha+1)x}[x]_q^{n-k}. \tag{3.2}$$

Proof. By (1.11) and (1.10), yields

$$\begin{aligned} W_{x;\lambda;q}^{(\alpha)}(t) &= \sum_{n=0}^{\infty} \mathcal{G}_{n;q}^{(\alpha)}(x;\lambda)\frac{t^n}{n!} = (2t)^\alpha \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!}(-\lambda)^n q^{n+x} e^{[n+x]_q t} \\ &= (2t)^\alpha q^x e^{[x]_q t} \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!}(-\lambda)^n q^n e^{[n]_q q^x t} \\ &= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} \mathcal{G}_{k;q}^{(\alpha)}(\lambda) q^{(k-\alpha+1)x}[x]_q^{n-k} \right] \frac{t^n}{n!}. \end{aligned} \tag{3.3}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of (3.3), we lead immediately to the desired (3.2).

Proposition 3.3 (Difference equation)**.**

$$\lambda q^{\alpha-1}\mathcal{G}_{n;q}^{(\alpha)}(x+1;\lambda)+\mathcal{G}_{n;q}^{(\alpha)}(x;\lambda)=2n\mathcal{G}_{n-1;q}^{(\alpha-1)}(x;\lambda)\quad(n\geqq 1). \tag{3.4}$$

Proof. It is easy to observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{([\alpha-1]_q)_n}{[n]_q!}(-\lambda)^n q^{n+x}e^{[n+x]_q t}=&\lambda q^{\alpha-1}\sum_{n=0}^{\infty}\frac{([\alpha]_q)_n}{[n]_q!}(-\lambda)^n q^{n+x+1}e^{[n+x+1]_q t}\\&+\sum_{n=0}^{\infty}\frac{([\alpha]_q)_n}{[n]_q!}(-\lambda)^n q^{n+x}e^{[n+x]_q t}.\end{aligned} \tag{3.5}$$

By (1.11) and (3.5), we obtain the desired (3.4).

Proposition 3.4 (Differential relationship)**.**

$$\frac{\partial}{\partial_x}\mathcal{G}_{n;q}^{(\alpha)}(x;\lambda)=\mathcal{G}_{n;q}^{(\alpha)}(x;\lambda)\log q+n\frac{\log q}{q-1}q^x\mathcal{G}_{n-1;q}^{(\alpha)}(x;\lambda q). \tag{3.6}$$

Proof. By (2.7), it is not difficult.

Proposition 3.5 (Integral formula)**.**

$$\begin{aligned}\int_a^b q^x\mathcal{G}_{n;q}^{(\alpha)}(x;\lambda q)\ \mathrm{d}x=\\ \frac{1-q}{n+1}\int_a^b\mathcal{G}_{n+1;q}^{(\alpha)}(x;\lambda)\ \mathrm{d}x+\frac{q-1}{\log q}\ \frac{\mathcal{G}_{n+1;q}^{(\alpha)}(b;\lambda)-\mathcal{G}_{n+1;q}^{(\alpha)}(a;\lambda)}{n+1}.\end{aligned} \tag{3.7}$$

Proof. It is easy to obtain (3.7) by (3.6).

Proposition 3.6 (Addition theorem)**.**

$$\mathcal{G}_{n;q}^{(\alpha)}(x+y;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_{k;q}^{(\alpha)}(x;\lambda)q^{(k-\alpha+1)y}[y]_q^{n-k}. \tag{3.8}$$

Proof. By (1.11), yields

(3.9)

$$
\begin{aligned}
W^{(\alpha)}_{x+y;\lambda;q}(t) &= \sum_{n=0}^{\infty} \mathcal{G}^{(\alpha)}_{n;q}(x+y;\lambda)\frac{t^n}{n!} = (2t)^{\alpha}\sum_{n=0}^{\infty}\frac{([\alpha]_q)_n}{[n]_q!}(-\lambda)^n q^{n+x+y} e^{[n+x+y]_q t} \\
&= (2t)^{\alpha} q^y e^{[y]_q t}\sum_{n=0}^{\infty}\frac{([\alpha]_q)_n}{[n]_q!}(-\lambda)^n q^{n+x} e^{[n+x]_q q^y t} \\
&= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\mathcal{G}^{(\alpha)}_{k;q}(x;\lambda) q^{(k-\alpha+1)y}[y]_q^{n-k}\right]\frac{t^n}{n!}.
\end{aligned}
$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of (3.9), we can arrive at formula (3.8) immediately.

Proposition 3.7 (Theorem of complement)**.**

$$
\mathcal{G}^{(\alpha)}_{n;q}(\alpha-x;\lambda) = \frac{(-1)^{n-\alpha}}{\lambda^{\alpha}} q^{\alpha-\binom{\alpha}{2}-n}\mathcal{G}^{(\alpha)}_{n;q^{-1}}(x;\lambda^{-1}), \tag{3.10}
$$

$$
\mathcal{G}^{(\alpha)}_{n;q}(\alpha+x;\lambda) = \frac{(-1)^{n-\alpha}}{\lambda^{\alpha}} q^{\alpha-\binom{\alpha}{2}-n}\mathcal{G}^{(\alpha)}_{n;q^{-1}}(-x;\lambda^{-1}). \tag{3.11}
$$

Proof. It follows that by (2.7).

Proposition 3.8 (Recursive formulas)**.**

(3.12)

$$
(n-\alpha)\mathcal{G}^{(\alpha)}_{n;q}(x;\lambda) = n[x]_q\mathcal{G}^{(\alpha)}_{n-1;q}(x;\lambda) - \frac{\lambda}{2}[\alpha]_q q^x \mathcal{G}^{(\alpha+1)}_{n;q}(x+1;\lambda),
$$

(3.13)

$$
[\alpha]_q q^{x-\alpha}\mathcal{G}^{(\alpha+1)}_{n;q}(x;\lambda) = 2n\left([\alpha]_q q^{x-\alpha} - [x]_q\right)\mathcal{G}^{(\alpha)}_{n-1;q}(x;\lambda) + 2(n-\alpha)\mathcal{G}^{(\alpha)}_{n;q}(x;\lambda).
$$

Proof. We differentiate both side of (1.11) with respect to the variable t

yields

(3.14)

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}W^{(\alpha)}_{x;\lambda;q}(t)=\sum_{n=0}^{\infty}n\mathcal{G}^{(\alpha)}_{n;q}(x;\lambda)\frac{t^{n-1}}{n!}\\
&=2\alpha(2t)^{\alpha-1}\sum_{n=0}^{\infty}\frac{([\alpha]_q)_n}{[n]_q!}(-\lambda)^n q^{n+x}e^{[n+x]_q t}+(2t)^{\alpha}[n+\\
&+x]_q\sum_{n=0}^{\infty}\frac{([\alpha]_q)_n}{[n]_q!}(-\lambda)^n q^{n+x}e^{[n+x]_q t}\\
&=\alpha\sum_{n=0}^{\infty}\mathcal{G}^{(\alpha)}_{n;q}(x;\lambda)\frac{t^{n-1}}{n!}+[x]_q\sum_{n=0}^{\infty}\mathcal{G}^{(\alpha)}_{n;q}(x;\lambda)\frac{t^{n}}{n!}-\\
&-\frac{\lambda}{2}[\alpha]_q q^x\sum_{n=0}^{\infty}\mathcal{G}^{(\alpha+1)}_{n;q}(x+1;\lambda)\frac{t^{n-1}}{n!}\\
&=\sum_{n=0}^{\infty}\left[\alpha\mathcal{G}^{(\alpha)}_{n;q}(x;\lambda)+n[x]_q\mathcal{G}^{(\alpha)}_{n-1;q}(x;\lambda)-\frac{\lambda}{2}[\alpha]_q q^x\mathcal{G}^{(\alpha+1)}_{n;q}(x+1;\lambda)\right]\frac{t^{n-1}}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of (3.14), we get the desired (3.12).

We derive easily equation (3.13) by (3.4) and (3.12). The proof is complete.

Remark 3.1. When $q\to 1$, then the formulas in Proposition 3.1–Proposition 3.8 will become the corresponding formulas of Apostol-Genocchi polynomials of higher order. Further, letting $q\to 1,\alpha=1$, then these formulas will become the corresponding formulas of Apostol-Genocchi polynomials.

Remark 3.2. When $\lambda=1$, then the formulas in Proposition 3.1–Proposition 3.8 will become the corresponding formulas of q-Genocchi polynomials of higher order. Further, letting $\lambda=1,\alpha=1$, then these formulas will become the corresponding formulas of q-Genocchi polynomials.

4 Some explicit relationships between the q-Genocchi polynomials of higher order and q-Goyal-Laddha-Hurwitz-Lerch Zeta function

In this section, we give several interesting relationship between the Genocchi polynomials and Hurwitz-Lerch Zeta function.

We differentiate both side of (1.11) with respect to the variable t, for $\alpha = l \in \mathbb{N}$.

(4.1)

$$\mathcal{G}_{n;q}^{(l)}(a;\lambda) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} W_{a;\lambda;q}^{(l)}(t)\Big|_{t=0} = 2^l \sum_{k=0}^{\infty} \frac{([l]_q)_k}{[k]_q!}(-\lambda)^k q^{k+a} \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left\{e^{[k+a]_q t} t^l\right\}\Big|_{t=0}$$

$$= 2^l \{n\}_l \sum_{k=0}^{\infty} \frac{([l]_q)_k}{[k]_q!}(-\lambda)^k q^{k+a} [k+a]_q^{\,n-l} = 2^l \{n\}_l \sum_{k=0}^{\infty} \frac{([l]_q)_k}{[k]_q!} \frac{(-\lambda)^k q^{k+a}}{\left[k+a\right]_q^{l-n}},$$

we obtain the following theorem.

Theorem 4.1. *The following relationship*

(4.2)

$$\mathcal{G}_{n;q}^{(l)}(a;\lambda) = 2^l \{n\}_l \Phi_{l;q}(-\lambda, l-n, a), \qquad (n, l \in \mathbb{N};\ n \geqq l;\ |\lambda| \leqq 1;\ a \in \mathbb{C}\backslash\mathbb{Z}_0^-),$$

holds true between the q-Apostol-Genocchi polynomials of higher order and q-Goyal-Laddha-Hurwitz-Lerch Zeta function.

Taking $l = 1$ in (4.2), yields

Corollary 4.1. *The following relationship*

$$\mathcal{G}_{n;q}(a;\lambda) = 2n\Phi_q(-\lambda, 1-n, a), \quad (n \in \mathbb{N};\ |\lambda| \leqq 1;\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-), \tag{4.3}$$

holds true between the q-Apostol-Genocchi polynomials and the q-Hurwitz-Lerch Zeta function.

Letting $q \to 1$ in (4.2), we have

Corollary 4.2. *The following relationship*

(4.4)
$$\mathcal{G}_n^{(l)}(a;\lambda) = 2^l \{n\}_l \Phi_l(-\lambda, l-n, a), \qquad (n, l \in \mathbb{N};\ n \geqq l;\ |\lambda| \leqq 1;\ a \in \mathbb{C}\backslash\mathbb{Z}_0^-),$$

holds true between the Apostol-Genocchi polynomials of higher order and Goyal-Laddha-Hurwitz-Lerch Zeta function.

Setting $l = 1$ in (4.4), we deduce the following interesting relationship

Corollary 4.3. *The following relationship*

(4.5)
$$\mathcal{G}_n(a;\lambda) = 2n\Phi(-\lambda, 1-n, a), \qquad (n \in \mathbb{N};\ |\lambda| \leqq 1;\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-),$$

holds true between the Apostol-Genocchi polynomials and Hurwitz-Lerch Zeta function.

Acknowledgements

The present investigation was supported, in part, by the *PhD Program Scholarship Fund of ECNU 2009 of China* under Grant # 2009041, *PCSIRT Project of the Ministry of Education of China* and *Innovation Program of Shanghai Municipal Education Committee of China*.

References

[1] T. M. Apostol, *On the Lerch Zeta function*, Pacific J. Math. **1** (1951), 161–167.

[2] G. Gasper, M. Rahman, *Basic Hypergeometric Series* (2nd edition), Cambridge University Press, 2004.

[3] S. P. Goyal, R. K. Laddha, *On the generalized Riemann Zeta function and the generalized Lambert transform*, Gan. ita Sandesh, **11** (1997), 99-108.

[4] S.-D. Lin, H. M.Srivastava, *Some families of the Hurwitz-Lerch Zeta function and associated fractional derivative and other integral representations.* Appl. Math. Comput., **154** (2004), 725–733.

[5] J. Sándor and B. Crstici, *Handbook of Number Theory II*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2004.

Qiu-Ming Luo
Department of Mathematics, East China Normal University
Dongchuan Road 500, Shanghai 200241, People's Republic of China
Department of Mathematics, Jiaozuo University
Henan Jiaozuo 454003, People's Republic of China
E-Mail: luomath2007@163.com