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A note on a differential superordination defined by a generalized Sălăgean operator[1]

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Abstract

The object of this paper is to obtain a certain superordination using a generalized form of the Sălăgean differential operator.

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1 Introduction

Let Ω be any set in the complex plane $\mathbb{C}$, let p be analytic in the unit disc U and let $\psi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. Many research mathematicians have determined properties of functions p that satisfy the differential subordination

$$\{\psi(p(z), zp'(z), z^2p''(z); z) \mid z \in U\} \subset \Omega.$$

In this article we consider the dual problem of determining properties of functions p that satisfy the differential superordination

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z) : \ z \in U\}.$$

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This problem was introduced in [3]. More information related to this subject can be obtained from [2].

We will denote by $\mathcal{H}(U)$ the set of analytic functions in the unit disc U, i.e. $U = \{z \in \mathbb{C} : \ |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we denote by

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : \ f(z) = a + a_n z^n + \dots\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}(U) : \ f(z) = z + a_{n+1} z^{n+1} + \dots\},$$

so $\mathcal{A} = \mathcal{A}_1$.

Definition 1.1. (see [3]) Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and let h be analytic in U. If p and $\psi(p(z), zp'(z); z)$ are univalent in U and satisfy the first-order differential superordination

$$h(z) \prec \psi(p(z), zp'(z); z) \tag{1.1}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (1.1). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.1) is said to be the best subordinant. The best subordinant is unique up to a rotation of U.

For Ω a set in $\mathbb{C}$, with ψ and p as given in Definition 1.1, suppose (1.1) is replaced by

$$\Omega \subset \{\psi(p(z), zp'(z); z) \mid z \in U\}. \tag{1.2}$$

Although this more general situation is a differential containment, the condition in (1.2) will also be referred to as a differential superordination, and the definitions of solution, subordinant and the best subordinant as given above can be extended to this generalization.

We denote by Q the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$. The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$ (see [3]).

We will use the following lemmas.

Lemma 1.1. *(see [3]) Let h be convex in U, with $h(0) = a$, $\gamma \neq 0$ with* $\mathrm{Re}\,\gamma \geq 0$, *and $p \in \mathcal{H}[a,n] \cap Q$. If $p(z) + \frac{zp'(z)}{\gamma}$ is univalent in U,*

$$h(z) \prec p(z) + \frac{zp'(z)}{\gamma} \tag{1.3}$$

then

$$q(z) \prec p(z), \tag{1.4}$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt.$$

The function q is convex and is the best subordinant.

Lemma 1.2. *(see [3]) Let q be convex in U and let h be defined by*

$$h(z) = q(z) + \frac{zq'(z)}{\gamma}, \quad z \in U \tag{1.5}$$

with $\mathrm{Re}\,\gamma \geq 0$. *If $p \in \mathcal{H}[a,n] \cap Q$, $p(z) + \frac{zp'(z)}{\gamma}$ is univalent in U, and*

$$q(z) + \frac{zq'(z)}{\gamma} \prec p(z) + \frac{zp'(z)}{\gamma}, \quad z \in U$$

then

$$q(z) \prec p(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt.$$

The function q is the best subordinant.

We will use also the following operator.

F.M. Al-Oboudi in [1] defined, for a function in $\mathcal{A}_n$, the following differential operator:

$$D^0 f(z) = f(z) \tag{1.6}$$

$$D^1_\lambda f(z) = D_\lambda f(z) = (1-\lambda) f(z) + \lambda z f'(z) \tag{1.7}$$

$$D^m_\lambda f(z) = D_\lambda (D^{m-1}_\lambda f(z)), \quad \lambda > 0. \tag{1.8}$$

When $\lambda = 1$, we get the Sălăgean operator [5]

$$D^m_\lambda f(z) = z + \sum_{k=n+1}^{\infty} [1 + (k-1)\lambda]^m a_k z^k \tag{1.9}$$

for $f \in \mathcal{A}_n$, $m \in \mathbb{N}$, $\lambda > 0$.

2 Main results

Theorem 2.1. *Let $h \in \mathcal{H}(U)$ be convex in U, with $h(0) = 1$.*

Let $f \in \mathcal{A}_n$, $n \in \mathbb{N}^*$ and suppose that $[D^{m+1}_\lambda f(z)]'$ is univalent and $[D^m_\lambda f(z)]' \in \mathcal{H}[1, n] \cap Q$.

If

$$h(z) \prec [D^{m+1}_\lambda f(z)]' \tag{2.1}$$

then

$$q(z) \prec [D^m_\lambda f(z)]' \tag{2.2}$$

where

$$q(z) = \frac{1}{n z^{1/\lambda n}} \int_0^z h(t) t^{\frac{1}{\lambda n} - 1} dt. \tag{2.3}$$

The function q is convex and is the best subordinant.

Proof. Let $f \in \mathcal{A}_n$. By using the properties of the operator D_λ^m we have

$$D_\lambda^{m+1} f(z) = D_\lambda(D_\lambda^m f(z)) = (1-\lambda) D_\lambda^m f(z) + \lambda z [D_\lambda^m f(z)]' \tag{2.4}$$

Differentiating (2.4) we obtain

$$[D_\lambda^{m+1} f(z)]' = [D_\lambda^m f(z)]' + \lambda z [D_\lambda^m f(z)]'' \tag{2.5}$$

If we let $p(z) = [D_\lambda^m f(z)]'$ then (2.5) becomes

$$[D_\lambda^{m+1} f(z)]' = p(z) + \lambda z p'(z) \tag{2.6}$$

By using Lemma 1.1, for $\gamma = \dfrac{1}{\lambda}$, we have

$$q(z) \prec p(z) = [D_\lambda^m f(z)]' \tag{2.7}$$

where

$$q(z) = \frac{1}{\lambda n z^{1/\lambda n}} \int_0^z h(t) t^{\frac{1}{\lambda n} - 1} dt.$$

The function q is the best dominant.

Theorem 2.2. *Let $h \in \mathcal{H}(U)$ be convex in U, with $h(0) = 1$ and let $f \in \mathcal{A}_n$, $[D_\lambda^m f(z)]'$ is univalent and $\dfrac{D_\lambda^m f(z)}{z} \in \mathcal{H}[1, n] \cap Q$.*

If

$$h(z) \prec [D_\lambda^m f(z)]' \tag{2.8}$$

then

$$q(z) \prec \frac{D_\lambda^m f(z)}{z} \tag{2.9}$$

where

$$q(z) = \frac{1}{n z^{1/n}} \int_0^z h(t) t^{\frac{1}{n} - 1} dt.$$

The function q is convex and is the best subordinant.

Proof. We set

$$p(z) = \frac{D_\lambda^m f(z)}{z} \tag{2.10}$$

and we have

$$D_\lambda^m f(z) = zp(z). \tag{2.11}$$

By differentiating (2.11) we obtain

$$[D_\lambda^m f(z)]' = p(z) + zp'(z),$$

and (2.8) becomes

$$h(z) \prec p(z) + zp'(z)$$

Using Lemma 1.1 we get

$$q(z) \prec p(z) = \frac{D_\lambda^m f(z)}{z}$$

where

$$q(z) = \frac{1}{nz^{1/n}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q is convex and is the best subordinant.

Theorem 2.3. *Let q be convex in U and let h be defined by*

$$h(z) = q(z) + \lambda z q'(z), \quad z \in U, \ \lambda > 0. \tag{2.12}$$

Let $f \in \mathcal{A}_n$ and suppose that $[D_\lambda^{m+1} f(z)]'$ is univalent in U, $[D_\lambda^m f(z)]' \in \mathcal{H}[1,n] \cap Q$ and

$$h(z) = q(z) + \lambda z q'(z) \prec [D_\lambda^{m+1} f(z)]' \tag{2.13}$$

then

$$q(z) \prec [D_\lambda^m f(z)]' \tag{2.14}$$

where

$$q(z) = \frac{1}{\lambda n z^{1/\lambda n}} \int_0^z h(t) t^{\frac{1}{\lambda n}-1} dt.$$

The function q is the best subordinant.

Proof. Let $f \in \mathcal{A}_n$. By using the properties of the operator $D_\lambda^m f$ we have

$$[D_\lambda^{m+1} f(z)]' = [D_\lambda^m f(z)]' + \lambda z [D_\lambda^m f(z)]''$$

and by denoting

$$p(z) = [D_\lambda^m f(z)]'$$

then we obtain

$$[D_\lambda^{m+1} f(z)]' = p(z) + \lambda z p'(z)$$

By using Lemma 1.2 we have

$$q(z) \prec [D_\lambda^m f(z)]'$$

where

$$q(z) = \frac{1}{\lambda n z^{1/n\lambda}} \int_0^z h(t) t^{\frac{1}{\lambda n} - 1} dt.$$

Theorem 2.4. *Let q be convex in U and let h be defined by*

$$h(z) = q(z) + z q'(z). \tag{2.15}$$

Let $f \in \mathcal{A}_n$ and suppose that $[D_\lambda^m f(z)]'$ is univalent in U, $\dfrac{D_\lambda^m f(z)}{z} \in \mathcal{H}[1, n] \cap Q$ and

$$h(z) = q(z) + z q'(z) \prec [D_\lambda^m f(z)]' \tag{2.16}$$

then

$$q(z) \prec \frac{D_\lambda^m f(z)}{z} \tag{2.17}$$

where

$$q(z) = \frac{1}{n z^{1/n}} \int_0^z h(t) t^{\frac{1}{n} - 1} dt.$$

The function q is the best subordinant.

Proof. If we let

$$p(z) = \frac{D_\lambda^m f(z)}{z}$$

we obtain

$$[D_\lambda^m f(z)]' = p(z) + zp'(z).$$

Then (2.16) becomes

$$q(z) + zq'(z) \prec p(z) + zp'(z).$$

By using Lemma 1.2 we get

$$q(z) \prec \frac{D_\lambda^m f(z)}{z}$$

where

$$q(z) = \frac{1}{nz^{1/n}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

Remark 2.1. *For the function*

$$h(z) = \frac{1 + (2\alpha - 1)z}{1 + z},$$

and $\lambda = 1$, similar results were obtained in [4].

References

[1] F.M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Inter. J. of Math. and Mathematical Sci.,1429-1436, 27 (2004).

[2] T. Bulboacă, *Differential subordinations and superordinations. Recent results*, Casa Cărţii de Ştiinţă, 2005.

[3] S.S. Miller, P.T. Mocanu, *Subordinants of Differential Superordinations*, Complex Variables, vol.48, no.10, 815-826, October, 2003.

[4] Gh. Oros, Georgia Irina Oros, *Differential superordination defined by Sălăgean operator*, General Mathematics, Sibiu, vol.12, no. 4, 3-10, 2004.

[5] G.S. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, 262-372, 1013(1983).

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