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Characterization theorem′s of Hermite polynomials[1]

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Abstract

The aim of this paper is to prove two equalities concerning the roots of the Hermite polynomial. For the proof we used multiple points Hermite interpolation.

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1 Introduction

Let $x \in (-\infty, \infty)$ and $H_n(x) = (-1)^n e^{x^2} \left(e^{-x^2}\right)^{(n)}$, $n \in \mathbb{Z}_+$.

The following formulas are known:

$$y''(x) - 2xy'(x) + 2ny(x) = 0 \ , \quad x \in \mathbb{R} \ , \quad y(x) = H_n(x) \tag{1.1}$$

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0 \ , \quad n \in \mathbb{N} \ , \quad n \geq 1 \tag{1.2}$$

$$\frac{dH_n(x)}{dx} = 2nH_{n-1}(x) \text{ for all } n \in \mathbb{N} \setminus \{0\} \ , \quad x \in \mathbb{R} \tag{1.3}$$

$$\lim_{x \to \infty} \frac{H_n(x)}{x^n} = 2^n. \tag{1.4}$$

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2 Main results

Let the polynomials $f(x) = H_n^2 - H_{n+1}(x)H_{n-1}(x)$. From (1.4) we obtain grad $f = 2n - 1$.

According to Hermite interpolation formula

$$f(x) = H_{2n-1}(x_1, x_1, x_2, x_2, ..., x_n, x_n;\ f/x) = \sum_{k=1}^{n} \varphi_k(x) A_k(f; x) \tag{2.1}$$

where

: $x_1, x_2, ..., x_n$ are the roots of $H_n(x)$

: $\varphi_k(x) = \left[\dfrac{H_n(x)}{(x - x_k)H'_n(x_k)}\right]^2$

: $A_k(f; x) = f(x_k) + (x - x_k)\left[f'(x_k) - \dfrac{H''_n(x_k)}{H'_n(x_k)} f(x_k)\right].$

Further, we investigate $A_k(f; x)$. From (1.2) and (1.3) we obtain:

$$f(x_k) = 2nH_{n-1}^2(x_k) \tag{2.2}$$

$$H'_{n+1}(x_k) = 0. \tag{2.3}$$

Using (1.1), (1.2), (2.2) and (2.3) we find

$$\frac{f'(x_k)}{f(x_k)} = \frac{H'_{n-1}(x_k)}{H_{n-1}(x_k)} = 2x_k,$$

$$\frac{H''_n(x_k)}{H'_n(x_k)} = 2x_k.$$

Therefore

$$A_k(f; x) = f(x_k)\left\{1 + (x - x_k)\left[\frac{f'(x_k)}{f(x_k)} - \frac{H''_n(x_k)}{H'_n(x_k)}\right]\right\} = 2nH_{n-1}^2(x_k).$$

We have

$$f(x) = 2n \sum_{k=1}^{n} \left[\frac{H_n(x)}{(x - x_k)H'_n(x_k)}\right]^2 \cdot H_{n-1}^2(x_k). \tag{2.4}$$

From (2.4) we obtain Turán inequality

$$f(x) = H_n^2(x) - H_{n-1}(x)H_{n+1}(x) = \begin{vmatrix} H_n(x) & H_{n+1}(x) \\ H_{n-1}(x) & H_n(x) \end{vmatrix} \geq 0.$$

Observe that

$$H_n^2(x) - H_{n-1}(x)H_{n+1}(x) = 2n\sum_{k=1}^{n}\left[\frac{H_n(x)}{(x-x_k)H_n'(x_k)}\right]^2 H_{n-1}^2(x_k),$$

$$1 - \frac{H_{n-1}(x)H_{n+1}(x)}{H_n^2(x)} = 2n\sum_{k=1}^{n}\left[\frac{H_{n-1}(x_k)}{(x-x_k)H_n'(x_k)}\right]^2, \quad x \neq x_k.$$

Using (1.2) and (1.3) we calculate

$$1 - \frac{1}{2n}\cdot\frac{H_n'(x)}{H_n(x)}\left[2x - 2n\frac{H_n'(x)}{H_n(x)}\right] = \frac{1}{2n}\sum_{k=1}^{n}\frac{1}{(x-x_k)^2},$$

$$1 - \frac{x}{n}\sum_{k=1}^{n}\frac{1}{x-x_k} + \left(\sum_{k=1}^{n}\frac{1}{x-x_k}\right)^2 = \frac{1}{2n}\sum_{k=1}^{n}\frac{1}{(x-x_k)^2},$$

$$1 - \frac{x}{n}\sum_{k=1}^{n}\frac{1}{x-x_k} + \sum_{k=1}^{n}\frac{1}{(x-x_k)^2} + 2\sum_{1\leq i<j\leq n}\frac{1}{(x-x_i)(x-x_j)} =$$
$$\frac{1}{2n}\sum_{k=1}^{n}\frac{1}{(x-x_k)^2},$$

$$1 + \left(1 - \frac{1}{2n}\right)\sum_{k=1}^{n}\frac{1}{(x-x_k)^2} + 2\sum_{1\leq i<j\leq n}\frac{1}{(x-x_i)(x-x_j)} -$$
$$-\frac{x}{n}\sum_{k=1}^{n}\frac{1}{x-x_k} = 0.$$

In conclusion, we proof the following theorem's:

Theorem 2.1. *If* $\{x_1, x_2, ..., x_n\} \subset \mathbb{R}$, $x_i \neq x_j$ *for* $i \neq j$, $i, j \in \{1, 2, ..., n\}$ *verifies*

(2.5)
$$1 + \left(1 - \frac{1}{2n}\right)\sum_{k=1}^{n}\frac{1}{(x-x_k)^2} + 2\sum_{1\leq i<j\leq n}\frac{1}{(x-x_i)(x-x_j)} - \frac{x}{n}\sum_{k=1}^{n}\frac{1}{x-x_k} = 0$$

then $H_n(x_i) = 0$, $i \in \{1, 2, ..., n\}$.

Theorem 2.2. *If* $\{x_1, x_2, ..., x_n\} \subset \mathbb{R}$, $x_i \neq x_j$ *for* $i \neq j$, $i, j \in \{1, 2, ..., n\}$ *verifies*

$$x_j = \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{1}{x_j - x_k}\ , \quad j = 1, 2, ..., n \tag{2.6}$$

then $H_n(x_i) = 0$, $i \in \{1, 2, ..., n\}$.

Proof. Let $P(x) = \prod_{k=1}^{n}(x - x_k)$. We obtain

$$\frac{P'(x)}{P(x)} - \frac{1}{x - x_j} = \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{1}{x - x_k}\ , \quad j \in \{1, 2, ..., n\},$$

$$\frac{(x - x_j)P'(x) - P(x)}{(x - x_j)P(x)} = \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{1}{x - x_k},$$

$$\lim_{x \to x_j} \frac{(x - x_j)P'(x) - P(x)}{(x - x_j)P(x)} = \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{1}{x_j - x_k},$$

$$\frac{P''(x_j)}{2P'(x_j)} = \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{1}{x_j - x_k}. \tag{2.7}$$

From (2.6) and (2.7) we have

$$2x_jP'(x_j) - P''(x_j) = 0\ , \quad j \in \{1, 2, ..., n\}. \tag{2.8}$$

Let $h(x) = 2xP'(x) - P''(x)$. From (2.8), we observe

$$h(x_j) = 0\ , \quad j = 1, 2, ..., n.$$

In conclusion exists $c_n \in \mathbb{R}$ such that $h(x) = c_nP(x)$, then

$$P''(x) - 2xP'(x) + c_nP(x) = 0. \tag{2.9}$$

Because $\{H_0, H_1, ..., H_n\}$ is base in Π_n, exists $a_k \in \mathbb{R}$, $k \in \{0, 1, ..., n\}$ such that

$$P(x) = \sum_{k=0}^{n} a_k H_k(x).$$

From (1.1) and (2.9) we obtain:

$$a_k = 0 \ , \quad k \in \{0, 1, 2, ..., n-1\}$$

$$c_n = 2n.$$

In conclusion, the polynomial P verifies following identity

$$P''(x) - 2xP'(x) + 2nP(x) = 0. \tag{2.10}$$

Using (1.1) and (2.10) we obtain

$$P(x) = \lambda_n H_n(x) \ , \quad \lambda_n \in \mathbb{R} \text{ namely } H_n(x_i) = 0 \ , \quad i \in \{1, 2, ..., n\}.$$

References

[1] G. Gasper, *On the extension of Turan′s inequality to Jacobi polynomials*, Duke Math. J. 38 (1971), 415-428.

[2] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Providence, R. I. 1985.

[3] I. Ţincu, *Proofs of Turan′s inequality*, Mathematical Analysis and Approximation Theory, Burg Verlog, 2002.

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