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Carlson - Shaffer operator and their applications to certain subclass of uniformly convex functions[1]

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Abstract

Making use of Carlson - Shaffer operator,we define a new subclass of uniformly convex functions with negative coefficients and obtain the coefficient bounds, extreme points and radius of starlikeness for functions belonging to the generalized class $TS(\lambda, \alpha, \beta)$. Furthermore, partial sums $f_k(z)$ of functions $f(z)$ in the class $S(\lambda, \alpha, \beta)$ are considered and sharp lower bounds for the ratios of real part of $f(z)$ to $f_k(z)$ and $f'(z)$ to $f'_k(z)$ are determined.

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1 Introduction

Let A denote the class of functions of the form

(1.1) $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

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which are analytic and univalent in the open disc $U = \{z : z \in \mathcal{C}\ |z| < 1\}$. Also denote by T the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0) \tag{1.2}$$

Following Gooodman [3, 4], Rønning [5, 6] introduced and studied the following subclasses
(i) A function $f \in A$ is said to be in the class $S_p(\alpha, \beta)$ of uniformly $\beta-$starlike functions if it satisfies the condition

$$\text{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right|,\ z \in U, \tag{1.3}$$

$-1 < \alpha \leq 1$ and $\beta \geq 0$.
(ii) A function $f \in A$ is said to be in the class $UCV(\alpha, \beta)$ of uniformly $\beta-$convex functions if it satisfies the condition

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right|,\ z \in U, \tag{1.4}$$

and $-1 < \alpha \leq 1$ and $\beta \geq 0$.
Indeed it follows from (1.3) and (1.4) that

$$f \in UCV(\alpha, \beta) \text{ is equivalent with } zf' \in S_p(\alpha, \beta). \tag{1.5}$$

For functions $f \in A$ given by (1.1) and $g(z) \in A$ given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or Convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U. \tag{1.6}$$

Let $\phi(a, c; z)$ be the incomplete beta function defined by

$$\phi(a, c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n,\ \ c \neq 0, -1, -2, \ldots \tag{1.7}$$

where $(x)_n$ is the Pochhammer symbol defined interms of the Gamma functions, by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & \text{n} = 0 \\ x(x+1)(x+2)\dots(x+n-1), & n \in N \end{cases} \tag{1.8}$$

Further, for $f \in A$

$$L(a,c)f(z) = \phi(a,c;z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n, \tag{1.9}$$

where $L(a,c)$ is called Carlson - Shaffer operator [2] and the operator $*$ stands for the hadamard product (or convolution product) of two power series as given by (1.6).

We notice that

$$L(a,a)f(z) = f(z), \;\; L(2,1)f(z) = zf'(z).$$

For $-1 \leq \alpha < 1$,$0 \leq \lambda \leq 1$ and $\beta \geq 0$, we let $S(\lambda, \alpha, \beta)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\begin{aligned} &\text{Re}\left\{ \frac{z(L(a,c)f(z))' + \lambda z^2 (L(a,c)f(z))''}{(1-\lambda)L(a,c)f(z) + \lambda(zL(a,c)f(z))'} - \alpha \right\} \\ &> \beta \left| \frac{z(L(a,c)f(z))' + \lambda z^2 (L(a,c)f(z))''}{(1-\lambda)L(a,c)f(z) + \lambda(zL(a,c)f(z))'} - 1 \right|, \;\; z \in U \end{aligned} \tag{1.10}$$

where $L(a,c)f(z)$ is given by (1.9). We also let $TS(\lambda, \alpha, \beta) = S(\lambda, \alpha, \beta) \cap T$.

By suitably specializing the values of λ, (a) and (c), the class $S(\lambda, \alpha, \beta)$ can be reduces to the class studied earlier by Rønning [5, 6]. Also choosing $\alpha = 0$ and $\beta = 1$ the class coincides with the classes studied in [10] and [11] respectively.

The main object of this paper is to study the coefficient bounds, extreme points and radius of starlikeness for functions belonging to the generalized class $TS(\lambda, \alpha, \beta)$. Furthermore, partial sums $f_k(z)$ of functions $f(z)$ in the class $S(\lambda, \alpha, \beta)$ are considered and sharp lower bounds for the ratios of real part of $f(z)$ to $f_k(z)$ and $f'(z)$ to $f'_k(z)$ are determined.

2 Basic Properties

In this section we obtain a necessary and sufficient condition for functions $f(z)$ in the classes $S(\lambda,\alpha,\beta)$ and $TS(\lambda,\alpha,\beta)$.

Theorem 2.1. *A function $f(z)$ of the form (1.1) is in $S(\lambda,\alpha,\beta)$ if*

$$\sum_{n=2}^{\infty}(1+\lambda(n-1))[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}|a_n|\leq 1-\alpha, \tag{2.1}$$

$-1\leq\alpha<1,\quad 0\leq\lambda\leq 1,\beta\geq 0.$

Proof. It sufficies to show that

$$\beta\left|\frac{z(L(a,c)f(z))'+\lambda z^2(L(a,c)f(z))''}{(1-\lambda)L(a,c)f(z)+\lambda(zL(a,c)f(z))'}-1\right|$$
$$-\text{Re}\left\{\frac{z(L(a,c)f(z))'+\lambda z^2(L(a,c)f(z))''}{(1-\lambda)L(a,c)f(z)+\lambda(zL(a,c)f(z))'}-1\right\}$$
$$\leq 1-\alpha$$

We have

$$\beta\left|\frac{z(L(a,c)f(z))'+\lambda z^2(L(a,c)f(z))''}{(1-\lambda)L(a,c)f(z)+\lambda(zL(a,c)f(z))'}-1\right|$$

$$-\text{Re}\left\{\frac{z(L(a,c)f(z))'+\lambda z^2(L(a,c)f(z))''}{(1-\lambda)L(a,c)f(z)+\lambda(zL(a,c)f(z))'}-1\right\}$$

$$\leq(1+\beta)\left|\frac{z(L(a,c)f(z))'+\lambda z^2(L(a,c)f(z))''}{(1-\lambda)L(a,c)f(z)+\lambda(zL(a,c)f(z))'}-1\right|$$

$$\leq\frac{(1+\beta)\sum\limits_{n=2}^{\infty}(n-1)[1+\lambda(n-1)]\frac{(a)_{n-1}}{(c)_{n-1}}|a_n|}{1-\sum\limits_{n=2}^{\infty}[1+\lambda(n-1)]\frac{(a)_{n-1}}{(c)_{n-1}}|a_n|}.$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{n=2}^{\infty}[1+\lambda(n-1)][n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}|a_n| \leq 1-\alpha,$$

and hence the proof is complete.

Theorem 2.2. *A necessary and sufficient condition for $f(z)$ of the form (1.2) to be in the class $TS(\lambda,\alpha,\beta)$, $-1 \leq \alpha < 1, 0 \leq \lambda \leq 1, \beta \geq 0$ is that*

$$\sum_{n=2}^{\infty}(1+\lambda(n-1))[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}\ a_n \leq 1-\alpha, \tag{2.2}$$

Proof. In view of Theorem 1, we need only to prove the necessity. If $f \in TS(\lambda,\alpha,\beta)$ and z is real then

$$\frac{1-\sum\limits_{n=2}^{\infty} n[1+\lambda(n-1)]\frac{(a)_{n-1}}{(c)_{n-1}}a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}[1+\lambda(n-1)]\frac{(a)_{n-1}}{(c)_{n-1}}a_n z^{n-1}} - \alpha \geq$$

$$\geq \beta \left| \frac{\sum\limits_{n=2}^{\infty}(n-1)[1+\lambda(n-1)]\frac{(a)_{n-1}}{(c)_{n-1}}|a_n|}{1-\sum\limits_{n=2}^{\infty}[1+\lambda(n-1)]\frac{(a)_{n-1}}{(c)_{n-1}}|a_n|} \right|$$

Letting $z \longrightarrow 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}(1+\lambda(n-1))[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}\ a_n \leq 1-\alpha.$$

Theorem 2.3. *Let $f(z)$ defined by (1.2) and $g(z)$ defined by $g(z) = z - \sum\limits_{n=2}^{\infty} b_n z^n$ be in the class $TS(\lambda,\alpha,\beta)$. Then the function $h(z)$ defined by*

$$h(z) = (1-\mu)f(z) + \mu g(z) = z - \sum_{n=2}^{\infty} q_n z^n,$$

where $q_n = (1-\mu)a_n + \mu b_n, \quad 0 \leq \mu < 1$ is also in the class $TS(\lambda,\alpha,\beta)$.

Proof. Let the function

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \qquad a_{n,j} \geq 0, \qquad j = 1, 2\,, \tag{2.3}$$

be in the class $TS(\lambda, \alpha, \beta)$. It is sufficient to show that the function $g(z)$ defined by

$$g(z) = \mu f_1(z) + (1-\mu) f_2(z), \qquad 0 \leq \mu \leq 1,$$

is in the class $TS(\lambda, \alpha, \beta)$. Since

$$g(z) = z - \sum_{n=2}^{\infty} [\mu a_{n,1} + (1-\mu) a_{n,2}] z^n,$$

an easy computation with the aid of Theorem 2.2 gives,

$$\begin{aligned} \sum_{n=2}^{\infty} [1+\lambda(n-1)][n(\beta+1)-(\alpha+\beta)] \frac{(a)_{n-1}}{(c)_{n-1}} \mu a_{n,1} & \\ + \sum_{n=2}^{\infty} [1+\lambda(n-1)][n(\beta+1)-(\alpha+\beta)] \frac{(a)_{n-1}}{(c)_{n-1}} (1-\mu) a_{n,2} & \\ \leq \mu(1-\alpha) + (1-\mu)(1-\alpha) & \\ \leq 1-\alpha, & \end{aligned}$$

which implies that $g \in TS(\lambda, \alpha, \beta)$. Hence $TS(\lambda, \alpha, \beta)$ is convex.

Theorem 2.4. *(Extreme points) Let $f_1(z) = z$ and*

$$f_n(z) = z - \frac{(1-\alpha)(c)_{n-1}}{(1+n\lambda-\lambda)[n(1+\beta)-(\alpha+\beta)](a)_{n-1}} z^n \text{ for } n = 2, 3, 4, \ldots. \tag{2.4}$$

Then $f(z) \in TS(\lambda, \alpha, \beta)$ if and only if $f(z)$ can be expressed in the form $f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z)$, where $\mu_n \geq 0$ and $\sum_{n=1}^{\infty} \mu_n = 1$.

The proof of Theorem 2.4, follows on lines similar to the proof of the theorem on extreme points given in Silverman [8].

Next we prove the following closure theorem.

Theorem 2.5. *(Closure theorem) Let the functions $f_j(z)$ $(j = 1, 2, \dots m)$ defined by (2.3) be in the classes $TS(\lambda, \alpha_j, \beta)$ $(j = 1, 2, \dots m)$ respectively. Then the function $h(z)$ defined by*

$$h(z) = z - \frac{1}{m}\sum_{n=2}^{\infty}\left(\sum_{j=1}^{m} a_{n,j}\right) z^n$$

is in the class $TS(\lambda, \alpha, \beta)$, where $\alpha = \min_{1\le j\le m}\{\alpha_j\}$ where $-1 \le \alpha_j < 1$.

Proof. Since $f_j(z) \in TS(\lambda, \alpha_j, \beta)$ $(j = 1, 2, 3, \dots m)$ by applying Theorem 2.2, to (2.3) we observe that

$$\sum_{n=2}^{\infty}(1+\lambda(n-1))[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}}\left(\frac{1}{m}\sum_{j=1}^{m} a_{n,j}\right)$$

$$= \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{n=2}^{\infty}(1+\lambda(n-1))[n(1+\beta)-(\alpha+\beta)]\frac{(a)_{n-1}}{(c)_{n-1}} a_{n,j}\right)$$

$$\le \frac{1}{m}\sum_{j=1}^{m}(1-\alpha_j) \le 1-\alpha$$

which in view of Theorem 2.2, again implies that $h(z) \in TS(\lambda, \alpha, \beta)$ and so the proof is complete.

Theorem 2.6. *Let $f \in TS(\lambda, \alpha, \beta)$. Then*

1. *f is starlike of order $\delta(0 \le \delta < 1)$ in the disc $|z| < r_1$; that is, Re $\left\{\frac{zf'(z)}{f(z)}\right\} > \delta$, $(|z| < r_1\ ;\ 0 \le \delta < 1)$, where*

$$r_1 = \inf_{n\le 2}\left\{\frac{(a)_{n-1}}{(c)_{n-1}}\left(\frac{1-\delta}{n-\delta}\right)\frac{(1+n\lambda-\lambda)[n(1+\beta)-(\alpha+\beta)]}{1-\alpha}\right\}^{\frac{1}{n-1}}.$$

2. *f is convex of order δ $(0 \le \delta < 1)$ in the disc $|z| < r_2$, that is Re $\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \delta$, $(|z| < r_2; 0 \le \delta < 1)$, where*

$$r_2 = \inf_{n\le 2}\left\{\frac{(a)_{n-1}}{(c)_{n-1}}\frac{(1-\delta)(1+n\lambda-\lambda)[n(1+\beta)-(\alpha+\beta)]}{n(n-\delta)}\right\}^{\frac{1}{n-1}}.$$

Each of these results are sharp for the extremal function $f(z)$ given by (2.4).

Proof. Given $f \in A$, and f is starlike of order δ, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \delta. \tag{2.5}$$

For the left hand side of (2.5) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n\ |z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_n\ |z|^{n-1}}.$$

The last expression is less than $1-\delta$ if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}a_n\ |z|^{n-1} < 1.$$

Using the fact, that $f \in TS(\lambda, \alpha, \beta)$ if and only if

$$\sum_{n=2}^{\infty}\frac{(1+\lambda(n-1))[n(1+\beta)-(\alpha+\beta)]}{1-\alpha}\frac{(a)_{n-1}}{(c)_{n-1}}a_n\ < 1.$$

We can say (2.5) is true if

$$\frac{n-\delta}{1-\delta}|z|^{n-1} < \frac{(1+\lambda(n-1))[n(1+\beta)-(\alpha+\beta)]}{1-\alpha}\frac{(a)_{n-1}}{(c)_{n-1}}.$$

Or, equivalently,

$$|z|^{n-1} < \frac{(1-\delta)(1+\lambda(n-1))[n(1+\beta)-(\alpha+\beta)]}{(n-\delta)(1-\alpha)}\frac{(a)_{n-1}}{(c)_{n-1}}$$

which yields the starlikeness of the family.

(ii) Using the fact that f is convex if and only if zf' is starlike, we can prove (ii), on lines similar to the proof of (i).

3 Partial Sums

Following the earlier works by Silverman [8] and Silvia [9] on partial sums of analytic functions. We consider in this section partial sums of functions in the class $TS(\lambda, \alpha, \beta)$ and obtain sharp lower bounds for the ratios of real part of $f(z)$ to $f_k(z)$ and $f'(z)$ to $f'_k(z)$.

Theorem 3.1. *Let $f(z) \in TS(\lambda, \alpha, \beta)$ be given by (1.1) and define the partial sums $f_1(z)$ and $f_k(z)$, by*

$$f_1(z) = z; \text{ and } f_k(z) = z + \sum_{n=2}^{k} a_n z^n, \ (k \in N/1) \tag{3.1}$$

Suppose also that

$$\sum_{n=2}^{\infty} d_n |a_n| \leq 1,$$

where

$$d_n := \frac{(1+\lambda(n-1))[n(\alpha+\beta)-(\alpha+\beta)]}{(1-\alpha)} \frac{(a)_{n-1}}{(c)_{n-1}}. \tag{3.2}$$

Then $f \in TS(\lambda, \alpha, \beta)$. Furthermore,

$$Re\left\{\frac{f(z)}{f_k(z)}\right\} > 1 - \frac{1}{d_{k+1}} \ \ z \in U, k \in N \tag{3.3}$$

and

$$Re\left\{\frac{f_k(z)}{f(z)}\right\} > \frac{d_{k+1}}{1+d_{k+1}}. \tag{3.4}$$

Proof. For the coefficients d_n given by (3.2) it is not difficult to verify that

$$d_{n+1} > d_n > 1. \tag{3.5}$$

Therefore we have

$$\sum_{n=2}^{k} |a_n| + d_{k+1} \sum_{n=k+1}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} d_n |a_n| \leq 1 \tag{3.6}$$

by using the hypothesis (3.2). By setting

$$g_1(z) = d_{k+1}\left\{\frac{f(z)}{f_k(z)} - \left(1-\frac{1}{d_{k+1}}\right)\right\}$$

$$= 1 + \frac{d_{k+1}\sum\limits_{n=k+1}^{\infty} a_n z^{n-1}}{1+\sum\limits_{n=2}^{k} a_n z^{n-1}} \tag{3.7}$$

and applying (3.6), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{d_{k+1}\sum\limits_{n=k+1}^{\infty}|a_n|}{2-2\sum\limits_{n=2}^{n}|a_n| - d_{k+1}\sum\limits_{n=k+1}^{\infty}|a_n|}$$

$$\leq 1, \quad z \in U, \tag{3.8}$$

which readily yields the assertion (3.3) of Theorem 3.1. In order to see that

$$f(z) = z + \frac{z^{k+1}}{d_{k+1}} \tag{3.9}$$

gives sharp result, we observe that for $z = re^{i\pi/k}$ that $\frac{f(z)}{f_k(z)} = 1 + \frac{z^k}{d_{k+1}} \to 1 - \frac{1}{d_{k+1}}$ as $z \to 1^-$. Similarly, if we take

$$g_2(z) = (1+d_{k+1})\left\{\frac{f_k(z)}{f(z)} - \frac{d_{k+1}}{1+d_{k+1}}\right\}$$

$$= 1 - \frac{(1+d_{n+1})\sum\limits_{n=k+1}^{\infty} a_n z^{n-1}}{1+\sum\limits_{n=2}^{\infty} a_n z^{n-1}} \tag{3.10}$$

and making use of (3.6), we can deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1+d_{k+1})\sum\limits_{n=k+1}^{\infty}|a_n|}{2-2\sum\limits_{n=2}^{k}|a_n| - (1-d_{k+1})\sum\limits_{n=k+1}^{\infty}|a_n|} \tag{3.11}$$

which leads us immediately to the assertion (3.4) of Theorem 3.1.

The bound in (3.4) is sharp for each $k \in N$ with the extremal function $f(z)$ given by (3.9). The proof of the Theorem 3.1, is thus complete.

Theorem 3.2. *If $f(z)$ of the form (1.1) satisfies the condition (2.1). Then*

$$Re\left\{\frac{f'(z)}{f_k'(z)}\right\} \geq 1 - \frac{k+1}{d_{k+1}}. \tag{3.12}$$

Proof. By setting

$$\begin{aligned} g(z) &= d_{k+1}\left\{\frac{f'(z)}{f_k'(z)} - \left(1 - \frac{k+1}{d_{k+1}}\right)\right\} \\ &= \frac{1 + \frac{d_{k+1}}{k+1}\sum\limits_{n=k+1}^{\infty} na_n z^{n-1} + \sum\limits_{n=2}^{\infty} na_n z^{n-1}}{1 + \sum\limits_{n=2}^{k} na_n z^{n-1}} \\ &= 1 + \frac{\frac{d_{k+1}}{k+1}\sum\limits_{n=k+1}^{\infty} na_n z^{n-1}}{1 + \sum\limits_{n=2}^{k} na_n z^{n-1}}. \end{aligned}$$

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\frac{d_{k+1}}{k+1}\sum\limits_{n=k+1}^{\infty} n|a_n|}{2 - 2\sum\limits_{n=2}^{k} n|a_n| - \frac{d_{k+1}}{k+1}\sum\limits_{n=k+1}^{\infty} n|a_n|}. \tag{3.13}$$

Now

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq 1$$

if

$$\sum_{n=2}^{k} n|a_n| + \frac{d_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n|a_n| \leq 1 \tag{3.14}$$

since the left hand side of (3.14) is bounded above by $\sum\limits_{n=2}^{k} d_n|a_n|$ if

$$\sum_{n=2}^{k}(d_n - n)|a_n| + \sum_{n=k+1}^{\infty} d_n - \frac{d_{k+1}}{k+1}n|a_n| \geq 0, \tag{3.15}$$

and the proof is complete.

The result is sharp for the extremal function $f(z) = z + \frac{z^{k+1}}{c_{k+1}}$.

Theorem 3.3. *If $f(z)$ of the form (1.1) satisfies the condition (2.1) then*

$$Re\left\{\frac{f_k'(z)}{f'(z)}\right\} \geq \frac{d_{k+1}}{k+1+d_{k+1}}. \tag{3.16}$$

Proof. By setting

$$\begin{aligned} g(z) &= [(k+1)+d_{k+1}]\left\{\frac{f_k'(z)}{f'(z)} - \frac{d_{k+1}}{k+1+d_{k+1}}\right\} \\ &= 1 - \frac{\left(1+\frac{d_{k+1}}{k+1}\right)\sum\limits_{n=k+1}^{\infty} na_n z^{n-1}}{1+\sum\limits_{n=2}^{k} na_n z^{n-1}} \end{aligned}$$

and making use of (3.15), we deduce that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\left(1+\frac{d_{k+1}}{k+1}\right)\sum\limits_{n=k+1}^{\infty} n|a_n|}{2-2\sum\limits_{n=2}^{k} n|a_n| - \left(1+\frac{d_{k+1}}{k+1}\right)\sum\limits_{n=k+1}^{\infty} n|a_n|} \leq 1,$$

which leads us immediately to the assertion of Theorem 3.3.

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