General Mathematics Vol. 15, Nr. 2–3(2007), 3-23

Elliptic analogue of the Hardy sums related to elliptic Bernoulli functions

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Abstract

In this paper, we define generalized Hardy-Berndt sums and elliptic analogue of the generalized Hardy-Berndt sums related to elliptic Bernoulli polynomials. We give relations between the Weierstrass $\wp(z)$-function, Hardy-Bernd sums, theta functions and generalized Dedekind eta function.

2000 Mathematical Subject Classification: Primary 11F20, 11B68; Secondary 14K25, 14H42.

Key words and phrases: Bernoulli polynomials and Bernoulli functions,, Dedekind sums, Dedekind-Rademacher sums, Hardy sums, Theta functions.

1 Introduction, definitions and notations

The Dedekind eta function, $\eta(z)$ is defined by

$$\eta(z) = e^{\frac{\pi i z}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}),$$

where $z \in \mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$. The behavior of this function under the modular group, $\Gamma(1)$ is given by the following functional equation:

Theorem 1. *([1]) Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(1)$,

$$\log \eta(Az) = \log \eta(z) + \frac{\pi i(a+d)}{12c} - \pi i(s(d,c) - \frac{1}{4}) + \frac{1}{2}\log(cz+d),$$

where $s(d,c)$ *is the Dedekind sum, which is defined as follows:*

$$s(d,c) = \sum_{n=1}^{c-1}((\frac{n}{c}))((\frac{dn}{c})),$$

and

$$((x)) = \begin{cases} x - [x] - \frac{1}{2},\ x \notin \mathbb{Z} \\ 0,\ x \in \mathbb{Z} \end{cases}$$

$[x]$ *denotes the greatest integer functions cf. (see also [1], [6], [3], [2], [7], [22], [11], [23], [25], [30]).*

Generalized Dedekind eta functions defined by [22]:

Let g and h be integers, and N be a positive integer. We define

$$\eta_{g,h}(z;N) = \alpha_{g,h}(N)\exp(\pi iz\overline{B}_2\left(\frac{g}{N}\right))_{m\equiv g(N),\ m>0}\left(1-\zeta_N^h q_N^m\right)_{m\equiv -g(N),\ m>0}\left(1-\zeta_N^{-h} q_N^m\right)$$

for $z \in \mathbb{H}$ where $\zeta_N = e^{\frac{2\pi i}{N}}$, $q_N = e^{\frac{2\pi iz}{N}}$ and

$$\alpha_{g,h}(N) = \begin{cases} \exp(\pi i\overline{B}_1\left(\frac{h}{N}\right))\left(1-\zeta_N^{-h}\right),\ \text{if } g \equiv 0,\ h \not\equiv mod N, \\ 1,\ \text{otherwise.} \end{cases}$$

$\overline{B}_1$ and $\overline{B}_2$ in the formulae are Bernoulli functions:

$$\overline{B}_1(x) = x - [x] - \frac{1}{2},\ \overline{B}_2(x) = (x-[x])^2 - (x-[x]) + \frac{1}{6}.$$

Here $\overline{B}_n(x)$ is the nth Bernoulli functions, which are defined by

$$\overline{B}_n(x) = B_n(x - [x]) = \begin{cases} 0, \text{ if } n = 1, \ x \in \mathbb{Z}, \\ \overline{B}_n(\{x\}), \text{ otherwise} \end{cases},$$

where $B_n(x)$ denotes Bernoulli polynomials:

$$\frac{te^{tx}}{e^t - 1} = \sum_{j=0}^{\infty} B_j(x) \frac{t^j}{j!},$$

cf. ([34], [35], [38], [31]). The functions $\eta_{g,h}(z; N)$ are holomorphic for $z \in \mathbb{H}$ and depend upon g, h modulo N. Furthermore, $\eta_{g,h}(z; N) = \eta_{-g,-h}(z; N)$ for each g and h, and $\eta_{g,h}(z; N) = \eta^2(z)$ for $(g, h) \equiv (0, 0) \, (mod N)$ (see for detail [22], [37], [19], [23], [25]).

Halbritter[12] and Hall et at.[13] defined generalized Dedekind sums as follows:

Definition 1. *Let a, b and c be positive integers, and x, y and z be real numbers.*

$$(1) \qquad S_{m,n}(a, b, c : x, y, z) = \sum_{k mod c} \overline{B}_m(a\frac{k + z}{c} - x)\overline{B}_n(b\frac{k + z}{c} - y).$$

The classical theta-functions, $\vartheta_n(0, q)(n = 2, 3, 4)$ are defined as follows ([38], [16], [24], [30])

$$\vartheta_2(0, q) = 2q^{\frac{1}{4}} \sum_{n=0} q^{n(n+1)} = 2q^{\frac{1}{4}} \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n})^2,$$

$$\vartheta_3(0, q) = 1 + 2 \sum_{n=1} q^{n^2} = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1})^2,$$

$$\vartheta_4(0, q) = 1 + 2 \sum_{n=1} (-1)^n q^{n^2} = \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n-1})^2,$$

$q=e^{\pi iz}$, $z\in\mathbb{C}$, $|q|<1$. Throughout of this paper, we denote $\vartheta_2(0,q)$, $\vartheta_3(0,q)$ and $\vartheta_4(0,q)$ by $\vartheta_2(q)$, $\vartheta_3(q)$ and $\vartheta_4(q)$, respectively.

Relations between theta functions and η-function are given by (see [16], [19], [23]):

$$\vartheta_2(z)=\frac{2\eta^2(2z)}{\eta(z)},\ \vartheta_3(z)=\frac{\eta^5(z)}{\eta^2(2z)\eta^2(\frac{z}{2})},\ \vartheta_4(z)=\frac{\eta^2(\frac{z}{2})}{\eta(z)}. \tag{2}$$

Let $\tau\in\mathbb{H}$, $e(x)=e^{2\pi ix}$. Jacobi's theta function is defined by ([17], [18])

$$\theta(x,\tau)=\sum_{m\in\mathbb{Z}}e\left(\frac{(m+\frac{1}{2})^2}{2}\tau+(m+\frac{1}{2})(x+\frac{1}{2})\right).$$

Observe that $\theta(x+1,\tau)=-\theta(x,\tau)$, $\theta(x+\tau,\tau)=-e(-\frac{\tau}{2}-x)\theta(x,\tau)$.

In [17] and [18], Machide defined the following generating function of the elliptic Bernoulli functions (*Kronecker's double series*), $B_m(y,x;\tau)$:

$$\begin{aligned}F(y,x;a,\tau)&=e(ax)f(-y+x\tau,a;\tau)\\&=\sum_{m=0}^{\infty}B_m(y,x;\tau)\frac{(2\pi i)^m}{m!}a^{m-1},\end{aligned}$$

where

$$f(x,a;\tau)=\frac{\frac{\partial\theta(x,\tau)}{\partial x}\theta(x+a,\tau)}{\theta(x,\tau)\theta(a,\tau)},$$

$x,a\in\mathbb{C}\setminus\mathbb{Z}+\tau\mathbb{Z}$.

We note that $B_m(y+1,x;\tau)=B_m(y,x+1;\tau)=B_m(y,x;\tau)$ cf. ([17], [18], [21]).

Proposition 1. *([18]) Let x and y be real numbers and $y\notin Z$. Then we have*

$$\lim_{\tau\to i\infty}B_m(y,x;\tau)=\begin{cases}\frac{1+e(y)}{2(1-e(y))}=\frac{i\cot(y\pi)}{2},\ m=1\ \text{and}\ x\in Z\\ \overline{B}_m(x),\ \text{otherwise}\end{cases},$$

especially

$$Re\left(\lim_{\tau\to i\infty} B_m(y,x;\tau)\right) = \overline{B}_m(x).$$

Here Rew means the real part of a complex number w.

In [18], Machide defined the elliptic Dedekind-Rademacher sums as follows:

Let a, $a^{'}$, b, $b^{'}$, c, $c^{'}$ be positive integers and x, $x^{'}$, y, $y^{'}$, z, $z^{'}$ be real numbers. Suppose that

$$a^{'}z^{'}-c^{'}x^{'} \notin < a^{'},c^{'} > \mathbb{Z} \text{ and } b^{'}z^{'}-c^{'}y^{'} \notin < b^{'},c^{'} > \mathbb{Z},$$

where $< a,b >$ is the greatest common divisor of a and b.

Set $(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c})=((a^{'},a),(b^{'},b),(c^{'},c))$, $(\overrightarrow{x},\overrightarrow{y},\overrightarrow{z})=((x^{'},x),(y^{'},y),(z^{'},z))$. The elliptic Dedekind-Rademacher sums are defined by [18]

$$(3)\quad S^{\tau}_{m,n}(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c};\overrightarrow{x},\overrightarrow{y},\overrightarrow{z})=\sum_{\substack{j \bmod c\\ l \bmod c^{'}}} B_m\left(a^{'}\frac{l+z^{'}}{c^{'}}-x^{'},a\frac{j+z}{c}-x;\frac{a^{'}}{a}\tau\right) \times B_n\left(b^{'}\frac{l+z^{'}}{c^{'}}-y^{'},b\frac{j+z}{c}-y;\frac{b^{'}}{b}\tau\right).$$

Observe that if $m\neq 1$, $n\neq 1$, or if $az-cx\notin < a,c > \mathbb{Z}$ and $bz-cy\notin < b,c > \mathbb{Z}$, then

$$\lim_{\tau\to i\infty} S^{\tau}_{m,n}(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c};\overrightarrow{x},\overrightarrow{y},\overrightarrow{z}) = S_{m,n}(a,b,c;x,y,z).$$

Hardy-Berndt sums derived from the transformation formulae for $\log\vartheta_n(z)$ ($n=2$, 3, 4), which are similar to Dedekind sum. For h, $k\in\mathbb{Z}$ with $k>0$, Hardy-Berndt sums are defined as follows ([7], [23], [26], [36]):

$$(4)\qquad S(h,k)=\sum_{j=1}^{k-1}(-1)^{j+1+[\frac{hj}{k}]},\ s_1(h,k)=\sum_{j=1}^{k}(-1)^{[\frac{hj}{k}]}((\frac{j}{k})),$$

$$s_2(h,k)=\sum_{j=1}^{k}(-1)^{j}((\frac{j}{k}))((\frac{hj}{k})),\ s_3(h,k)=\sum_{j=1}^{k}(-1)^{j}((\frac{hj}{k})),$$

$$s_4(h,k)=\sum_{j=1}^{k-1}(-1)^{[\frac{hj}{k}]},\ s_5(h,k)=\sum_{j=1}^{k}(-1)^{j+[\frac{hj}{k}]}((\frac{j}{k})).$$

The relations between Hardy sums and Dedekind sums are given as follows:

Theorem 2. *([36]) Let $(h,k)=1$. Then if $h+k$ is odd,*

$$S(h,k)=8s(h,2k)+8s(2h,k)-20s(h,k),$$

if h is even,

$$s_1(h,k)=2s(h,k)-4s(h,2k),$$

if k is even,

$$s_2(h,k)=-s(h,k)+2s(2h,k),$$

if k is odd,

$$s_3(h,k)=2s(h,k)-4s(2h,k),$$

if h is odd,

$$s_4(h,k)=-4s(h,k)+8s(h,2k),$$

if $h+k$ is even,

$$s_5(h,k)=-10s(h,k)+4s(2h,k)+4s(h,2k)$$

and each one of $S(h,k)$ $(h+k$ even), $s_1(h,k)$ $(h$ odd), $s_2(h,k)$ $(k$ odd), $s_3(h,k)$ $(k$ even), $s_4(h,k)$ $(h$ even) and $s_5(h,k)$ $(h+k$ odd) is zero.

Recently, relations between Hardy-Berndt sums and theta function were studied cf. ([6], [7], [29], [23], [24], [26], [36]).

The manin motivations of this paper are given as follows:

In Section 2, we give some identities related to the Weierstrass $\wp(z)$-function, Hardy-Bernd sums, theta functions and generalized Dedekind eta function. We give relations between the Weierstrass $\wp(z)$-function, Hardy-Bernd sums, theta functions and generalized Dedekind eta function.

In Section 3, we define generalized Hardy-Bernd sums. By using these generalizations, we construct elliptic analogue of the generalized Hardy-Berndt sums related to elliptic Bernoulli polynomials.

2 Generalized Dedekind Eta function

In this section, we give relations between the Weierstrass $\wp$-function, Dedekind sums, Hardy-Berndt sums and generalized Dedekind eta functions. In [37], Tzeng and Miao defined the following relations related to the generalized Dedekind eta function:

$$\eta^2\left(2\tau\right)=\frac{1}{2}\eta_{01}\left(\tau,2\right)\eta^2\left(\tau\right),$$

$$\eta^2\left(\frac{\tau+1}{2}\right)=\eta_{10}\left(\tau,2\right)\eta^2\left(\tau\right),$$

$$\eta^2\left(\frac{\tau+1}{2}\right)=e^{\frac{\pi i}{12}}\eta_{11}\left(\tau,2\right)\eta^2\left(\tau\right),$$

$$\eta\left(3\tau\right)=\frac{1}{\sqrt{3}}\eta_{10}\left(\tau,3\right)\eta\left(\tau\right),$$

$$\eta\left(\frac{\tau}{3}\right)=\eta_{10}\left(\tau,3\right)\eta\left(\tau\right),$$

$$\eta\left(\frac{\tau+1}{3}\right)=e^{\frac{\pi i}{36}}\eta_{11}\left(\tau,3\right)\eta\left(\tau\right),$$

$$\eta\left(\frac{\tau+2}{3}\right)=e^{\frac{\pi i}{18}}\eta_{11}\left(\tau,3\right)\eta\left(\tau\right).$$

By the above equations and (2), we have

(5) $$\vartheta_3\left(z\right)=\frac{\eta^2\left(z\right)}{\eta_{10}\left(z\right)\eta\left(z+1\right)},\ \text{cf.[20]},$$

and

$$\log\vartheta_3\left(z\right)=\log\eta\left(z\right)-\log\eta_{10}\left(z\right)-\frac{\pi i}{12},\ \text{cf.[20]}.$$

By using (5), we obtain

(6) $$\log\eta_{10}(Az)=\log\eta(Az)-\log\vartheta_3\left(Az\right)-\frac{\pi i}{12},$$

where $A\in\Gamma(1)$.

The Weierstrass $\wp$-function is defined as follows:

Let $\Lambda_\tau=\mathbb{Z}+\tau\mathbb{Z}$, $\tau\in\mathbb{H}$ be a lattice and $z\in\mathbb{C}$

$$\wp(z;\Lambda_\tau)=\frac{1}{z^2}+\sum_{0\neq w\in\Lambda_\tau}\left(\frac{1}{(z-w)^2}-\frac{1}{w^2}\right)\ \text{cf. ([39], [22], [38])}.$$

Relation between the function $\wp$ and ϑ_3 is given by

(7) $$y(z)=\wp(\frac{1}{2})-\wp(\frac{z}{2})=\pi^2\vartheta_3^4\left(z\right)\ \text{cf. ([15], [24])}.$$

By using (6) and (7), we have the following theorem:

Theorem 3.

(8) $$\log\eta_{10}(Az)=\log\eta(Az)-\frac{\log y(Az)-2\log\pi}{4}-\frac{\pi i}{12}$$

By using(8), we obtain

$$\log \eta_{10}(A\frac{z}{2}) = \log \eta(A\frac{z}{2}) - \frac{\log y(A\frac{z}{2}) - 2\log \pi}{4} - \frac{\pi i}{12}$$

Substituting $z = \frac{\tau - d}{c}$ into Theorem 1, we have

(9) $$\log \eta(\frac{z}{2}) = \log \eta(\frac{a - \frac{1}{\tau}}{2c}) - \frac{\pi i(a + d - 6c)}{24c} + \pi i s(d, 2c) - \frac{1}{2}\log(\tau),$$

and

(10) $$\log \eta(2z) = \log \eta(\frac{2a - \frac{1}{2\tau}}{c}) - \frac{\pi i(2a + 2d - 3c)}{12c} + \pi i s(2d, c) - \frac{1}{2}\log(\tau).$$

By substituting (9) and (10) into (2), if $c + d$ is odd, then we obtain

(11) $$\log \vartheta_3(A\tau) = \log \vartheta_3(\tau) + \frac{\pi i}{4}(S(d, c) - 4) - \frac{1}{2}\log \tau,$$

where $S(d, c)$ denotes Hardy-Berndt sum.

By using (7), (8) and (11), we arrive at the following theorem:

Theorem 4. *If $a + d$ is odd, then we have*

$$\log \eta_{10}(A\tau) = \log \eta_{10}(\tau) - \pi i\frac{S(d, c) + 4s(d, c) - 3}{4} + \log \tau.$$

If $a + d$ is even, then we have

$$\log \eta_{10}(A\tau) = \log \eta_{10}(\tau) - \pi i\frac{s_5(d, c) + 2s(d, c)}{2} - \frac{3\pi i}{4} + \log \tau.$$

3 Elliptic analogue of generalized Hardy-Berndt sums

Recently, elliptic Bernoulli polynomials, elliptic analogue of the generalized Dedekind-Rademacher, Dedekind-Apostol and Hardy-Berndt sums have studied by many mathematicians (for detail see [10], [6], [7], [9], [13], [17], [18], [21], [22], [25], [26], [12], [5]) In this section, we introduce generalized Hardy-Berndt sums and elliptic analogue of the generalized Hardy-Berndt sums related to elliptic Bernoulli polynomials.

Hardy-Berndt sums in (4), are redefined as follows:

Let h and k be integers with $k > 0$, the Hardy-Berndt sums are defined as follows

$$
\begin{aligned}
(12)\qquad S(h,k) &= 4\sum_{j=1}^{k-1}((\frac{(h+k)j}{2k})), \\
s_1(h,k) &= \sum_{j=1}^{k}((\frac{j}{k}))\left\{2((\frac{hj}{k}))-4((\frac{hj}{2k}))\right\}, \\
s_2(h,k) &= -4\sum_{j=1}^{k-1}((\frac{hj}{2k})), \\
s_5(h,k) &= \sum_{j=1}^{k}((\frac{j}{k}))\left\{2((\frac{hj}{k}))-4((\frac{(h+k)j}{2k}))\right\},
\end{aligned}
$$

for detail see cf. ([8], [32], [27], [24]). By using (12) and Bernoulli functions, we arrive at the fallowing definition ([8], [27]):

Definition 2. *Let h and k be integers with $k > 0$, the Hardy sums are defined as follows:*

$$
\begin{aligned}
S(h,k:m) &= 4\sum_{j=1}^{k-1}\overline{B}_m(\frac{(h+k)j}{2k}), \\
s_1(h,k:m) &= \sum_{j=1}^{k-1}\overline{B}_1(\frac{j}{k})(2\overline{B}_m(\frac{hj}{k})-4\overline{B}_m(\frac{hj}{2k})) \\
&= 2s(h,k:m)-4\sum_{j=1}^{k-1}\overline{B}_1(\frac{j}{k})\overline{B}_m(\frac{hj}{2k}),
\end{aligned}
$$

$$s_2(h,k:m)=\sum_{j=1}^{k-1}(-1)^j\overline{B}_1(\frac{j}{k})\overline{B}_m(\frac{hj}{k}),$$
$$s_3(h,k:m)=-4\sum_{j=1}^{k-1}(-1)^j\overline{B}_m(\frac{hj}{k}),$$

$$s_4(h,k:m)=-4\sum_{j=1}^{k-1}\overline{B}_m(\frac{hj}{k}),$$
$$s_5(h,k:m)=\sum_{j=1}^{k-1}\overline{B}_1(\frac{j}{k})\left(2\overline{B}_m(\frac{hj}{k})-4\overline{B}_m(j\frac{h+k}{2k})\right)$$
$$=2s(h,k:m)-4\sum_{j=1}^{k-1}\overline{B}_1(\frac{j}{k})\overline{B}_m(\frac{(h+k)j}{2k}),$$

where $s(h,k:m)$ is generalized Dedekind sums, which is defined by

$$s(h,k:m)=\sum_{j \bmod k}\frac{j}{k}\overline{B}_m(\frac{hj}{k}) \text{ cf. ([1], [3], [2], [4], [32], [30], [25]).}$$

By using Definition 2 and (1), we define

$$S_{2,m,n}(a,b,c:x,y,z)=\sum_{k \bmod c}(-1)^k\overline{B}_m(a\frac{k+z}{c}-x)\overline{B}_n(b\frac{k+z}{c}-y). \tag{13}$$

By using (3) and (13), we arrive at the following Theorem:

Theorem 5. *Let a, a', b, b', c, c' be positive integers and x, x', y, y', z, z' be real numbers. Suppose that*

$$a'z'-c'x'\notin<a',c'>\mathbb{Z} \text{ and } b'z'-c'y'\notin<b',c'>\mathbb{Z},$$

where $< a, b >$ is the greatest common divisor of a and b.

$$S^{\tau}_{2,m,n}(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}; \overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}) = \sum_{\substack{j mod c \\ l mod c^{'}}} (-1)^{j+l} \overline{B}_m \left(a^{'} \frac{l+z^{'}}{c^{'}} - x^{'}, a\frac{j+z}{c} - x; \frac{a^{'}}{a}\tau\right) \times \overline{B}_n \left(b^{'} \frac{l+z^{'}}{c^{'}} - y^{'}, b\frac{j+z}{c} - y; \frac{b^{'}}{b}\tau\right).$$

Remark 1. *If $m \neq 1$, $n \neq 1$, or if $az - cx \notin < a, c > \mathbb{Z}$ and $bz - cy \notin < b, c > \mathbb{Z}$, then*

$$\lim_{\tau \to i\infty} S^{\tau}_{2,m,n}(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}; \overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}) = S_{2,m,n}(a, b, c; x, y, z),$$

Let a, b and c be positive integers, and x, y and z be real numbers.

$$S_{2,m,n}(a, b, c : x, y, z) = \sum_{k mod c} (-1)^k \overline{B}_m(a\frac{k+z}{c} - x)\overline{B}_n(b\frac{k+z}{c} - y).$$

If $m = a = 1$ and $x = y = z = 0$, then $S_{2,m,n}(a, b, c : x, y, z)$ reduces to $s_2(b, c : m)$.

By using Definition 2 and (1), we define

$$S_{5,m,n}(a, b, c : x, y, z) = 2 \sum_{k mod c} \overline{B}_m(a\frac{k+z}{c} - x)\overline{B}_n(b\frac{k+z}{c} - y) - 4 \sum_{k mod c} \overline{B}_m(a\frac{k+z}{c} - x)\overline{B}_n((b+c)\frac{k+z}{2c} - y),$$

or

(14) $S_{5,m,n}(a, b, c : x, y, z) = 2S_{m,n}(a, b, c : x, y, z) - 4Y_{5,m,n}(a, b, c : x, y, z)$,

where $S_{m,n}(a, b, c : x, y, z)$ denotes an analogue of generalized Dedekind-Rademacher sums and

$$Y_{5,m,n}(a, b, c : x, y, z) = \sum_{k mod c} \overline{B}_m(a\frac{k+z}{c} - x)\overline{B}_n((b+c)\frac{k+z}{2c} - y).$$

By using (3) and (14), we arrive at the following Theorem:

Theorem 6. *Let a, $a^{'}$, b, $b^{'}$, c, $c^{'}$ be positive integers and x, $x^{'}$, y, $y^{'}$, z, $z^{'}$ be real numbers. Suppose that*

$$a^{'}z^{'} - c^{'}x^{'} \notin < a^{'}, c^{'} > \mathbb{Z} \text{ and } b^{'}z^{'} - c^{'}y^{'} \notin < b^{'}, c^{'} > \mathbb{Z},$$

where $< a, b >$ is the greatest common divisor of a and b.

$$S_{5,m,n}^{\tau}(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c};\overrightarrow{x},\overrightarrow{y},\overrightarrow{z}) = 2 \sum_{\substack{j \bmod c \\ l \bmod c^{'}}} \overline{B}_m\left(a^{'}\frac{l+z^{'}}{c^{'}} - x^{'}, a\frac{j+z}{c} - x; \frac{a^{'}}{a}\tau\right) \times \overline{B}_n\left(b^{'}\frac{l+vz^{'}}{c^{'}} - y^{'}, b\frac{j+z}{c} - y; \frac{b^{'}}{b}\tau\right) - 4 \sum_{\substack{j \bmod c \\ l \bmod c^{'}}} \overline{B}_m\left(a^{'}\frac{l+z^{'}}{c^{'}} - x^{'}, a\frac{j+z}{c} - x; \frac{a^{'}}{a}\tau\right) \times \overline{B}_n\left(b^{'}\frac{l+z^{'}}{c^{'}} - y^{'}, (b+c)\frac{j+z}{2c} - y; \frac{b^{'}}{b}\tau\right).$$

Remark 2. *If $m \neq 1$, $n \neq 1$, or if $az - cx \notin < a, c > \mathbb{Z}$ and $bz - cy \notin < b, c > \mathbb{Z}$, then*

$$\lim_{\tau \to i\infty} S_{5,m,n}^{\tau}(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c};\overrightarrow{x},\overrightarrow{y},\overrightarrow{z}) = S_{5,m,n}(a, b, c; x, y, z).$$

Let a, b and c be positive integers, and x, y and z be real numbers. If $m = a = 1$ and $x = y = z = 0$, then $S_{5,m,n}(a, b, c : x, y, z)$ reduces to $s_5(b, c : n)$. Observe that elliptic analogue of the $s_1(h, k : n)$ is similar to that of $s_5(b, c : n)$.

Now, we define generalized Hardy-Berndt sum's $s_j(h,k:n)$, $j=1,3,4$ and $S(h,k:n)$ as follows:

$$S_{4,0,n}(0,b,c;0,y,z) = -4\sum_{kmodc}\overline{B}_n(b\frac{k+z}{2c}-y), \tag{15}$$

$$S_{1,m,n}(a,b,c:x,y,z) = 2S_{m,n}(a,b,c:x,y,z) - 4Y_{1,m,n}(a,b,c:x,y,z), \tag{16}$$

where $S_{m,n}(a,b,c:x,y,z)$ denotes an analogue of generalized Dedekind-Rademacher sums and

$$Y_{1,m,n}(a,b,c:x,y,z) = \sum_{kmodc}\overline{B}_m(a\frac{k+z}{c}-x)\overline{B}_n(b\frac{k+z}{2c}-y),$$

and

$$\begin{aligned} S_{3,0,n}(0,b,c;0,y,z) &= -4\sum_{kmodc}(-1)^k\overline{B}_n(b\frac{k+z}{c}-y), \\ S_{H,0,n}(0,b,c;0,y,z) &= -4\sum_{kmodc}\overline{B}_n((b+c)\frac{k+z}{2c}-y). \end{aligned} \tag{17}$$

Note that substituting $x=y=z=0$ in the above, then $S_{k,m,n}(a,b,c:x,y,z)$, $k=1,3,4$ and $S_{H,0,n}(0,b,c;0,y,z)$ reduce to $s_j(h,k:n)$, $j=1,3,4$ and $S(h,k:n)$, respectively.

By using (3) and (15), we construct elliptic analogue of $s_j(h,k:n)$, $j=1,3,4$ and $S(h,k:n)$ sums by the following theorem:

Theorem 7. *Let a, a', b, b', c, c' be positive integers and x, x', y, y', z, z' be real numbers. Suppose that*

$$a'z' - c'x' \notin <a',c'> \mathbb{Z} \text{ and } b'z' - c'y' \notin <b',c'> \mathbb{Z},$$

where $<a,b>$ *is the greatest common divisor of* a *and* b.

$$S^{\tau}_{H,0,n}(\overrightarrow{0},\overrightarrow{b},\overrightarrow{c};\overrightarrow{0},\overrightarrow{y},\overrightarrow{z})=\sum_{\substack{j mod c\\ l mod c^{'}}}\overline{B}_n\left(b^{'}\frac{l+z^{'}}{c^{'}}-y^{'},(b+c)\frac{j+z}{2c}-y;\frac{b^{'}}{b}\tau\right),$$

$$S^{\tau}_{1,m,n}(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c};\overrightarrow{x},\overrightarrow{y},\overrightarrow{z})=2\sum_{\substack{j mod c\\ l mod c^{'}}}\overline{B}_m\left(a^{'}\frac{l+z^{'}}{c^{'}}-x^{'},a\frac{j+z}{c}-x;\frac{a^{'}}{a}\tau\right)$$
$$\times\overline{B}_n\left(b^{'}\frac{l+z^{'}}{c^{'}}-y^{'},b\frac{j+z}{c}-y;\frac{b^{'}}{b}\tau\right)$$
$$-2\sum_{\substack{j mod c\\ l mod c^{'}}}\overline{B}_m\left(a^{'}\frac{l+z^{'}}{c^{'}}-x^{'},a\frac{j+z}{c}-x;\frac{a^{'}}{a}\tau\right)$$
$$\times\overline{B}_n\left(b^{'}\frac{l+z^{'}}{c^{'}}-y^{'},b\frac{j+z}{2c}-y;\frac{b^{'}}{b}\tau\right),$$

$$S^{\tau}_{3,0,n}(\overrightarrow{0},\overrightarrow{b},\overrightarrow{c};\overrightarrow{0},\overrightarrow{y},\overrightarrow{z})=\sum_{\substack{j mod c\\ l mod c^{'}}}(-1)^{j+l}\overline{B}_n\left(b^{'}\frac{l+z^{'}}{c^{'}}-y^{'},b\frac{j+z}{c}-y;\frac{b^{'}}{b}\tau\right),$$

$$S^{\tau}_{4,0,n}(\overrightarrow{0},\overrightarrow{b},\overrightarrow{c};\overrightarrow{0},\overrightarrow{y},\overrightarrow{z})=\sum_{\substack{j mod c\\ l mod c^{'}}}\overline{B}_n\left(b^{'}\frac{l+z^{'}}{c^{'}}-y^{'},b\frac{j+z}{2c}-y;\frac{b^{'}}{b}\tau\right).$$

Remark 3. *If* $m\neq 1$, $n\neq 1$, *or if* $az-cx\notin <a,c>\mathbb{Z}$ *and*

$bz-cy\notin <b,c>\mathbb{Z}$, *then*

$$\begin{aligned}
\lim_{\tau\to i\infty} S^{\tau}_{k,0,n}(\overrightarrow{0},\overrightarrow{b},\overrightarrow{c};\overrightarrow{0},\overrightarrow{y},\overrightarrow{z}) &= S_{k,0,n}(0,b,c;0,y,z),\ k=3,4,\\
\lim_{\tau\to i\infty} S^{\tau}_{H,0,n}(\overrightarrow{0},\overrightarrow{b},\overrightarrow{c};\overrightarrow{0},\overrightarrow{y},\overrightarrow{z}) &= S_{H,0,n}(0,b,c;0,y,z),\\
\lim_{\tau\to i\infty} S^{\tau}_{1,m,n}(\overrightarrow{a},\overrightarrow{b},\overrightarrow{c};\overrightarrow{x},\overrightarrow{y},\overrightarrow{z}) &= S_{1,m,n}(a,b,c;x,y,z).
\end{aligned}$$

Acknowledgement 1 *This paper was supported by the Scientific Research Project Administration of Akdeniz University.*

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