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Some Results on Subclasses of Janowski λ-Spirallike Functions of Complex Order

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Abstract

We give some results of Janowski λ-spirallike functions of complex order in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the unit disc $\mathbb{D} = \{z : |z| < 1\}$.

For a function $f(z)$ belonging to the class $\mathcal{A}$ we say that $f(z)$ is Janowski λ-spirallike functions of complex order in $\mathbb{D}$ if and only if

$$\mathrm{Re}\left\{1 + \frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0 \tag{2}$$

for some real $\lambda, |\lambda| < \frac{\pi}{2}, b \neq 0$, complex. We denote this class by $\mathcal{S}^{\lambda}(b)$. It was introduced and studied by Al-Oboudi and Haidan [1].

Let Ω be the family of functions $\omega(z)$ regular in the unit disc $\mathbb{D} = \{z : |z| < 1\}$ and satisfying the conditions $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in \mathbb{D}$.

For arbitrary fixed numbers $A, B, -1 \leq B < A \leq 1$, denote by $\mathcal{P}(A, B)$ the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{3}$$

regular in $\mathbb{D}$, and such that $p(z) \in \mathcal{P}(A, B)$ if and only if

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \tag{4}$$

for some functions $\omega(z) \in \Omega$ and every $z \in \mathbb{D}$. This class was introduced by W. Janowski [5].

Next we consider the following class of functions defined in $\mathbb{D}$. Let $\mathcal{S}^{\lambda}(A, B, b)$ denote the family of functions the equality (1) regular in $\mathbb{D}$, such that $f(z) \in \mathcal{S}^{\lambda}(A, B, b)$ if and only if

$$1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{z f'(z)}{f(z)} - 1 \right) = \frac{1 + A\omega(z)}{1 + B\omega(z)} = p(z), \tag{5}$$

where $b \neq 0,\ b$ is a complex number, for some functions $\omega(z) \in \Omega$ and all $z \in \mathbb{D}$, and $p(0) = 1,\ Rep(z) > 0$ in $\mathbb{D}$. The class $\mathcal{S}^{\lambda}(A, B, b)$ is called Janowski λ-spirallike functions of complex order.

We note that by giving special values to A, B, b and λ, then we obtain the following subclasses.

1. For $A = 1,\ B = -1;\ z\frac{f'(z)}{f(z)} \prec \frac{1+(-1+2be^{-i\lambda}\cos\lambda)z}{1-z}$
2. For $A = 1,\ B = -1,\ b = 1,\ \lambda = 0;\ z\frac{f'(z)}{f(z)} \prec \frac{1+z}{1-z}$

3. For $A = 1-2\beta,\ B = -1,\ 0 \le \beta < 1;\ z\frac{f'(z)}{f(z)} \prec \frac{1+(-1+2(1-\beta)be^{-i\lambda}\cos\lambda)z}{1-z}$

4. For $A = 1-2\beta,\ B = -1,\ b = 1,\ \lambda = 0;\ z\frac{f'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}$

5. For $A = 1,\ B = 0;\ z\frac{f'(z)}{f(z)} \prec 1 + be^{-i\lambda}\cos\lambda z$

6. For $A = 1,\ B = 0,\ b = 1,\ \lambda = 0;\ z\frac{f'(z)}{f(z)} \prec 1 + z$

7. For $A = \beta,\ B = 0,\ 0 \le \beta < 1;\ z\frac{f'(z)}{f(z)} \prec 1 + \beta be^{-i\lambda}\cos\lambda z$

8. For $A = \beta,\ B = 0,\ b = 1,\ \lambda = 0,\ 0 \le \beta < 1;\ z\frac{f'(z)}{f(z)} \prec 1 + \beta z$

9. For $A = 1,\ B = -1 + \frac{1}{M},\ M > \frac{1}{2};$

$$z\frac{f'(z)}{f(z)} \prec \frac{1 + \left(\left(-1+\frac{1}{M}\right) + \left(2-\frac{1}{M}\right)be^{-i\lambda}\cos\lambda\right)z}{1+\left(-1+\frac{1}{M}\right)z}$$

10. For $A = 1,\ B = -1 + \frac{1}{M},\ M > \frac{1}{2},\ b = 1,\ \lambda = 0;$

$$z\frac{f'(z)}{f(z)} \prec \frac{1 + \left(\left(-1+\frac{1}{M}\right) + \left(2-\frac{1}{M}\right)\right)z}{1+\left(-1+\frac{1}{M}\right)z}$$

11. For $A = \beta,\ B = -\beta,\ 0 \le \beta < 1;\ z\frac{f'(z)}{f(z)} \prec \frac{1+\left(-\beta+2\beta be^{-i\lambda}\cos\lambda\right)z}{1-\beta z}$

12. For $A = \beta,\ B = -\beta,\ b = 1,\ \lambda = 0,\ 0 \le \beta < 1;\ z\frac{f'(z)}{f(z)} \prec \frac{1+\beta z}{1-\beta z}$

2 Theorems

From the definition of the classes $\mathcal{P}(A,B)$ and $\mathcal{S}^{\lambda}(A,B,b)$ we easily obtain the following theorems.

Theorem 1. *$f(z) = z + a_2 z^2 + a_3 z^3 + ...$ belongs to $S^\lambda(A, B, b)$ if and only if*

$$e^{i\lambda}\left(z\frac{f'(z)}{f(z)} - 1\right) \prec \begin{cases} \frac{(A-B)b\cos\lambda z}{1+Bz}, & B \neq 0 \\ Ab\cos\lambda z, & B = 0 \end{cases}$$

Proof. We prove first the necessity of the condition.

Let $B \neq 0$ and

$$e^{i\lambda}\left(z\frac{f'(z)}{f(z)} - 1\right) \prec \frac{(A-B)b\cos\lambda z}{1+Bz}.$$

It follows that using subordination principle

$$e^{i\lambda}\left(z\frac{f'(z)}{f(z)} - 1\right) = \frac{(A-B)b\cos\lambda\omega(z)}{1+B\omega(z)},$$

and then

$$\frac{e^{i\lambda}}{b\cos\lambda}\left(z\frac{f'(z)}{f(z)} - 1\right) = \frac{(A-B)\omega(z)}{1+B\omega(z)}.$$

This equality can be written in the form

$$1 + \frac{e^{i\lambda}}{b\cos\lambda}\left(z\frac{f'(z)}{f(z)} - 1\right) = \frac{1+A\omega(z)}{1+B\omega(z)}.$$

This means that $f(z) \in \mathcal{S}^\lambda(A, B, b)$.

Let $B = 0$ and

$$e^{i\lambda}\left(z\frac{f'(z)}{f(z)} - 1\right) \prec Ab\cos\lambda z.$$

It follows that

$$e^{i\lambda}\left(z\frac{f'(z)}{f(z)} - 1\right) = Ab\cos\lambda\omega(z).$$

This equality can be written in the form

$$\frac{e^{i\lambda}}{b\cos\lambda}\left(z\frac{f'(z)}{f(z)} - 1\right) = A\omega(z)$$

and then

$$1+\frac{e^{i\lambda}}{b\cos\lambda}\left(z\frac{f'(z)}{f(z)}-1\right)=1+A\omega(z)=\frac{1+A\omega(z)}{1+B\omega(z)}.$$

This shows that $f(z)\in\mathcal{S}^{\lambda}(A,B,b)$.

The condition is also sufficient. Let $f(z)\in\mathcal{S}^{\lambda}(A,B,b)$ and $B\neq 0$. Then

$$1+\frac{e^{i\lambda}}{b\cos\lambda}\left(z\frac{f'(z)}{f(z)}-1\right)=p(z)$$

for some $p(z)\in\mathcal{P}(A,B)$. On the other hand the boundary function $p_0(z)$ of $\mathcal{P}(A,B)$ with respect to this equality has the form

$$p_0(z)=\frac{1+A\omega(z)}{1+B\omega(z)}.$$

Therefore we have the equality

$$1+\frac{e^{i\lambda}}{b\cos\lambda}\left(z\frac{f'(z)}{f(z)}-1\right)=\frac{1+A\omega(z)}{1+B\omega(z)}$$

for every boundary function. After simple calculations we deduce

$$e^{i\lambda}\left(z\frac{f'(z)}{f(z)}-1\right)=\frac{(A-B)b\cos\lambda\omega(z)}{1+B\omega(z)}.$$

If we apply the subordination principle [1] to this equality we obtain

$$e^{i\lambda}\left(z\frac{f'(z)}{f(z)}-1\right)\prec\frac{(A-B)b\cos\lambda z}{1+Bz}.$$

Let $f(z)\in\mathcal{S}^{\lambda}(A,B,b)$ and $B=0$. Then

$$1+\frac{e^{i\lambda}}{b\cos\lambda}\left(z\frac{f'(z)}{f(z)}-1\right)=p(z)$$

for some $p(z)\in\mathcal{P}(A,B)$ and so we obtain

$$e^{i\lambda}\left(z\frac{f'(z)}{f(z)}-1\right)\prec Ab\cos\lambda z.$$

The assertion is also proved.

Theorem 2. *If $f(z) \in \mathcal{S}^{\lambda}(A,B,b)$ then for all $z \in \mathbb{D}$ we have*

$$\left|1-\left(\frac{f(z)}{z}\right)^{\frac{B}{(A-B)e^{-i\lambda}b\cos\lambda}}\right| < 1.$$

This inequality is called Marx-Strohhacker inequality for the class $\mathcal{S}^{\lambda}(A,B,b)$, and if the special values to $b \neq 0$ are given obtain new Marx-Strohhacker type inequalities for the subclasses of starlike functions, which one mentioned in the special cases.

Proof. We define the function $\omega(z)$ by

$$\frac{f(z)}{z} = (1+B\omega(z))^{\frac{(A-B)e^{-i\lambda}b\cos\lambda}{B}} \tag{7}$$

where choose the determination of the power such that $(1+B\omega(z))^{\frac{(A-B)e^{-i\lambda}b\cos\lambda}{B}}$ has the value 1 at the origin. Then $\omega(z)$ is analytic in $\mathbb{D}$ and satisfies $\omega(0)=0$, and if we take logarithmic derivative we obtain

$$e^{i\lambda}z\frac{f'(z)}{f(z)} - e^{i\lambda} = \frac{(A-B)b\cos\lambda z\omega'(z)}{1+B\omega(z)}. \tag{8}$$

From the previous equality, using Theorem 1, it follows that $|\omega(z)|<1$ for all $z \in \mathbb{D}$. Indeed, assuming the contrary, there exists $z_1 \in \mathbb{D}$ with $|\omega(z_1)| = 1$ such that $|\omega(z)|$ attains its maximum value on the circle $|z| = |z_1| < 1$ at the point z_1.

Using Jack's lemma [4] in this equality we obtain

$$e^{i\lambda}z_1\frac{f'(z_1)}{f(z_1)} - e^{i\lambda} = \frac{(A-B)b\cos\lambda k\omega(z_1)}{1+B\omega(z_1)} = F(\omega(z_1)) \notin F(\mathbb{D})$$

because $|\omega(z_1)| = 1$ and $k \geq 1$. But this contradicts Theorem 1, and therefore we have $|\omega(z)| < 1$ for every $z \in \mathbb{D}$. Now using (6) we obtain

$$\left|1-\left(\frac{f(z)}{z}\right)^{\frac{B}{(A-B)e^{-i\lambda}b\cos\lambda}}\right| = |B\omega(z)| < |B|.$$

Therefore the theorem is proved.

Theorem 3. *If $f(z) = z + a_2z^2 + a_3z^3 + ...$ belongs to $\mathcal{S}^\lambda(A, B, b)$ then*

$$G(r, -A, -B, |b|) \leq |f(z)| \leq G(r, A, B, |b|), \tag{9}$$

where

$$G(r, A, B, |b|) = \begin{cases} \frac{r(1+Br)^{\frac{(A-B)\cos\lambda(|b|+Reb\cos\lambda)}{2B}}}{(1-Br)^{\frac{(A-B)\cos\lambda(|b|-Reb\cos\lambda)}{2B}}}, & B \neq 0, \\ re^{A|b|\cos\lambda r}, & B = 0. \end{cases}$$

Remark 1. *This bound is sharp, because the extremal function is*

$$f_*(z) = \begin{cases} z(1 + Bz)^{\frac{(A-B)be^{-i\lambda}\cos\lambda}{B}}, & B \neq 0, \\ ze^{Abe^{-i\lambda}}\cos\lambda z, & B = 0. \end{cases}$$

Proof. Let $f(z) \in \mathcal{S}^\lambda(A, B, b)$ and $B \neq 0$. The set of the values of $\left(z\frac{f'(z)}{f(z)}\right)$ is the closed disc with the center

$$C(r) = \left(\frac{1 - B^2r^2 - B(A - B)b\cos^2\lambda r^2}{1 - B^2r^2}, \frac{B(A - B)b\cos\lambda\sin\lambda r^2}{1 - B^2r^2}\right)$$

and the radius $\rho(r) = \frac{(A-B)|b|\cos\lambda r}{1-B^2r^2}$. Therefore we can write

$$\left|z\frac{f'(z)}{f(z)} - \frac{1 - B^2r^2 - B(A - B)b\cos^2\lambda r^2}{1 - B^2r^2}\right| \leq \frac{(A - B)|b|\cos\lambda r}{1 - B^2r^2}. \tag{12}$$

This inequality can be written in the form

$$M_1(r) \leq Re\left(z\frac{f\prime(z)}{f(z)}\right) \leq M_2(r), \tag{13}$$

where

$$M_1(r) = \frac{1 - (A - B)|b|\cos\lambda r - (B^2 + B(A - B)Reb\cos^2\lambda)r^2}{1 - B^2r^2},$$

$$M_2(r) = \frac{1 + (A - B)|b|\cos\lambda r - (B^2 + B(A - B)Reb\cos^2\lambda)r^2}{1 - B^2r^2}.$$

On the other hand

$$Re\left(z\frac{f'(z)}{f(z)}\right) = r\frac{\partial}{\partial r}\log|f(z)|. \tag{14}$$

By considering (10) and (11) we can write $M_1(r) \le r\frac{\partial}{\partial r}log|f(z)| \le M_2(r)$ then we obtain desired result by integration.

If we take $B = 0$ in the inequality (10) then the proof of Theorem 3 is complete.

For example if we take $A = 1, B = -1, \lambda = 0, b = 1$; we obtain

$$\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2}.$$

This is the well known which is the distortion theorem of starlike functions [3].

Corollary 1. *The radius of starlikeness of the class $\mathcal{S}^\lambda(A, B, b)$ is*

$$r_s = \frac{(A-B)|b|\cos\lambda - \sqrt{(A-B)^2|b|^2\cos^2\lambda + 4B^2 + 4B(A-B)Reb\cos^2\lambda}}{2\left[-B^2 - B(A-B)Reb\cos^2\lambda\right]}.$$

This radius is sharp, because the extremal function is

$$f_*(z) = z(1+Bz)^{\frac{(A-B)be^{-i\lambda}\cos\lambda}{B}}.$$

Proof. From (10) we have

(15)

$$Re\left(z\frac{f'(z)}{f(z)}\right) \ge \frac{1-(A-B)|b|\cos\lambda r - (B^2 + B(A-B)Reb\cos^2\lambda)r^2}{1-B^2r^2}.$$

For $r < r_s$ the right hand side of the preceding inequality is positive, which implies

$$r_s = \frac{(A-B)|b|\cos\lambda - \sqrt{(A-B)^2|b|^2\cos^2\lambda + 4B^2 + 4B(A-B)Reb\cos^2\lambda}}{2\left[-B^2 - B(A-B)Reb\cos^2\lambda\right]}.$$

We note also that the inequality (12) becomes an equality for the function

$$f_*(z) = z(1+Bz)^{\frac{(A-B)be^{-i\lambda}\cos\lambda}{B}}.$$

It follows that

$$r_s = \frac{(A-B)|b|\cos\lambda - \sqrt{(A-B)^2|b|^2\cos^2\lambda + 4B^2 + 4B(A-B)Reb\cos^2\lambda}}{2\left[-B^2 - B(A-B)Reb\cos^2\lambda\right]},$$

and the proof is complete. For $A = 1, B = -1, b = 1, \lambda = 0$; we obtain $r_s = 1$.

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