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Certain inequalities concerning some complex and positive functionals

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Abstract

In this paper we study an inequality for the complex and positive functionals. Some applications for the Carlson′s inequality and for complex matrices on given.

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1 Introduction

Let X be a complex algebra and $F : X \times X \to \mathbb{C}$ a complex functional with the following properties

i) $F(\alpha x_1 + \beta x_2, y) = \alpha F(x_1, y) + \beta F(x_2, y)$for all $x_1, x_2, y \in X$ and $\alpha, \beta \in \mathbb{C}$

ii) $F(x,y) = \overline{F(y,x)}$ for all $x, y \in X$

iii) $F(x,x) \geq 0$ for all $x \in X$.

Appling the Cauchy - Schwartz - Buniakowski inequality we have

$$|F(yx,z)|^2 \leq F(yx,yx) \cdot F(z,z) \tag{1}$$

for all $x, y, z \in X$.

Let $y_0 = \lambda w_1 + \frac{1}{\lambda} w_2$ be an element of X where $w_1, w_2 \in X$, $\lambda \in \mathbb{C}$, $Re(\lambda) \neq 0$, $Im(\lambda) \neq 0$, $|y_0| \neq 0$.

We have

$$\begin{aligned} F(y_0 x, y_0 x) &= F\left(\lambda w_1 x + \frac{1}{\lambda} w_2 x, \lambda w_1 x + \frac{1}{\lambda} w_2 x\right) = \\ &= |\lambda|^2 F(w_1 x, w_1 x) + \frac{1}{|\lambda|^2} F(w_2 x, w_2 x) + 2Re\left(\frac{\lambda}{\overline{\lambda}} F(w_1 x, w_2 x)\right). \end{aligned} \tag{2}$$

We can formulate the next lemma

Lemma 1. *If* $|F(w_1 x, w_1 x)| \neq 0$, $|F(w_2 x, w_2 x)| \neq 0$ *and* $Re\left(F(w_1 x, w_2 x)\right) \neq 0$, *there exist a complex number* $\lambda = p + qi$, *such that*

$$|\lambda|^2 = \sqrt{\frac{F(w_2 x, w_2 x)}{F(w_1 x, w_1 x)}} \tag{3}$$

$$Re\left(\frac{\lambda}{\overline{\lambda}} F(w_1 x, w_2 x)\right) = 0 \tag{4}$$

$$p^2 \geq q^2 \tag{5}$$

Proof. We denote $F(w_1x, w_2x) = a + bi$, $a \neq 0$ and from (4) we obtain

$$a\left(\frac{p}{q}\right)^2 - 2b\frac{p}{q} - a = 0$$

with $b^2 + a^2 > 0$ and $x_1x_2 = 1$ (x_1, x_2 roots). Hence $|x_1| \geq 1$ or $|x_2| \geq 1$.

Then exists $p, q \in \mathbb{R}$, $p \neq 0$, $q \neq 0$ such that $\left|\frac{p}{q}\right| \geq 1$ or $|p| \geq |q|$ satisfying (3), (4), (5).

With this λ, from (1) and (2) we obtain

$$|F(y_0x, z)|^2 \leq 2\sqrt{F(w_1x, w_1x) \cdot F(w_2x, w_2x)} \cdot F(z, z)$$

so

$$|F(y_0x, z)|^4 \leq 4F^2(z, z) \cdot F(w_1x, w_1x) \cdot F(w_2x, w_2x). \tag{6}$$

We name (6) the Carlson - type inequality for complex and positive functionals, because of (6) we obtain for example the classical Carlson integral inequality.

2 Applications

1. Let X be the complex algebra of the complex and integrable functions defined on $[a, \infty)$, $a > 0$. We consider

$$F(f, g) = \int_a^\infty f(t)\overline{g(t)}dt$$

where $f, g \in X$.

F verify the conditions i), ii), iii) of introduction, evidently.

Now we consider $x(t)$, $y_0(t)$, $z(t) \in X$, non-nulls, and $w_1, w_2 : [a,\infty) \to (0,\infty)$ two continuously differentiable functions such that

$$w_2'(t)w_1(t) - w_2(t)w_1'(t) \geq m > 0.$$

It is clear that

$$F(w_1x, w_1x) = \int_a^\infty w_1^2(t)|x(t)|^2 dt \geq 0,$$
$$F(w_2x, w_2x) = \int_a^\infty w_2^2(t)|x(t)|^2 dt \geq 0,$$
$$F(w_1x, w_2x) = \int_a^\infty w_1(t)w_2(t)|x(t)|^2 dt \geq 0,$$

and hence, using (4) we have $p^2 = q^2$.

Of (6) we obtain

$$\left|\int_a^\infty y_0(t)x(t)\overline{z(t)}dt\right|^4 \leq 4\left(\int_a^\infty z(t)\overline{z(t)}dt\right)^2 \cdot \int_a^\infty w_1^2(t)|x(t)|^2 dt \cdot \int_a^\infty w_2^2(t)|x(t)|^2 dt. \tag{7}$$

Since $|y_0(t)| \neq 0$ we choose $z(t) = \dfrac{1}{\overline{y_0(t)}}$ in (7) and we get

$$\left|\int_a^\infty x(t)dt\right|^4 \leq 4\left(\int_a^\infty \frac{dt}{|y_0(t)|^2}\right)^2 \cdot \int_a^\infty w_1^2(t)|x(t)|^2 dt \cdot \int_a^\infty w_2^2(t)|x(t)|^2 dt. \tag{8}$$

Clearly

$$\int \frac{dt}{|y_0(t)|^2} = \int \frac{dt}{\left|\lambda w_1(t) + \frac{1}{\lambda} w_2(t)\right|^2} = \int \frac{\frac{|\lambda|^2}{w_1^2(t)}}{\left|\lambda^2 + \frac{w_2(t)}{w_1(t)}\right|^2} dt.$$

Since $\lambda = p + qi$, $p^2 = q^2$, we have

$$\left|\lambda^2 + \frac{w_2(t)}{w_1(t)}\right|^2 = |\lambda|^4 + \frac{w_2^2(t)}{w_1^2(t)}.$$

Hence

$$\int \frac{dt}{|y_0(t)|^2} = \int \frac{\frac{1}{|\lambda|^2 w_1^2(t)}}{1 + \left(\frac{w_2(t)}{|\lambda|^2 w_1(t)}\right)^2} dt \leq$$

(9)

$$\leq \frac{1}{m} \int \frac{\left(\frac{w_2(t)}{|\lambda|^2 w_1(t)}\right)'}{1 + \left(\frac{w_2(t)}{|\lambda|^2 w_1(t)}\right)^2} dt = \frac{1}{m} \text{ arctg } \frac{w_2(t)}{|\lambda|^2 w_1(t)}$$

and we have the following result

Theorem 1. *Let $x(t) : [a, \infty) \to \mathbb{C}$, $a > 0$, an integrable function and $w_1(t)$, $w_2(t) : [a, \infty) \to (0, \infty)$ two continuously differentiable functions with $w_2'(t)w_1(t) - w_2(t)w_1'(t) \geq m > 0$, $\lim\limits_{t\to\infty} \frac{w_2(t)}{w_1(t)} = \infty$.*

Then

(10)

$$\left|\int_a^\infty x(t)dx\right|^4 \leq 4\left(\frac{\pi}{2m} - \frac{1}{m} \ arctg \ \frac{w_2(a)}{c \cdot w_1(a)}\right)^2 \cdot$$
$$\cdot \int_a^\infty w_1^2(t)|x(t)|^2 dt \cdot \int_a^\infty w_2^2(t)|x(t)|^2 dt$$

where

$$c = c(w_1, w_2) = \sqrt{\frac{\int_a^\infty w_2^2(t)|x(t)|^2 dt}{\int_a^\infty w_1^2(t)|x(t)|^2 dt}}$$

and $\int\limits_a^\infty w_1^2(t)|x(t)|^2dt > 0.$

Proof. Of (8) and (9) we obtain

$$\left|\int\limits_a^\infty x(t)dt\right|^4 \le 4\left(\frac{\pi}{2m} - \frac{1}{m}\ \text{arctg}\ \frac{w_2(a)}{|\lambda|^2 w_1(a)}\right)^2 \cdot$$
$$\cdot \int\limits_a^\infty w_1^2(t)|x(t)|^2dt \cdot \int\limits_a^\infty w_2^2(t)|x(t)|^2dt$$

where

$$|\lambda|^2 = \sqrt{\frac{\int\limits_a^\infty w_2^2(t)|x(t)|^2dt}{\int\limits_a^\infty w_1^2(t)|x(t)|^2dt}}$$

in conformity with (3).

Remark 1. *When* $w_1(t) = 1$, $w_2(t) = t$ *then the inequality (10) reduces to*

$$\left|\int\limits_a^\infty x(t)dt\right|^4 \le 4\left(\frac{\pi}{2} - \ \text{arctg}\ \frac{a}{c(1,t)}\right)^2 \cdot \int\limits_a^\infty |x(t)|^2dt \cdot \int\limits_a^\infty t^2|x(t)|^2dt. \tag{11}$$

When $a \to 0$, inequality (11) reduces to the well known Carlson′s integral inequality

$$\left|\int\limits_0^\infty x(t)dt\right|^4 \le \pi^2 \int\limits_0^\infty |x(t)|^2dt \cdot \int\limits_0^\infty t^2|x(t)|^2dt \tag{12}$$

(see [7]).

Hence (10) and (11) are an improvement of (12).

2. We consider now the complex algebra of square matrices with complex elements $X = \mathcal{M}_n(\mathbb{C})$. If A is a $n \times n$ matrix, we write tr A to denote the trace of A.

If

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} , \quad a_{ij} \in \mathbb{C}$$

we denote

$$A^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} & \dots & \overline{a_{n1}} \\ \overline{a_{12}} & \overline{a_{22}} & \dots & \overline{a_{n2}} \\ \dots & \dots & \dots & \dots \\ \overline{a_{1n}} & \overline{a_{2n}} & \dots & \overline{a_{nn}} \end{pmatrix} .$$

Let F be a complex functional defined by

$$F(x, y) = \mathrm{tr}(y^* x) \ , \quad F : X \times X \to \mathbb{C}$$

which verify i), ii), iii), evidently.

Using (6) we get

$$|\mathrm{tr}(z^* y_0 x)|^4 \le 4\mathrm{tr}^2(z^* z) \cdot \mathrm{tr}(x^* w_1^* w_1 x) \cdot \mathrm{tr}(x^* w_2^* w_2 x) \tag{13}$$

where $x, z, w_1, w_2 \in X$ and $y_0 = \lambda w_1 + \dfrac{1}{\lambda} w_2$, $y_0 \in X$, with λ of (3), (4), (5).

If $y_0^* y_0 = I_n$ then choosing in (13) $z = y_0$ we obtain

$$|\mathrm{tr} x|^4 \le 4\mathrm{tr}^2(y_0^* y_0) \cdot \mathrm{tr}(x^* w_1^* w_1 x) \cdot \mathrm{tr}(x^* w_2^* w_2 x) \tag{14}$$

and we have the next

Theorem 2. *Let x, w_1, w_2 be some matrices of $\mathcal{M}_n(\mathbb{C})$. If*

$$\left(\lambda w_1 + \frac{1}{\lambda} w_2\right)^* \left(\lambda w_1 + \frac{1}{\lambda} w_2\right) = I_n$$

with λ complex number which verify

$$|\lambda|^2 = \sqrt{\frac{tr(x^* w_2^* w_2 x)}{tr(x^* w_1^* w_1 x)}} \tag{15}$$

$$Re\left(\frac{\lambda}{\overline{\lambda}} \cdot tr(x^* w_2^* w_1 x)\right) = 0 \tag{16}$$

$$(Re(\lambda))^2 \geq (Im(\lambda))^2, \tag{17}$$

then we have the following inequality of Carlson′s type

$$|tr\ x|^4 \leq 4n^2 \cdot tr(x^* w_1^* w_1 x) \cdot tr(x^* w_2^* w_2 x). \tag{18}$$

Proof. Using the inequality (14) and (3), (4), (5) we get (15), evidently.

Remark 1. *For $w_1 = w_2 = \dfrac{1}{2p} w$, where $p = Re(\lambda)$ and $w^* w = I_n$, we obtain*

$$\left(\lambda w_1 + \frac{1}{\lambda} w_2\right)^* \left(\lambda w_1 + \frac{1}{\lambda} w_2\right) = \frac{1}{4p^2} \frac{(\lambda^2+1)(\overline{\lambda}^2+1)}{\lambda\overline{\lambda}} w^* w =$$

$$= \frac{1}{4p^2} \frac{(\lambda^2+1)(\overline{\lambda}^2+1)}{\lambda\overline{\lambda}} I_n.$$

Since (15), (16), (17) we have $|\lambda|^2 = 1$ and $(\mathrm{Re}(\lambda))^2 = (\mathrm{In}(\lambda))^2$. Hence $|\lambda|^2 = 2p^2 = 1$ and

$$\left(\lambda w_1 + \frac{1}{\lambda} w_2\right)^* \left(\lambda w_1 + \frac{1}{\lambda} w_2\right) = I_n.$$

Therefore from (18) it follows that

$$|\mathrm{tr}x|^4 \leq \frac{n^2}{4p^4}\mathrm{tr}^2(x^*x).$$

This implies

$$|\mathrm{tr}x|^2 \leq n \cdot \mathrm{tr}(x^*x). \tag{19}$$

Remark 2. *We observe the fact that from (19) we get the well known inequality*

$$|a_1 + a_2 + ... + a_n|^2 \leq n(|a_1|^2 + |a_2|^2 + ... + |a_n|^2)$$

for $a_i \in \mathbb{C}$, $i = \overline{1, 2}$.

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