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# Convolutions for certain analytic functions

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**Abstract**

Applying the coefficient inequalities of functions $f(z)$ belonging to the subclasses $\mathcal{SD}(\alpha,\beta)$ and $\mathcal{KD}(\alpha,\beta)$ of certain analytic functions in the open unit disk $\mathbb{U}$, two subclasses $\mathcal{SD}^*(\alpha,\beta)$ and $\mathcal{KD}^*(\alpha,\beta)$ are introduced. In this present paper, some interesting convolution properties of functions $f(z)$ in the classes $\mathcal{SD}^*(\alpha,\beta)$ and $\mathcal{KD}^*(\alpha,\beta)$ are discussed by using Shwarz inequality.

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## 1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be a member of the class $\mathcal{SD}(\alpha, \beta)$ if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha\left|\frac{zf'(z)}{f(z)} - 1\right| + \beta \qquad (z \in \mathbb{U})$$

for some $\alpha$ $(\alpha \geqq 0)$ and $\beta$ $(0 \leqq \beta < 1)$. Also $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{KD}(\alpha, \beta)$ if it satisfies $zf'(z) \in \mathcal{SD}(\alpha, \beta)$, that is,

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha\left|\frac{zf''(z)}{f'(z)}\right| + \beta \qquad (z \in \mathbb{U})$$

for some $\alpha$ $(\alpha \geqq 0)$ and $\beta$ $(0 \leqq \beta < 1)$. The classes $\mathcal{SD}(\alpha, \beta)$ and $\mathcal{KD}(\alpha, \beta)$ were defined by Shams, Kulkarni and Jahangiri [3]. We try to derive some properties of functions $f(z)$ belonging to the classes $\mathcal{SD}(\alpha, \beta)$ and $\mathcal{KD}(\alpha, \beta)$.

**Remark 1.** *For $f(z) \in \mathcal{SD}(\alpha, \beta)$, we write $w(z) = zf'(z)/f(z) = u + iv$. If $\alpha > 1$, then $w$ lies in the domain which is the part of the complex plane which contains $w = 1$ and is bounded by the elliptic domain such that*

$$\left(u - \frac{\alpha^2 - \beta}{\alpha^2 - 1}\right)^2 + \frac{\alpha^2}{\alpha^2 - 1}v^2 < \frac{\alpha^2(\beta - 1)^2}{(\alpha^2 - 1)^2}.$$

*If $\alpha = 1$, then $w$ lies in the domain which is the part of the complex plane which contains $w = 1$ and is bounded by the parabolic domain such that*

$$u > \frac{v^2}{2(1 - \beta)} + \frac{1 + \beta}{2}.$$

*If $0 \leqq \alpha < 1$, then $w$ lies in the domain which is the part of the complex plane which contains $w = 1$ and is bounded by the hyperbolic domain such that*

$$\left(u - \frac{\alpha^2 - \beta}{1 - \alpha^2}\right)^2 - \frac{\alpha^2}{1 - \alpha^2}v^2 > \frac{\alpha^2(\beta - 1)^2}{(1 - \alpha^2)^2}.$$

We recall here the following lemmas due to Shams, Kulkarni and Jahangiri [3], which provide the sufficient conditions for a function $f(z) \in \mathcal{A}$ to belong to the classes $\mathcal{SD}(\alpha, \beta)$ and $\mathcal{KD}(\alpha, \beta)$, respectively.

**Lemma 1.** *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\sum_{n=2}^{\infty} \{(1+\alpha)(n-1)+(1-\beta)\}|a_n| \leqq 1-\beta \tag{1}$$

*for some* $\alpha$ $(\alpha \geqq 0)$ *and* $\beta$ $(0 \leqq \beta < 1)$*, then* $f(z) \in \mathcal{SD}(\alpha, \beta)$*.*

**Lemma 2.** *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\sum_{n=2}^{\infty} n\{(1+\alpha)(n-1)+(1-\beta)\}|a_n| \leqq 1-\beta \tag{2}$$

*for some* $\alpha$ $(\alpha \geqq 0)$ *and* $\beta$ $(0 \leqq \beta < 1)$*, then* $f(z) \in \mathcal{KD}(\alpha, \beta)$*.*

By Lemma 1, the class $\mathcal{SD}^*(\alpha, \beta)$ is considered as the subclass of $\mathcal{SD}(\alpha, \beta)$ consisting of $f(z)$ satisfying the inequality (1) for some $\alpha$ $(\alpha \geqq 0)$ and $\beta$ $(0 \leqq \beta < 1)$. From Lemma 2, The class $\mathcal{KD}^*(\alpha, \beta)$ is also considered as the subclass of $\mathcal{KD}(\alpha, \beta)$ consisting of $f(z)$ satisfying the inequality (2) for some $\alpha$ $(\alpha \geqq 0)$ and $\beta$ $(0 \leqq \beta < 1)$.

## 2 Convolution properties of the classes $\mathcal{SD}^*(\alpha, \beta)$ and $\mathcal{KD}^*(\alpha, \beta)$

For functions $f_j(z) \in \mathcal{A}$ $(j = 1, 2, \cdots, m)$ given by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \qquad (z \in \mathbb{U}),$$

the Hadamard product (or convolution) of $f_1(z)$, $f_2(z), \cdots, f_m(z)$ is defined by

$$G_m(z) = (f_1 * f_2 * \cdots * f_m)(z) = z + \sum_{n=2}^{\infty} \left( \prod_{j=1}^{m} a_{n,j} \right) z^n.$$

The convolution was studied by Owa and Srivastava [2]. Lately, it was studied by Nishiwaki and Owa [1]. In this present paper, we discuss some convolutions for $f_j(z)$ belonging to $\mathcal{SD}^*(\alpha, \beta)$ and $\mathcal{KD}^*(\alpha, \beta)$, respectively. Our first result is

**Theorem 1.** *If* $f_j(z) \in \mathcal{SD}^*(\alpha, \beta_j)$ *for each* $j = 1, 2, \cdots, m$, *then* $G_m(z) \in \mathcal{SD}^*(\alpha, \beta^*)$ *with*

$$\beta^* = 1 - \frac{(1+\alpha) \prod_{j=1}^{m} (1-\beta_j)}{\prod_{j=1}^{m} \{(1+\alpha) + (1-\beta_j)\} - \prod_{j=1}^{m} (1-\beta_j)}.$$

**Proof.** We consider $G_2(z) \in \mathcal{SD}^*(\alpha, \beta^*)$ for $f_1(z)$ and $f_2(z)$. Letting $f(z) \in \mathcal{SD}^*(\alpha, \beta_j)$,

$$\sum_{n=2}^{\infty} \left\{ \frac{(1+\alpha)(n-1) + (1-\beta_j)}{1-\beta_j} \right\} |a_{n,j}| \leqq 1 \qquad (j = 1, 2).$$

Applying the Shwarz inequality, we have the following inequality

$$\sum_{n=2}^{\infty} \sqrt{\left\{ \frac{(1+\alpha)(n-1)+(1-\beta_1)}{1-\beta_1} \right\} \left\{ \frac{(1+\alpha)(n-1)+(1-\beta_2)}{1-\beta_2} \right\}} \sqrt{|a_{n,1}||a_{n,2}|} \leqq 1.$$

Then we will determine the largest $\beta^*$ such that

$$\sum_{n=2}^{\infty} \frac{(1+\alpha)(n-1) + (1-\beta^*)}{1-\beta^*} |a_{n,1}||a_{n,2}| \leqq 1,$$

that is,

$$\sum_{n=2}^{\infty} \frac{(1+\alpha)(n-1)+(1-\beta^*)}{1-\beta^*}|a_{n,1}||a_{n,2}|$$

$$\leqq \sum_{n=2}^{\infty} \sqrt{\left\{\frac{(1+\alpha)(n-1)+(1-\beta_1)}{1-\beta_1}\right\}\left\{\frac{(1+\alpha)(n-1)+(1-\beta_2)}{1-\beta_2}\right\}}\sqrt{|a_{n,1}||a_{n,2}|}.$$

Therefore, we need to find the largest $\beta^*$ such that

$$\frac{(1+\alpha)(n-1)+(1-\beta^*)}{1-\beta^*}\sqrt{|a_{n,1}||a_{n,2}|}$$

$$\leqq \sqrt{\left\{\frac{(1+\alpha)(n-1)+(1-\beta_1)}{1-\beta_1}\right\}\left\{\frac{(1+\alpha)(n-1)+(1-\beta_2)}{1-\beta_2}\right\}}$$

for all $n \geqq 2$. Thus we get

$$\frac{(1+\alpha)(n-1)+(1-\beta^*)}{1-\beta^*} \leqq \left\{\frac{(1+\alpha)(n-1)+(1-\beta_1)}{1-\beta_1}\right\}\left\{\frac{(1+\alpha)(n-1)+(1-\beta_2)}{1-\beta_2}\right\}$$

which implies

$$\beta^* \leqq 1-\frac{(1+\alpha)(1-\beta_1)(1-\beta_2)(n-1)}{\{(1+\alpha)(n-1)+(1-\beta_1)\}\{(1+\alpha)(n-1)+(1-\beta_2)\}-(1-\beta_1)(1-\beta_2)}.$$

The right hand side of the above inequality is a increasing function for all $n \geqq 2$. This means

$$\beta^* = \min_{n\geqq 2}\left\{1-\frac{(1+\alpha)(1-\beta_1)(1-\beta_2)(n-1)}{\{(1+\alpha)(n-1)+(1-\beta_1)\}\{(1+\alpha)(n-1)+(1-\beta_2)\}-(1-\beta_1)(1-\beta_2)}\right\}$$

$$(3)= \quad 1-\frac{(1+\alpha)(1-\beta_1)(1-\beta_2)}{\{(1+\alpha)+(1-\beta_1)\}\{(1+\alpha)+(1-\beta_2)\}-(1-\beta_1)(1-\beta_2)},$$

so that $G_2(z) \in \mathcal{SD}^*(\alpha, \beta^*)$. Therefore, the theorem is true for $m=2$. Let us suppose that $G_{m-1}(z) \in \mathcal{SD}^*(\alpha, \beta_0)$ and $f_m(z) \in \mathcal{SD}^*(\alpha, \beta_m)$, where

$$\beta_0 = 1-\frac{(1+\alpha)\prod_{j=1}^{m-1}(1-\beta_j)}{\prod_{j=1}^{m-1}\{(1+\alpha)+(1-\beta_j)\}-\prod_{j=1}^{m-1}(1-\beta_j)}.$$

Then replacing $\beta_1$ by $\beta_0$ and $\beta_2$ by $\beta_m$ in the inequality (3), we see

$$\beta^* = 1 - \frac{(1+\alpha)(1-\beta_0)(1-\beta_m)}{\{(1+\alpha)+(1-\beta_0)\}\{(1+\alpha)+(1-\beta_m)\} - (1-\beta_0)(1-\beta_m)}$$

$$= 1 - \frac{(1+\alpha)\prod_{j=1}^{m}(1-\beta_j)}{\prod_{j=1}^{m}\{(1+\alpha)+(1-\beta_j)\} - \prod_{j=1}^{m}(1-\beta_j)}.$$

For the integer $m$, the theorem is also true. Using the mathematical induction, we complete the proof of the theorem.

From Theorem 1, we get

**Corollary 1.** *If* $f_j(z) \in \mathcal{KD}^*(\alpha, \beta_j)$ *for each* $j = 1, 2, \cdots, m$, *then* $G_m(z) \in \mathcal{KD}^*(\alpha, \beta^*)$ *with*

$$\beta^* = 1 - \frac{(1+\alpha)\prod_{j=1}^{m}(1-\beta_j)}{2^{m-1}\prod_{j=1}^{m}\{(1+\alpha)+(1-\beta_j)\} - \prod_{j=1}^{m}(1-\beta_j)}.$$

**Proof.** Using the same way as the proof in Theorem 1, we obtain $G_2(z) \in \mathcal{KD}^*(\alpha, \beta^*)$ with

$$\beta^* = 1 - \frac{(1+\alpha)(1-\beta_1)(1-\beta_2)}{2\{(1+\alpha)+(1-\beta_1)\}\{(1+\alpha)+(1-\beta_2)\} - (1-\beta_1)(1-\beta_2)}. \tag{4}$$

Let us suppose that $G_{m-1}(z) \in \mathcal{SD}^*(\alpha, \beta_0)$ and $f_m(z) \in \mathcal{SD}^*(\alpha, \beta_m)$, where

$$\beta_0 = 1 - \frac{(1+\alpha)\prod_{j=1}^{m-1}(1-\beta_j)}{2^{m-2}\prod_{j=1}^{m-1}\{(1+\alpha)+(1-\beta_j)\} - \prod_{j=1}^{m-1}(1-\beta_j)}.$$

Then replacing $\beta_1$ by $\beta_0$ and $\beta_2$ by $\beta_m$ in the inequality (4), we see

$$\beta^* = 1 - \frac{(1+\alpha)(1-\beta_0)(1-\beta_m)}{2\{(1+\alpha)+(1-\beta_0)\}\{(1+\alpha)+(1-\beta_m)\} - (1-\beta_0)(1-\beta_m)}$$

$$= 1 - \frac{(1+\alpha)\prod_{j=1}^{m}(1-\beta_j)}{2^{m-1}\prod_{j=1}^{m}\{(1+\alpha)+(1-\beta_j)\} - \prod_{j=1}^{m}(1-\beta_j)}.$$

The corollary is true for the integer $m$. Using the mathematical induction, this complete the proof of the corollary.

Considering $\mathcal{SD}^*(\alpha_j, \beta)$ instead of $\mathcal{SD}^*(\alpha, \beta_j)$ in Theorem 1, we derive

**Theorem 2.** *If $f_j(z) \in \mathcal{SD}^*(\alpha_j, \beta)$ for each $j = 1, 2, \cdots, m$, then $G_m(z) \in \mathcal{SD}^*(\alpha^*, \beta)$ with*

$$\alpha^* = \frac{\prod_{j=1}^{m}\{(1+\alpha_j)+(1-\beta)\} - (1-\beta)^m}{(1-\beta)^{m-1}} - 1.$$

**Proof.** By means of Theorem 1, we easily see that $G_2(z) \in \mathcal{SD}^*(\alpha^*, \beta)$ with

$$\alpha^* = \frac{\{(1+\alpha_1)+(1-\beta)\}\{(1+\alpha_2)+(1-\beta)\} - (1-\beta)^2}{1-\beta} - 1.$$

This gives us that $G_m(z) \in \mathcal{SD}^*(\alpha^*, \beta)$ with

$$\alpha^* = \frac{\prod_{j=1}^{m}\{(1+\alpha_j)+(1-\beta)\} - (1-\beta)^m}{(1-\beta)^{m-1}} - 1$$

from the mathematical induction. We prove the theorem.

**Corollary 2.** *If $f_j(z) \in \mathcal{KD}^*(\alpha_j, \beta)$ for each $j = 1, 2, \cdots, m$, then $G_m(z) \in \mathcal{KD}^*(\alpha^*, \beta)$ with*

$$\alpha^* = \frac{2^{m-1}\prod_{j=1}^{m}\{(1+\alpha_j)+(1-\beta)\} - (1-\beta)^m}{(1-\beta)^{m-1}} - 1.$$

**Proof.** By means of Theorem 1, we easily know that $G_2(z) \in \mathcal{KD}^*(\alpha^*, \beta)$ with

$$\alpha^* = \frac{2\{(1+\alpha_1)+(1-\beta)\}\{(1+\alpha_2)+(1-\beta)\}-(1-\beta)^2}{1-\beta} - 1.$$

Therefore, applying the mathematical induction, we see that $G_m(z) \in \mathcal{KD}^*(\alpha^*, \beta)$ with

$$\alpha^* = \frac{2^{m-1}\prod_{j=1}^{m}\{(1+\alpha_j)+(1-\beta)\}-(1-\beta)^m}{(1-\beta)^{m-1}} - 1.$$

The corollary is proved.

By virtue of Theorem 1 and Corollary 1, we derive

**Theorem 3.** *If $f_j(z) \in \mathcal{SD}^*(\alpha, \beta_j)$ for each $j = 1, 2, \cdots, m$ and $f_i(z) \in \mathcal{KD}^*(\alpha, \beta_i)$ for each $i = 1, 2, \cdots, k$, then $G_{m+k}(z) \in \mathcal{SD}^*(\alpha, \beta^*)$ with*

$$\beta^* = 1 - \frac{(1+\alpha)\prod_{j=1}^{m+k}(1-\beta_j)}{2^k\prod_{j=1}^{m+k}\{(1+\alpha)+(1-\beta_j)\}-\prod_{j=1}^{m+k}(1-\beta_j)}.$$

**Proof.** By using the same method as in the proof of Theorem 1, let $f_1(z) \in \mathcal{SD}^*(\alpha, \beta_1)$ and $f_2(z) \in \mathcal{KD}^*(\alpha, \beta_2)$, then $G_2(z) \in \mathcal{SD}^*(\alpha, \beta^*)$ with

$$\beta^* = 1 - \frac{(1+\alpha)(1-\beta_1)(1-\beta_2)}{2\{(1+\alpha)+(1-\beta_1)\}\{(1+\alpha)+(1-\beta_2)\}-(1-\beta_1)(1-\beta_2)}. \tag{5}$$

On the other hand, if $f_j(z) \in \mathcal{SD}^*(\alpha, \beta_j)$ $(j = 1, 2, \cdots, m)$, then $G_m(z) \in \mathcal{SD}^*(\alpha, \tilde{\beta}_1)$ with

$$\tilde{\beta}_1 = 1 - \frac{(1+\alpha)\prod_{j=1}^{m}(1-\beta_j)}{\prod_{j=1}^{m}\{(1+\alpha)+(1-\beta_j)\}-\prod_{j=1}^{m}(1-\beta_j)},$$

and also if $f_i(z) \in \mathcal{KD}^*(\alpha, \beta_i)$ $(i = 1, 2, \cdots, k)$, then $G_k(z) \in \mathcal{KD}^*(\alpha, \tilde{\beta}_2)$ with

$$\tilde{\beta}_2 = 1 - \frac{(1+\alpha)\prod_{i=1}^{k}(1-\beta_i)}{2^{k-1}\prod_{i=1}^{k}\{(1+\alpha)+(1-\beta_i)\} - \prod_{i=1}^{k}(1-\beta_i)}$$

from Theorem 1 and Corollary 1, respectively. Then replacing $\beta_1$ by $\tilde{\beta}_1$ and $\beta_2$ by $\tilde{\beta}_2$ from inequality (5), we have

$$\beta^* = 1 - \frac{(1+\alpha)(1-\tilde{\beta}_1)(1-\tilde{\beta}_2)}{2\{(1+\alpha)+(1-\tilde{\beta}_1)\}\{(1+\alpha)+(1-\tilde{\beta}_2)\} - (1-\tilde{\beta}_1)(1-\tilde{\beta}_2)}$$

$$= 1 - \frac{(1+\alpha)\prod_{j=1}^{m+k}(1-\beta_j)}{2^k\prod_{j=1}^{m+k}\{(1+\alpha)+(1-\beta_j)\} - \prod_{j=1}^{m+k}(1-\beta_j)}.$$

This complete the proof of the theorem.

**Corollary 3.** *If $f_j(z) \in \mathcal{SD}^*(\alpha, \beta_j)$ for each $j = 1, 2, \cdots, m$ and $f_i(z) \in \mathcal{KD}^*(\alpha, \beta_i)$ for each $i = 1, 2, \cdots, k$, then $G_{m+k}(z) \in \mathcal{KD}^*(\alpha, \beta^*)$ with*

$$\beta^* = 1 - \frac{(1+\alpha)\prod_{j=1}^{m+k}(1-\beta_j)}{2^{k-1}\prod_{j=1}^{m+k}\{(1+\alpha)+(1-\beta_j)\} - \prod_{j=1}^{m+k}(1-\beta_j)}.$$

**Proof.** By using the same method as in the proof of Theorem 1, let $f_1(z) \in \mathcal{SD}^*(\alpha, \beta_1)$ and $f_2(z) \in \mathcal{KD}^*(\alpha, \beta_2)$, then $G_2(z) \in \mathcal{KD}^*(\alpha, \beta^*)$ with

$$(6) \qquad \beta^* = 1 - \frac{(1+\alpha)(1-\beta_1)(1-\beta_2)}{\{(1+\alpha)+(1-\beta_1)\}\{(1+\alpha)+(1-\beta_2)\} - (1-\beta_1)(1-\beta_2)}.$$

On the other hand, if $f_j(z) \in \mathcal{SD}^*(\alpha, \beta_j)$ $(j = 1, 2, \cdots, m)$, then $G_m(z) \in \mathcal{SD}^*(\alpha, \tilde{\beta_1})$ with

$$\tilde{\beta_1} = 1 - \frac{(1+\alpha)\prod_{j=1}^{m}(1-\beta_j)}{\prod_{j=1}^{m}\{(1+\alpha)+(1-\beta_j)\} - \prod_{j=1}^{m}(1-\beta_j)},$$

and also if $f_i(z) \in \mathcal{KD}^*(\alpha, \beta_i)$ $(i = 1, 2, \cdots, k)$, then $G_k(z) \in \mathcal{KD}^*(\alpha, \tilde{\beta_2})$ with

$$\tilde{\beta_2} = 1 - \frac{(1+\alpha)\prod_{i=1}^{k}(1-\beta_i)}{2^{k-1}\prod_{i=1}^{k}\{(1+\alpha)+(1-\beta_i)\} - \prod_{i=1}^{k}(1-\beta_i)}$$

from Theorem 1 and Corollary 1, respectively. Then replacing $\beta_1$ by $\tilde{\beta_1}$ and $\beta_2$ by $\tilde{\beta_2}$ from inequality (6), we have

$$\begin{aligned}\beta^* &= 1 - \frac{(1+\alpha)(1-\tilde{\beta_1})(1-\tilde{\beta_2})}{\{(1+\alpha)+(1-\tilde{\beta_1})\}\{(1+\alpha)+(1-\tilde{\beta_2})\} - (1-\tilde{\beta_1})(1-\tilde{\beta_2})} \\ &= 1 - \frac{(1+\alpha)\prod_{j=1}^{m+k}(1-\beta_j)}{2^{k-1}\prod_{j=1}^{m+k}\{(1+\alpha)+(1-\beta_j)\} - \prod_{j=1}^{m+k}(1-\beta_j)}\end{aligned}$$

which proves the corollary.

Using $\mathcal{SD}^*(\alpha_j, \beta)$ and $\mathcal{KD}^*(\alpha_i, \beta)$ instead of $\mathcal{SD}^*(\alpha, \beta_j)$ and $\mathcal{KD}^*(\alpha, \beta_i)$ in Theorem 3, we derive

**Theorem 4.** *If $f_j(z) \in \mathcal{SD}^*(\alpha_j, \beta)$ for each $j = 1, 2, \cdots, m$ and $f_i(z) \in \mathcal{KD}^*(\alpha_i, \beta)$ for each $i = 1, 2, \cdots, k$, then $G_{m+k}(z) \in \mathcal{SD}^*(\alpha^*, \beta)$ with*

$$\alpha^* = \frac{2^k \prod_{j=1}^{m+k}\{(1+\alpha_j)+(1-\beta)\} - (1-\beta)^{m+k}}{(1-\beta)^{m+k-1}} - 1.$$

**Proof.** By the same way as Theorem 3, we obtain

$$\alpha^* = \frac{2\{(1+\tilde{\alpha_1})+(1-\beta)\}\{(1+\tilde{\alpha_2})+(1-\beta)\}-(1-\beta)^2}{1-\beta} - 1,$$

where

$$\tilde{\alpha_1} = \frac{\prod_{j=1}^{m}\{(1+\alpha_j)+(1-\beta)\}-(1-\beta)^m}{(1-\beta)^{m-1}} - 1$$

and

$$\tilde{\alpha_2} = \frac{2^{k-1}\prod_{i=1}^{k}\{(1+\alpha_i)+(1-\beta)\}-(1-\beta)^k}{(1-\beta)^{k-1}} - 1,$$

which implies that

$$\alpha^* = \frac{2^{k}\prod_{j=1}^{m+k}\{(1+\alpha_j)+(1-\beta)\}-(1-\beta)^{m+k}}{(1-\beta)^{m+k-1}} - 1.$$

This complete the proof of the theorem.

**Corollary 4.** *If $f_j(z) \in \mathcal{SD}^*(\alpha_j, \beta)$ for each $j = 1, 2, \cdots, m$ and $f_i(z) \in \mathcal{KD}^*(\alpha_i, \beta)$ for each $i = 1, 2, \cdots, k$, then $G_{m+k}(z) \in \mathcal{KD}^*(\alpha^*, \beta)$ with*

$$\alpha^* = \frac{2^{k-1}\prod_{j=1}^{m+k}\{(1+\alpha_j)+(1-\beta)\}-(1-\beta)^{m+k}}{(1-\beta)^{m+k-1}} - 1.$$

**Proof.** Using the same way as Theorem 3, we obtain

$$\alpha^* = \frac{\{(1+\tilde{\alpha_1})+(1-\beta)\}\{(1+\tilde{\alpha_2})+(1-\beta)\}-(1-\beta)^2}{1-\beta} - 1,$$

where

$$\tilde{\alpha_1} = \frac{\prod_{j=1}^{m}\{(1+\alpha_j)+(1-\beta)\}-(1-\beta)^m}{(1-\beta)^{m-1}} - 1$$

and

$$\tilde{\alpha_2} = \frac{2^{k-1}\prod_{i=1}^{k}\{(1+\alpha_i)+(1-\beta)\}-(1-\beta)^k}{(1-\beta)^{k-1}} - 1,$$

which implies that

$$\alpha^* = \frac{2^{k-1}\prod_{j=1}^{m+k}\{(1+\alpha_j)+(1-\beta)\}-(1-\beta)^{m+k}}{(1-\beta)^{m+k-1}} - 1.$$

We complete the proof of the theorem.

**Theorem 5.** *If $f_j(z) \in \mathcal{SD}^*(\alpha_j, \beta_j)$ for each $j = 1, 2, \cdots, m$, then $G_m(z) \in \mathcal{SD}^*(\alpha^*, \beta^*)$ with*

$$\alpha^* = \prod_{j=1}^{m}\{(1+\alpha_j)+(1-\beta_j)\} - \prod_{j=1}^{m}(1-\beta_j) - 1$$

*and*

$$\beta^* = 1 - \prod_{j=1}^{m}(1-\beta_j).$$

**Proof.** Let $f_1(z) \in \mathcal{SD}^*(\alpha_1, \beta_1)$ and $f_2(z) \in \mathcal{SD}^*(\alpha_2, \beta_2)$. Then we know that $G_2(z) \in \mathcal{SD}^*(\alpha^*, \beta^*)$ if

$$\frac{1+\alpha^*}{1-\beta^*} \leqq \frac{\{(1+\alpha_1)(n-1)+(1-\beta_1)\}\{(1+\alpha_2)(n-1)+(1-\beta_2)\}-(1-\beta_1)(1-\beta_2)}{(1-\beta_1)(1-\beta_2)(n-1)}$$

$$= \frac{(1+\alpha_1)(1+\alpha_2)(n-1)+\{(1-\beta_1)(1+\alpha_2)+(1+\alpha_1)(1-\beta_2)\}}{(1-\beta_1)(1-\beta_2)}$$

is satisfied. The right hand side of the above inequality is a increasing function for $n \geqq 2$. This means that

$$\frac{1+\alpha^*}{1-\beta^*} = \frac{\{(1+\alpha_1)+(1-\beta_1)\}\{(1+\alpha_2)+(1-\beta_2)\}-(1-\beta_1)(1-\beta_2)}{(1-\beta_1)(1-\beta_2)}.$$

Therefore, considering

$$\alpha^* = \{(1+\alpha_1)+(1-\beta_1)\}\{(1+\alpha_2)+(1-\beta_2)\}-(1-\beta_1)(1-\beta_2)-1$$

and

$$\beta^* = 1 - (1 - \beta_1)(1 - \beta_2),$$

we prove that $G_2(z) \in \mathcal{SD}^*(\alpha^*, \beta^*)$. Let us suppose that $G_k(z) \in \mathcal{SD}^*(\alpha_0, \beta_0)$ and $f_{k+1} \in \mathcal{SD}^*(\alpha_{k+1}, \beta_{k+1})$, where

$$\alpha_0 = \prod_{j=1}^{k}\{(1+\alpha_j) + (1-\beta_j)\} - \prod_{j=1}^{k}(1-\beta_j) - 1$$

and

$$\beta_0 = 1 - \prod_{j=1}^{k}(1-\beta_j).$$

Then we get

$$\frac{1+\alpha^*}{1-\beta^*} = \frac{\prod_{j=1}^{k+1}\{(1+\alpha_j) + (1-\beta_j)\} - \prod_{j=1}^{k+1}(1-\beta_j)}{\prod_{j=1}^{k+1}(1-\beta_j)}.$$

The theorem is true for the integer $m = k+1$. From the mathematical induction, we prove the theorem.

**Corollary 5.** *If $f_j(z) \in \mathcal{KD}^*(\alpha_j, \beta_j)$ for each $j = 1, 2, \cdots, m$, then $G_m(z) \in \mathcal{KD}^*(\alpha^*, \beta^*)$ with*

$$\alpha^* = 2^{m-1}\prod_{j=1}^{m}\{(1+\alpha_j) + (1-\beta_j)\} - \prod_{j=1}^{m}(1-\beta_j) - 1$$

*and*

$$\beta^* = 1 - \prod_{j=1}^{m}(1-\beta_j).$$

# References

[1] J. Nishiwaki and S. Owa, *Certain classes of analytic functions concerned with uniformly starlike and convex functions*, Appl. Math. Comp. **187**(1)(2007), 350-355.

[2] S. Owa and H. M. Srivastava, *Some generalized convolution properties associated with certain subclasses of analytic functions*, J. Inequl. Pure. Appl. Math. **3** No3. Article42. (2002), 1-13.

[3] S. Shams, S. R. Kulkarni and J. M. Jahangiri, *Classes of uniformly starlike and convex functions*, Internat. J. Math. Math. Sci. **55**(2004), 2959-2961.

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