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Fixed Points of Ciric Quasi-contractive Operators in Generalized Convex Metric Spaces [1]

Arif Rafiq

Abstract

We establish a general theorem to approximate fixed points of Ciric quasi-contractive operators on a generalized convex metric space through the Mann type iteration process with errors in the sense of Xu [11]. Our result generalizes and improves upon, among others, the corresponding results of [1, 8].

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1 Introduction and Preliminaries

Takahashi [10] introduced the notion of convex metric spaces and studied the fixed-point theory for nonexpansive mappings in such a setting.

Definition 1 *[10] Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a convex structure on X if, for each $(x, y, \lambda) \in X \times X \times [0, 1]$ and $u \in X$,*

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y). \tag{T}$$

The metric space X together with W is called a convex metric space.

Definition 2 *[10] Let X be a convex metric space. A nonempty subset A of X is said to be convex if $W(x, y, \lambda) \in A$ whenever $(x, y, \lambda) \in A \times A \times [0, 1]$.*

Takahashi [10] has shown that, in convex metric spaces, open spheres and closed spheres are convex.

All normed spaces and their convex subsets are convex metric spaces. However there are many examples of convex metric spaces which are not embedded in any normed space(see Takahashi [10]).

A Banach space, or any convex subset of it, is a convex metric space with $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$. More generally, if X is a linear space with a translation invariant metric satisfying $d(\lambda x + (1 - \lambda)y, 0) \leq \lambda d(x, 0) + (1 - \lambda)d(y, 0)$, then X is a convex metric space.

From (T) it follows that

$$\begin{aligned} d(x, W(x, y, \lambda)) &= (1 - \lambda)d(x, y), \\ d(y, W(x, y, \lambda)) &= \lambda d(x, y). \end{aligned}$$

Let C be a nonempty convex subset of a normed space E and $T : C \to C$ be a mapping.

The Mann iteration process is defined by the sequence $\{x_n\}$ (see [7]):

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - b_n)\, x_n + b_n T x_n, \ n \geq 0, \end{cases} \tag{1.1}$$

where $\{b_n\}$ is a sequence in $[0, 1]$.

In 1998, Xu [11] introduced more satisfactory error terms in the sequence defined by:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \ n \geq 0, \end{cases} \tag{1.2}$$

where $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences in $[0, 1]$ such that $a_n + b_n + c_n = 1$ and $\{u_n\}$ is a bounded sequence in C. Clearly, this iteration process contains the process (1.1), as its special cases.

We recall the following definitions in a metric space (X, d). A mapping $T : X \to X$ is called an a-contraction if

$$d(Tx, Ty) \leq a d(x, y) \text{ for all } x, y \in X, \tag{1.3}$$

where $a \in (0, 1)$.

The map T is called Kannan mapping [4] if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X. \tag{1.4}$$

A similar definition is due to Chatterjea [2]: there exists $c \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)] \text{ for all } x, y \in X. \tag{1.5}$$

Combining these three definitions, Zamfirescu [12] proved the following important result.

Theorem 1 *Let (X, d) be a complete metric space and $T : X \to X$ a mapping for which there exists the real numbers a, b and c satisfying $a \in (0, 1)$, $b, c \in (0, \frac{1}{2})$ such that for any pair x, $y \in X$, at least one of the following conditions holds:*

(z_1) $d(Tx, Ty) \leq ad(x, y)$,

(z_2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$,

(z_3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \ n = 0, \ 2, ...$$

converges to p for any arbitrary but fixed $x_0 \in X$.

The conditions $(z_1) - (z_3)$ can be written in the following equivalent form

$$d(Tx, Ty) \leq \tag{QC}$$

$$\leq h \max\{d(x, y), \ \frac{d(x, Tx) + d(y, Ty)}{2}, \ \frac{d(x, Ty) + d(y, Tx)}{2}\},$$

$\forall x, y \in X$; $0 < h < 1$, has been obtained by Ciric [3] in 1974.

A mapping satisfying (QC) is commonly called Ciric quasi contraction. It is obvious that each of the conditions $(z_1) - (z_3)$ implies (QC).

An operator T satisfying the contractive conditions $(z_1) - (z_3)$ in the theorem 1 is called $Z-$operator.

In 2000, Berinde [1] introduced a new class of operators on a normed space E satisfying

$$\|Tx - Ty\| \leq \delta \|x - y\| + L \|Tx - x\| , \tag{1.6}$$

for any $x, y \in E$, $0 \leq \delta < 1$ and $L \geq 0$.

It may be noted that (1.6) is equivalent to

$$\|Tx - Ty\| \leq \delta \|x - y\| + L \min\{\|Tx - x\| , \|Ty - y\|\}, \tag{1.7}$$

for any $x, y \in E$, $0 \leq \delta < 1$ and $L \geq 0$.

Thus

$$\|Tx - Ty\| \leq \delta \|x - y\| + L \max\{\|Tx - x\| , \|Ty - y\|\}, \tag{1.8}$$

for any $x, y \in E$, $0 \leq \delta < 1$ and $L \geq 0$.

He proved that this class is wider than the class of Zamfirescu operators and used the Mann iteration process to approximate fixed points of this class of operators in a normed space given in the form of following theorem:

Theorem 2 *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying* (1.6)*. Let $\{x_n\}_{n=0}^{\infty}$ be defined through the iterative process* (1.1)*. If $F(T) \neq \varphi$ and $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Inspired and motivated by the above said facts, we are introducing the following new concepts.

Definition 3 *Let (X, d) be a metric space. A mapping $W : X \times X \times X \times [0,1] \times [0,1] \times [0,1] \to X$ is said to be a generalized convex structure on X if for each $(x, y, z; a, b, c) \in X \times X \times X \times [0,1] \times [0,1] \times [0,1]$ and $u \in X$,*

$$d(u, W(x, y, z; a, b, c)) \leq ad(u, x) + bd(u, y) + cd(u, z); \tag{1.9}$$

$a + b + c = 1$. The metric space X together with W is called a generalized convex metric space.

Definition 4 *Let X be a generalized convex metric space. A nonempty subset A of X is said to be generalized convex if $W(x, y, z; a, b, c) \in A$ whenever $(x, y, z; a, b, c) \in A \times A \times A \times [0,1] \times [0,1] \times [0,1]$.*

Clearly every generalized convex metric space is a convex metric space.

Clearly every generalized convex set is a convex set.

It can be easily seen that open spheres and closed spheres are generalized convex.

All normed spaces and their generalized convex subsets are generalized convex metric spaces.

Clearly a Banach space, or any generalized convex subset of it, is a generalized convex metric space with $W(x, y, z; a, b, c) = ax + by + cz$. More generally, if X is a linear space with a translation invariant metric satisfying $d(ax + by + cz, 0) \leq ad(x, 0) + bd(y, 0) + cd(z, 0)$, then X is a generalized convex metric space.

It is clear from (1.9) that

$$d[x, W(x, y, z; a, b, c)] \leq bd(x, y) + cd(x, z), \tag{1.10}$$

$$d[y, W(x, y, z; a, b, c)] \leq ad(x, y) + cd(y, z),$$

$$d[z, W(x, y, z; a, b, c)] \leq ad(x, z) + bd(y, z).$$

Let C be a nonempty closed convex subset of a generalized convex metric space X and $T : C \to C$ be a mapping.

Algorithm 1 *The sequence* $\{x_n\}$ *defined by*

$$\begin{cases} x_0 \in C, \\ x_{n+1} = W(x_n, Tx_n, u_n; a_n, b_n, c_n), \ n \geq 0, \end{cases} \tag{1.11}$$

where $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences in $[0, 1]$ such that $a_n + b_n + c_n = 1$ and $\{u_n\}$ is a bounded sequence in C, is called Xu-Mann [11] type iteration process.

Algorithm 2 *The sequence* $\{x_n\}$ *defined by*

$$\begin{cases} x_0 \in C, \\ x_{n+1} = W(Tx_n, x_n, \alpha_n), \ n \geq 0, \end{cases} \tag{1.12}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ is called Mann [7] type iteration process.

In this paper, a convergence theorem of Rhoades [8] regarding the approximation of fixed points of some quasi contractive operators in uniformly convex Banach spaces using the Mann iteration process [7], is extended to generalized convex metric spaces using the Mann type iteration process (1.11) with errors in the sense of Xu [11].

2 Main Results

The following lemma is now well known.

Lemma 1 *Let $\{r_n\}, \{s_n\}, \{t_n\}$ and $\{k_n\}$ be sequences of nonnegative numbers satisfying*

$$r_{n+1} \leq (1 - s_n) r_n + s_n t_n + k_n \quad \textit{for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} s_n = \infty$, $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=1}^{\infty} k_n < \infty$ hold, then $\lim_{n\to\infty} r_n = 0$.

Theorem 3 *Let C be a nonempty closed convex subset of a generalized convex metric space X. Let $T : C \to C$ be an operator satisfying the condition*

$$(CR) \qquad d(Tx, Ty) \leq$$
$$\leq h \max\{d(x, y), \ \frac{d(x, Tx) + d(y, Ty)}{2}, \ d(x, Ty), d(y, Tx)\},$$

$\forall x,\ y \in C$; $0 < h < 1$ *(has been obtained by Ciric [3] in 1974). Let $\{x_n\}$ be defined by the iterative process* (1.11). *If $\sum_{n=1}^{\infty} b_n = \infty$ and $c_n = o(b_n)$, then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. By theorem 1, we know that T has a unique fixed point in C, say w. Consider $x, y \in C$. Since T is a operator satisfying condition (CR), then if

$$\begin{aligned} d\left(Tx, Ty\right) &\leq \frac{h}{2}\left[d(x, Tx) + d(y, Ty)\right] \\ &\leq \frac{h}{2}\left[d(x, Tx) + d(y, x) + d(x, Tx) + d\left(Tx, Ty\right)\right], \end{aligned}$$

implies

$$(1 - \frac{h}{2}) d\left(Tx, Ty\right) \leq \frac{h}{2} d(x, y) + h d(x, Tx),$$

which yields (using the fact that $0 < h < 1$)

$$d\left(Tx, Ty\right) \leq \frac{\frac{h}{2}}{1 - \frac{h}{2}} d(x, y) + \frac{h}{1 - \frac{h}{2}} d(x, Tx). \tag{2.1}$$

If

$$d\left(Tx,Ty\right)\leq hd(x,Ty),$$

implies

$$d\left(Tx,Ty\right)\leq \frac{h}{1-h}d(x,Tx), \tag{2.2}$$

and also for

$$\begin{aligned} d\left(Tx,Ty\right) &\leq hd(y,Tx) \\ &\leq hd(x,y)+hd(x,Tx).2.3 \end{aligned} \tag{2.3}$$

Denote

$$\begin{aligned} \delta &= \max\left\{h, \frac{\frac{h}{2}}{1-\frac{h}{2}}\right\}=h, \\ L &= \max\left\{h, \frac{h}{1-\frac{h}{2}}, \frac{h}{1-h}\right\}=\frac{h}{1-h}. \end{aligned}$$

Thus, in all cases,

$$\begin{aligned} d\left(Tx,Ty\right) &\leq \delta d(x,y)+Ld(x,Tx) \\ &= hd(x,y)+\frac{h}{1-h}d(x,Tx), 2.4 \end{aligned} \tag{2.4}$$

holds for all $x,y\in C$.

Also from (CR) with $y=w=Tw$, we have

$$\begin{aligned} d(Tx,w) &\leq h\max\{d(x,w),\ \frac{d(x,Tx)}{2},\ d(x,w),d(w,Tx)\} \\ &= h\max\{d(x,w),\ \frac{d(x,Tx)}{2}, d(w,Tx)\} \\ &\leq h\max\{d(x,w),\ \frac{d(x,w)+d(w,Tx)}{2},\ d(w,Tx)\} \\ &\leq h\max\{d(x,w),\ d(w,Tx)\}. \end{aligned}$$

If $d(Tx, w) \leq hd(w, Tx)$, is impossible or implies $d(w, Tx) = 0$. Thus

$$d(Tx, w) \leq hd(x, w). \tag{2.4a}$$

Assume that

$$M = \sup_{n \geq 0} d(u_n, w).$$

Using (1.9) and (1.11), we have

$$\begin{aligned} d(x_{n+1}, w) &= d(W(x_n, Tx_n, u_n; a_n, b_n, c_n), w) \\ &\leq a_n d(x_n, w) + b_n d(Tx_n, w) + c_n d(u_n, w) \\ &\leq (1 - b_n) d(x_n, w) + b_n d(Tx_n, w) + Mc_n.2.5 \end{aligned} \tag{2.5}$$

Now (2.4) or (2.4a) gives

$$d(Tx_n, w) \leq hd(x_n, w). \tag{2.6}$$

From (2.5-2.6), we obtain

$$d(x_{n+1}, w) \leq [1 - (1 - h)b_n] d(x_n, w) + Mc_n.$$

By Lemma 1, we get that $\lim_{n \to \infty} d(x_n, w) = 0$. Consequently $x_n \to w \in F$ and this completes the proof.

Corollary 1 *Let C be a nonempty closed convex subset of a convex metric space X. Let $T : C \to C$ be an operator satisfying* (2.4). *Let $\{x_n\}$ be defined by the iterative process* (1.12). *If $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}$ converges strongly to the unique fixed point of T.*

The contractive condition (1.3) makes T a continuous function on X while this is not the case with the contractive conditions $(1.4 - 1.5)$, (2.4) and (2.4a).

The Chatterjea's and the Kannan's contractive conditions (1.5) *and* (1.4) *are both included in the class of Zamfirescu operators and so their convergence theorems for the Mann iteration process with errors are obtained in theorem 3*.

Theorem 4 *of Rhoades [8] in the context of Mann iteration on a uniformly convex Banach space has been extended in corollary 1*.

In corollary 1, t*heorem* 8 *of Rhoades [8] is generalized to the setting of convex metric spaces.*

References

[1] V. Berinde, Iterative approximation of fixed points, Baia Mare: Efemeride, 2000.

[2] S. K. Chatterjea, *Fixed point theorems,* C. R. Acad. Bulgare Sci., **25** (1972), 727-730.

[3] L. B. Ciric, A generalization of Banach principle, Proc. Amer. Math. Soc., **45** (1974), 727-730.

[4] R. Kannan, *Some results on fixed points,* Bull. Calcutta Math. Soc., **10** (1968), 71-76.

[5] R. Kannan, *Some results on fixed points III,* Fund. Math., **70** (1971),169-177.

[6] R. Kannan, *Construction of fixed points of class of nonlinear mappings,* J. Math. Anal. Appl., **41** (1973), 430-438.

[7] W. R. Mann, *Mean value methods in iterations* , Proc. Amer. Math. Soc., **4** (1953), 506-510.

[8] B. E. Rhoades, *Fixed point iteration using infinite matrices*, Trans. Amer. Math. Soc., **196** (1974), 161-176.

[9] B. E. Rhoades,*Comments on two fixed point iteration method*, J. Math. Anal. Appl., **56 (2)** (1976), 741-750.

[10] Takahashi W., A convexity in metric spaces and nonexpansive mappings, Kodai Math. Sem. Rep. **22** (1970), 142-149.

[11] Y. Xu, *Ishikawa and Mann iteration process with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl., **224** (1998), 91-101.

[12] T. Zamfirescu, *Fixed point theorems in metric spaces,* Arch. Math. (Basel), **23** (1972), 292-298.

Dept. of Mathematics,
COMSATS Institute of Information Technology,
Islamabad, Pakistan
E-mail address:*arafiq@comsats.edu.pk*