General Mathematics Vol. 14, No. 3 (2006), 103–114

The multipoint model of the nuclear reactor dynamics, via weakly Picard operator [1]

Ion Marian Olaru and **Vasilica Olaru**

Dedicated to Associated Professor Silviu Crăciunaş on his 60^{th} birthday.

Abstract

In this paper we present a study about multipoint model for the systems describable by differential equations with time delay variable (like the nuclear reactor model), using the weakly Picard operator technique. First we study the Cauchy problem atached. Next we will determine the solution set for the multipoint problem and for this solution set, we will study the continuity with respect to the model data, used the weakly Picard operator method. Finally we will search out a physics representation for these results.

2000 Mathematical Subject Classification: 34 K10, 47H10.

Keywords: weakly Picard operators, reactor control system, fission reactors.

---

[1] *Received 10 March, 2006*

Accepted for publication (in revised form) 18 July, 2006

1 Weakly Picard operators

The notion of weakly Picard operator was introduced by professor I.A. Rus in 1983. Some of the basic concepts and results can be found in the works cited list. As it follows, I shall present some concepts and results which are requisite in this paper.

Let (X, d) be a metric space and $A : X \longrightarrow X$ an operator. We shall use the following notations:

$F_A := \{x \in X \mid A(x) = x\}$- the fixed point set of A.

$I(A) := \{Y \subset X \mid A(Y) \subset, Y \neq \emptyset\}$-the family of the nonempty invariant subsets of A.

$$A^{n+1} := A \circ A^n \; A^0 = 1_X, \;\; A^1 = A, \;\; n \in \mathbb{N}.$$

Definition 1.1.*[1],[2] An operator A is weakly Picard operator (WPO) if the sequence*

$$(A^n(x))_{n \in N}$$

converges , for all $x \in X$ and the limit (which depend on x) is a fixed point of A.

Definition 1.2.*[1],[2] If the operator A is WPO and $F_A = \{x^*\}$ then by definition A is Picard operator.*

Definition 1.3.*[1],[2] If A is WPO, then we consider the operator*

$$A^\infty : X \to X, A^\infty(x) = \lim_{n \to \infty} A^n(x).$$

We remark that $A^\infty(X) = F_A$.

Definition 1.4.*[1],[2] Let be A an WPO and $c > 0$.The operator A is c-WPO if*

$$d(x, A^\infty(x)) \leq c \cdot d(x, A(x)).$$

We have the following characterization of the WPOs:

Theorem 1.1. *[1],[2]Let (X, d) be a metric space and $A : X \to X$ an operator .The operator A is WPO (c-WPO) if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda \in \Lambda} X_\lambda$$

such that

(a) $X_\lambda \in I(A)$

(b) $A \mid X_\lambda : X_\lambda \to X_\lambda$ *is a Picard (c-Picard) operator, for all* $\lambda \in \Lambda$.

For the class of c-WPOs we have the following data dependence result:

Theorem 1.2. *[1],[2] Let (X, d) be a metric space and $A_i : X \to X, i = \overline{1, 2}$ an operator.We suppose that :*

(i) the operator A_i is $c_i - WPO$, $i = \overline{1, 2}$.

(ii) there exists $\eta > o$ such that

$$d(A_1(x), A_2(x)) \leq \eta, (\forall)x \in X.$$

Then

$$H(F_{A_1}, F_{A_2}) \leq \eta \; max\{c_1, c_2\}.$$

Here stands for Hausdorff-Pompeiu functional.

Lemma 1.1. *Let (X, d) be a metric space, $A, B \in P(X)$ and $\varepsilon > 0$. Then for each $a \in A$, there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \varepsilon$$

2 Main results

In [6] occurs the following multipoint model of the nuclear reactor dynamics

(1)
$$N_k' = \frac{\rho_k - \beta_k}{l} N_k + \sum_{i=1}^{m_k} \lambda_{ik} C_{ik} +$$
$$\sum_{j=1}^{M} \frac{\alpha_{kj}}{l} \int_0^{\infty} \varphi_{kj}(\tau) N_j(t-\tau) d\tau,$$

(2)
$$C_{ik}' = \frac{\beta_{ik}}{l_k} N_k - \lambda_{ik} C_{ik}$$

where: $k = \overline{1, M}$, $i = \overline{1, m_k}$, $\beta_k = \sum\limits_{i=1}^{m_k} \beta_{ik}$, M is the radiation zones number of the reactor, α_{kj} is the neutron coupling coefficient between k and j zones, and φ_{kj} is the time distribution function of j to k neutron zone transition probability, N_k is the neutron density, C_{ik}, λ_{ik} , β_{ik} are the concentration, the disintegration constant and the source of radiation fraction of retarded neutron i group, for each k zone, respectively. We study bellow the $M = 2$ case, the general case been enable in similarly treatment.

We denote by:

$$A(t) = \begin{pmatrix} \frac{\rho_1(t)-\beta_1}{l} & 0 \\ 0 & \frac{\rho_2(t)-\beta_2}{l} \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$$

$$N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} \in M_{2\times 1}(\mathbb{R})$$

$$C = \begin{pmatrix} C_{11} & 0 \\ \vdots & \vdots \\ C_{m_1 1} & 0 \\ 0 & C_{12} \\ \vdots & \vdots \\ 0 & C_{m_2 2} \end{pmatrix} \in M_{(m_1+m_2)\times 2}(\mathbb{R})$$

$$\Lambda = \begin{pmatrix} \lambda_{11} & \dots & \lambda_{m_1 1} & 0 & \dots & 0 \\ 0 & \dots & 0 & \lambda_{12} & \dots & \lambda_{m_2 2} \end{pmatrix}$$

$$\Lambda \in M_{(m_1+m_2)\times 2}(\mathbb{R})$$

$$\Omega = \begin{pmatrix} \frac{\beta_{11}}{l} & 0 \\ \vdots & \vdots \\ \frac{\beta_{m_1 1}}{l} & 0 \\ 0 & \frac{\beta_{12}}{l} \\ \vdots & \vdots \\ 0 & \frac{\beta_{m_2 2}}{l} \end{pmatrix} \in M_{(m_1+m_2)\times 2}(\mathbb{R})$$

$$B(\tau) = \begin{pmatrix} \frac{\alpha_{11}}{l}\varphi_{11}(\tau) & \frac{\alpha_{12}}{l}\varphi_{12}(\tau) \\ \frac{\alpha_{21}}{l}\varphi_{21}(\tau) & \frac{\alpha_{22}}{l}\varphi_{22}(\tau) \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$$

It follow that the equations (1) and (2) are equivalents with:

$$(3) \qquad \begin{cases} N'(t) = A(t)N(t) + \Lambda C(t) \quad + \\ \qquad + \int\limits_0^\infty B(\tau)N(t-\tau) \\ C'(t) = \Omega I_2 N(t) - \Lambda^t C^t(t) \end{cases}$$

where:

(a) $N \in C((-\infty, T], M_{2\times 2}(\mathbb{R})) \cap C^1([0,T])$;

(b) $C \in C([0,T], M_{(m_1+m_2)\times 2}(\mathbb{R})) \cap C^1[0,T]$;

Next we denote by

$$X = B((-\infty, T], M_{2\times 1}(\mathbb{R})) =$$

$$= \{x \in C((-\infty, T], M_{2\times 1}(\mathbb{R})) \mid x \textit{ is bounded}\}$$

and

$$Y = C([0,T], M_{(m_1+m_2)\times 2}(\mathbb{R}))$$

For a matrix $A = (a_{ij})_{i=\overline{1,m}, j=\overline{1,n}}$, we define the norm of A by relation

$$|A| = \max_{i=\overline{1,m}, j=\overline{1,n}} |a_{ij}|.$$

For $(x,y) \in X \times Y$ we define

$$\|(x,y)\| = \max\{\|x\|_1, \|y\|_2\}$$

where

$$\|x\|_1 = \sup_{t\in(-\infty,T]} |x(t)|$$

$$\|y\|_2 = \sup_{t\in[0,T]} |y(t)|$$

Then $(X \times Y, \|\cdot,\cdot\|)$, $(X, \|\cdot\|_1)$, $(Y, \|\cdot\|_2)$ are the Banach spaces.

The Cauchy problem atached to (3) is :

$$(4) \qquad \begin{cases} N'(t) = A(t)N(t) + \Lambda C(t) + \int\limits_0^\infty B(\tau)N(t-\tau) & , \quad t \in [0,T] \\ C'(t) = \Omega I_2 N(t) - \Lambda^t C^t(t) & , \quad t \in [0,T] \\ N(t) = \varphi(t) & , \quad t \in (-\infty, 0] \\ C(0) = A \end{cases}$$

where:

(a) $\varphi \in C((-\infty,0])$;

(b) $A \in M_{(m_1+m_2)\times 2}$

For the Cauchy problem attached to (3) we have the following results:

Theorem 2.1. *We suppose that*

$$\max\{L_1, L_2\} < 1,$$

where:

$$L_1 = \int_0^T |A(s)|ds + |\Lambda|T + T\int_0^\infty |B(\tau)|d\tau;$$

$$L_2 = T(|\Omega I_2| + |\Lambda|).$$

Then the Cauchy problem attached to equation(3) has a unique solution in $X \times Y$.

Proof: The equation (4) is equivalent with:

$$N(t) = \begin{cases} \varphi(0) + \int_0^t A(s)N(s)ds + \int_0^t \Lambda C(s)ds + \\ + \int_0^t \int_0^\infty B(\tau)N(s-\tau)d\tau ds, t \in [0,T] \\ \varphi(t), t \in (-\infty, 0] \end{cases} \tag{5}$$

$$C(t) = A + \int_0^t \Omega I_2 N(s)ds - \Lambda^t C^t(s)ds, \tag{6}$$

for $t \in [0,T]$. On $X \times Y$ we define the operator E by relation:

$$E : X \times Y \longrightarrow X \times Y,$$

$$E(N,C) = (E_1(N,C), E_2(N,C))$$

where

$$E_1 : X \times Y \longrightarrow X,$$

$E_1(N,C)(t)$ =the second part of relation (5)

and

$$E_2 : X \times Y \longrightarrow Y,$$

$E_2(N,C)(t) =$the second part of relation (6)

Let be $t \in [0,T]$. Then

$$|E_1(N,C)(t) - E_1(N_1,C_1)(t)| \leq$$

$$\leq L_1 \|(N,C) - (N_1,C_1)\|$$

with

$$L_1 = \int_0^T |A(s)|ds + |\Lambda|T + T\int_0^\infty |B(\tau)|d\tau$$

For $t \in (-\infty, 0]$ we have that

$$|E_1(N,C)(t) - E_1(N_1,C_1)(t)| = 0$$

It follow that

$$\|E_1(N,C) - E_1(N_1,C_1)\|_1 \leq L_1 \|(N,C) - (N_1,C_1)\|$$

For $t \in [0,T]$. Then

$$|E_2(N,C)(t) - E_2(N_1,C_1)(t)| \leq$$

$$\leq L_2 \|(N,C) - (N_1,C_1)\|$$

with

$$L_2 = T(|\Omega I_2| + |\Lambda|)$$

It follow that

$$\|E_2(N,C) - E_2(N_1,C_1)\|_2 \leq L_2 \|(N,C) - (N_1,C_1)\|$$

and from here we obtain

$$\|E(N,C) - E(N_1,C_1)\| \leq$$

$$\leq \max\{L_1, L_2\} \|(N,C) - (N_1,C_1)\|$$

From Banach principle we obtain the conclusion of Theorem.

Next we denote by R, S the following operators

$$R : X \times Y \longrightarrow X,$$

$$S : X \times Y \longrightarrow Y,$$

$$R(N,C)(t) = A(t)N(t) + \Lambda C(t) + \int_0^\infty B(\tau)N(t-\tau)d\tau$$

$$S(N,C)(t) = \Omega I_2 N(t) - \Lambda^t C^t(t)$$

Then the equations (1) and (2) is equivalent with:

$$N(t) = \begin{cases} N(0) + \int_0^t R(N,C)(s)ds \quad , & t \in [0,T] \\ N(t) \quad , & t \in (-\infty, 0] \end{cases} \tag{7}$$

$$C(t) = C(0) + \int_0^t S(N,C)(s)ds, \; t \in [0,T]. \tag{8}$$

Theorem 2.2. *We suppose that the conditions from Theorem 2.1 are satisfies.Then*

(a) the equations (7)+(8) has a infinity of solutions;

(b) If F_1, F_2 are the solutions set for the equation (7)+(8) with data R_1, S_1, R_2, S_2, and in addition we suppose that there exists η_1, η_2 such that

$$|R_1(N,C) - R_2(N,C)| \leq \eta_1 \tag{9}$$

$$|R_1(N,C) - S_2(N,C)| \leq \eta_2 \tag{10}$$

then

$$H(F_1, F_2) \leq T \max\{\eta_1, \eta_2\} \max\{c_1, c_2\}$$

Proof. a) On $X \times Y$ we define the operator E by relation:

$$E : X \times Y \longrightarrow X \times Y,$$

$$E(N, C) = (E_1(N, C), E_2(N, C))$$

where

$$E_1 : X \times Y \longrightarrow X,$$

$E_1(N, C)(t)$ =the second part of relation (7)

and

$$E_2 : X \times Y \longrightarrow Y,$$

$E_2(N, C)(t)$ =the second part of relation (8).

We have the following partition

$$X = \bigcup_{\varphi \in B((-\infty,0], M_{2\times 2}(\mathbb{R}))} X_\varphi$$

$$Y = \bigcup_{A \in M_{(m_1+m_2)\times 2}(\mathbb{R})} X_A,$$

where:

$$X_\varphi = \{N \in X \mid N \mid_{(-\infty,0]} = \varphi\}$$

and

$$X_A = \{C \in Y \mid \; N(0) = A\}.$$

Using the Theorem 2.1 we obtain that the operator $E \mid_{X_\varphi \times X_A}$ is Picard. Using the Theorem 1.1 we have that the operator E is c- weakly Picard operator with $c = max\{L_1, L_2\}$. In consequences result that the equation (7)+(8) has a infinity of solutions.

b)

$$|E_1^1(N, C)(t) - E_1^2(N, C)(t)| \leq$$

$$\leq \int_0^t |R_1(N,C)(s) - R_2(N,C)(s)|ds \leq \eta_1 T$$

$$|E_2^1(N,C)(t) - E_2^2(N,C)(t)| \leq$$

$$\leq \int_0^t |S_1(N,C)(s) - S_2(N,C)(s)|ds \leq \eta_2 T$$

It follow that

$$|E^1(N,C) - E^2(N,C)| \leq T \max\{\eta_1, \eta_2\}$$

From here via Theorem 1.2 we obtain the conclusion of Theorem.

3 Conclusion

From the above theorem, via Lema1.1 for $\varepsilon = \max\{\eta_1, \eta_2\}$, we get the following physics interpretation: if perturbation characterized by the inequalities (9) and (10) appear in our physics system, describable by multipoint model, then for each solution $(N_1, C_1) \in F_1$ here exists $(N_2, C_2) \in F_2$ such that

$$\|N_1 - N_2\|_1 \leq T \max\{\eta_1, \eta_2\}\{\max\{c_1, c_2\} + 1\}$$

$$\|C_1 - C_2\|_2 \leq T \max\{\eta_1, \eta_2\}\{\max\{c_1, c_2\} + 1\}$$

So for all $t \in [0, T]$ we have that

$$|N_1(t) - N_2(t)| \leq T \max\{\eta_1, \eta_2\}\{\max\{c_1, c_2\} + 1\}$$

$$|C_1(t) - C_2(t)| \leq T \max\{\eta_1, \eta_2\}\{\max\{c_1, c_2\} + 1\}$$

References

[1] I.A. Rus, *Weakly Picard operators and applications*,Seminar on fixed Point theory, Cluj-Napoca, 2,2001.

[2] I.A. Rus, *Functional-Differential Equations of mixed Type via weakly Picard operators*, Seminar on fixed Point theory, Cluj-Napoca, 3,2002,pp335-345.

[3] I.A.Rus, *Differential equation,integral equation and dynamic systems*, Tansilvania Press, Cluj-Napoca, 1996(in Romanian).

[4] I.A. Rus, S Muresan, *Data dependence of the fixed points set of weakly Picard operators*, Studia Univ."Babes-Bolyai", Mathematica, Volume XLIII, Number 1, March 1998,pp79-83.

[5] I.A.Rus, *Generalized contractions*, Cluj University Press, 2001

[6] Vladimir Răsvan, *Teoria Stabilităţii*, Editura ştiinţifică şi enciclopedică, Bucureşti, 1987.

Department of Mathematics,
University Lucian Blaga,
str. Dr. I.Ratiu, No.5-7,
550024 - Sibiu, Romania,
E-mail address: *olaruim@yahoo.com*

School of Iacobeni,
Sibiu, Romania,
E-mail address: *olarv2@yahoo.com*