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About the Second-Order Equation with Variable Coefficients [1]

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Dedicated to Associated Professor Silviu Crăciunaş on his 60^{th} birthday

Abstract

From the point of view of physical applications, as well as from the viewpoint of theory, it is very important to know how to solve the problem concerned with the presence of zeros of the solution $y(x)$ of the equation $p_0(x)y'' + p_1(x)y' + p_2(x)y = 0$ in the interval (a, b), i.e., the values of $x \in (a, b)$, for which the solution $y(x)$ turns into zero. That is the subject of the paper.

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Let the equation

(1) $$p_0(x)y'' + p_1(x)y' + p_2(x)y = 0, \quad x \in (a, b),$$

for which the solution $y(x)$ turns into zero. Let us consider an elementary second-order equation with constant coefficients

$$y'' + qy = 0, \quad q = \text{const}.$$

If $q \le 0$, then every solution of this equation can vanish throughout the interval $-\infty < x < +\infty$ at no more then one point. For $q > 0$ every solution

$$y = C_1 \cos\sqrt{q}x + C_2 \sin\sqrt{q}x = A\sin\left(\sqrt{q}x + \delta\right)$$

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has an infinite number of zeros, the distance between which is $\dfrac{\pi}{\sqrt{q}}$. i.e. the smaller the larger q is.

Definition 1. The solution $y(x)$ of a differential equation is said to be *nonoscillating* in a given interval if in that interval it has not more than one zero; otherwise the solution is *oscillating*.

Thus an equation of the form $y'' + qy = 0$ ($q =$ const) has solutions nonoscillating in any interval if $q \leq 0$, and the solutions oscillating in a sufficiently large interval if $q > 0$.

Let us generalize this result to a second-order equation with variable coefficients. We assume that the coefficients of the equation are real and study only real solutions of such equations. We consider an equation of the form

$$y'' + q(x)y = 0, \tag{2}$$

to which any equation of the form (1) can be reduced.

Theorem 1.1 *If* $q(x) \leq 0$ *everywhere in the interval* (a, b)*, then all solutions of the equation*

$$y'' + q(x)y = 0$$

are nonoscillating in the interval (a, b)*.*

Here is a geometrical interpretation of the theorem. We assume that some solution $y_1(x) \not\equiv 0$ of equation (2) has at least two zeros

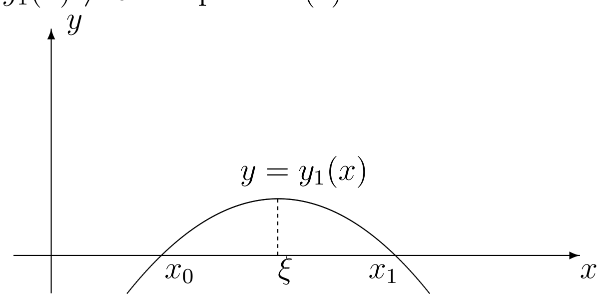

Fig. 1

on the interval (a, b). Let them be x_0 and x_1, $x_0 < x_1$, and let the function $y_1(x)$ have no other zeros on the interval (x_0, x_1) (fig. 1). Then $y_1(x)$ being a continuous function, retains a constant sign in the interval (x_0, x_1). We assume, for definitness, that $y_1(x) > 0$ in (x_0, x_1) (otherwise we would have taken a solution $-y_1(x)$).

At a certain point $\xi \in (x_0, x_1)$ the function $y_1(x)$ possesses a positive maximum; consequently, in some neighbourhood of the point ξ we have $y''(x) < 0$. On the other hand, if $q(x) \leq 0$ on (a, b), then it follows from equation (2) that $y_1''(x) \geq 0$ everywhere in (x_0, x_1). The contradiction obtained indicates that our assumption is wrong and all the solutions of the equation are nonoscillating.

Theorem 1.2 (Sturm′s separation theorem) (see [2]) *If x_0 and x_1 are two succesive zeros of the solution $y_1(x)$ of the differential equation*

$$y'' + q(x)y = 0, \tag{2}$$

then there is exactly one zero between x_0 and x_1 in any other linearly independent solution $y_2(x)$ of the some equation; in short, the zeros of two linearly independent solutions of equation (2) separate each other.

Theorem 1.3 (comparison theorem) *Suppose we have two equations*

$$y'' + q_1(x)y = 0 \tag{4}$$

and

$$z'' + q_2(x)z = 0. \tag{5}$$

If $q_1(x) \geq q_2(x)$ in the interval (a, b), then there is at least two zeros of any solution $y(x)$ of equation (4).

When the comparison theorem is used, an equation with constant coefficients is usually taken as one of the equation (4) or (5).

Given an equation

$$y'' + q(x)y = 0, \tag{6}$$

in which $q(x) > 0$ on the interval $[a, b]$ and the function $q(x)$ is continuous on it. Assume that $M = \max\limits_{a \leq x \leq b} q(x)$ and $m = \min\limits_{a \leq x \leq b} q(x)$. Let $M > m$ so that $q(x) \not\equiv$ const. on $[a, b]$. Taking an equation $y'' + my = 0$ as equation (4), and the given equation (6) as (5), we get the following result: the distance between two successive zeros of the solution of equation (6) as (4), and an equation $y'' + My = 0$ as (5), we infer that the distance between two successive zeros of the solution of equation (6) is not smaller then $\pi/\sqrt{M}$.

This theorem estimates from above and from below the distances between the zeros of the oscillating solutions of differential equations. We can also show that if $\lim\limits_{x \to \infty} q(x) = q > 0$, then any solution of equation (6) is infinitely oscillating, and the distance between the successive zeros tends to $\frac{\pi}{\sqrt{q}}$. For example, for Bessel′s equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad x > 0,$$

setting $y = x^{-1/2}z$, we obtain

$$z'' + \left(1 - \frac{\nu^2 - 1/4}{x^2}\right) z = 0.$$

For a sufficiently large x the expression $1 - \frac{\nu^2 - 1/4}{x^2}$ can be made arbitrarily close to unity. Therefore, for sufficiently large values of x the distance between successive zeros of the solutions of Bessel′s equation is arbitrarily close to π.

Application (see [1]). Let the equation

$$y'' + xy = 0, \quad x > 0, \tag{7}$$

which is encountered in various applications in quantum mechanics, and cannot be integrated by elementary methods. It can be showed that with an infinite growth of x the successive zeros of every solution of equation (7) tend to each other indefinitely.

References

[1] Mioara Boncuţ, Amelia Bucur, *Capitole de matematici speciale*, Ed. ”Alma Mater”, Sibiu, 2002 (in romanian)

[2] M.L. Krasnov, *Ordinary Differential Equations*, Mir Publishers, Moscow, 1987.

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