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Integral Representations and its Applications in Clifford Analysis

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

In this paper, we mainly study the integral representations for functions f with values in a universal Clifford algebra $C(V_{n,n})$, where $f \in \Lambda(f, \overline{\Omega})$,

$$\Lambda(f, \overline{\Omega}) = \Big\{ f | f \in C^{\infty}(\overline{\Omega}, C(V_{n,n})), \max_{x \in \overline{\Omega}} \left| D^j f(x) \right| =$$

$$= O(M^j)(j \to +\infty), \text{for some } M, 0 < M < +\infty \Big\} .$$

The integral representations of $T_i f$ are also given. Some properties of $T_i f$ and Πf are shown. As applications of the higher order Pompeiu formula, we get the solutions of the Dirichlet problem and the inhomogeneous equations $D^k u = f$.

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1 Introduction and Preliminaries

Integral representation formulas of Cauchy-Pompeiu type expressing complex valued, quaternionic and Clifford algebra valued functions have been well developed in [1-9, 12-19, 21, 24, 25 etc.]. These integral representation formulas serve to solve boundary value problems for partial differential equations. In [2, 3], H. Begehr gave the different integral representation formulas for functions with values in a Clifford algebra $C(V_{n,0})$, the integral operators provide particular weak solutions to the inhomogeneous equations $\partial^k\omega = f$, $\triangle^k\omega = g$ and $\partial\triangle^k\omega = h$. In [5, 24], the higher order Cauchy-Pompeiu formulas for functions with values in a universal Clifford algebra $C(V_{n,n})$ are obtained. In [16], G.N. Hile gave the detailed properties of the T-operator by following the techniques of Vekua. In [14, 15], K. Gürlebeck gave many properties of the Π-operator. In [18], H. Malonek and B. Müller gave some properties of the vectorial integral operator $\vec{\Pi}$. In [7, 19, 21], the integral representations related with the Helmholtz operator are given, the weak solutions of the inhomogeneous equations $L^k u = f$ and $L_*^k u = f$, $k \geq 1$, are obtained, where $Lu = Du + uh$ and $L_* u = uD - hu$, $h = \sum\limits_{i=1}^{n} h_i e_i$, D is the Dirac operator. In this paper, we shall continue to study the properties of Cauchy-Pompeiu operator, higher order Cauchy-Pompeiu operator and Π operator for $f \in \Lambda(f, \overline{\Omega})$, where

$$\Lambda(f,\overline{\Omega}) = \Big\{ f | f \in C^{\infty}(\overline{\Omega}, C(V_{n,n})), \max_{x\in\overline{\Omega}} \left| D^j f(x) \right| = \\ = O(M^j)(j \to +\infty), \text{for some}\, M, 0 < M < +\infty \Big\},$$

the integral representations of $T_i f$ are given, some properties of $T_i f$ and Πf are shown. As applications, we get the solutions of the Dirichlet problem and the inhomogeneous equations $D^k u = f$ which are not in weak sense as in [2, 25].

Let $V_{n,s}(0 \le s \le n)$ be an n–dimensional $(n \ge 1)$ real linear space with basis $\{e_1, e_2, \cdots, e_n\}$, $C(V_{n,s})$ be the 2^n–dimensional real linear space with basis

$$\{e_A, A = \{h_1, \cdots, h_r\} \in \mathcal{P}N, 1 \le h_1 < \cdots < h_r \le n\},$$

where N stands for the set $\{1, \cdots, n\}$ and $\mathcal{P}N$ denotes the family of all order-preserving subsets of N in the above way. We denote $e_\emptyset$ as e_0 and e_A as $e_{h_1 \cdots h_r}$ for $A = \{h_1, \cdots, h_r\} \in \mathcal{P}N$. The product on $C(V_{n,s})$ is defined by

(1)
$$\begin{cases} e_A e_B = (-1)^{\#((A \cap B)\setminus S)}(-1)^{P(A,B)} e_{A \triangle B}, & \text{if } A, B \in \mathcal{P}N, \\ \lambda\mu = \sum\limits_{A \in \mathcal{P}N} \sum\limits_{B \in \mathcal{P}N} \lambda_A \mu_B e_A e_B, & \text{if } \lambda = \sum\limits_{A \in \mathcal{P}N} \lambda_A e_A,\ \mu = \sum\limits_{B \in \mathcal{P}N} \mu_B e_B. \end{cases}$$

where S stands for the set $\{1, \cdots, s\}$, $\#(A)$ is the cardinal number of the set A, the number $P(A, B) = \sum\limits_{j \in B} P(A, j)$, $P(A, j) = \#\{i, i \in A, i > j\}$, the symmetric difference set $A \triangle B$ is also order-preserving in the above way, and $\lambda_A \in \mathcal{R}$ is the coefficient of the e_A–component of the Clifford number λ. We also denote λ_A as $[\lambda]_A$, for abbreviaty, we denote $\lambda_{\{i\}}$ as $[\lambda]_i$. It follows at once from the multiplication rule (1) that e_0 is the identity element written now as 1 and in particular,

$$(2) \qquad \begin{cases} e_i^2 = 1, & \text{if } i = 1, \cdots, s, \\ e_j^2 = -1, & \text{if } j = s+1, \cdots, n, \\ e_i e_j = -e_j e_i, & \text{if } 1 \le i < j \le n, \\ e_{h_1} e_{h_2} \cdots e_{h_r} = e_{h_1 h_2 \cdots h_r}, & \text{if } 1 \le h_1 < h_2 \cdots, < h_r \le n. \end{cases}$$

Thus $C(V_{n,s})$ is a real linear, associative, but non-commutative algebra and it is called the universal Clifford algebra over $V_{n,s}$.

Frequent use will be made of the notation $\mathcal{R}^n_z$ where $z \in \mathcal{R}^n$, which means to remove z from $\mathcal{R}^n$. In particular $\mathcal{R}^n_0 = \mathcal{R}^n \setminus \{0\}$.

Let Ω be an open non empty subset of $\mathcal{R}^n$, since we shall only consider the case of $s = n$ in this paper, we shall only consider the operator D which is written as

$$D = \sum_{k=1}^{n} e_k \frac{\partial}{\partial x_k} : C^{(r)}(\Omega, C(V_{n,n})) \to C^{(r-1)}(\Omega, C(V_{n,n})).$$

Let f be a function with value in $C(V_{n,n})$ defined in Ω, the operator D acts on the function f from the left and from the right being governed by the rule

$$D[f] = \sum_{k=1}^{n} \sum_{A} e_k e_A \frac{\partial f_A}{\partial x_k}, \quad [f]D = \sum_{k=1}^{n} \sum_{A} e_A e_k \frac{\partial f_A}{\partial x_k},$$

An involution is defined by

$$(3) \qquad \begin{cases} \overline{e_A} = (-1)^{\sigma(A)+\#(A\bigcap S)} e_A, & \text{if } A \in \mathcal{P}N, \\ \overline{\lambda} = \sum\limits_{A\in\mathcal{P}N} \lambda_A \overline{e_A}, & \text{if } \lambda = \sum\limits_{A\in\mathcal{P}N} \lambda_A e_A, \end{cases}$$

where $\sigma(A) = \#(A)(\#(A)+1)/2$. From (1) and (3), we have

$$(4) \qquad \begin{cases} \overline{e_i} = e_i, & \text{if } i = 0, 1, \cdots, s, \\ \overline{e_j} = -e_j, & \text{if } j = s+1, \cdots, n, \\ \overline{\lambda\mu} = \overline{\mu}\overline{\lambda}, & \text{for any } \lambda, \mu \in C(V_{n,s}). \end{cases}$$

The $C\,(V_{n.n})$–valued $(n-1)$–differential form

$$\mathrm{d}\sigma = \sum_{k=1}^{n} (-1)^{k-1} e_k \mathrm{d}\widehat{x}_k^N$$

is exact, where

$$\mathrm{d}\widehat{x}_k^N = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{k-1} \wedge \mathrm{d}x^{k+1} \wedge \cdots \wedge \mathrm{d}x^n.$$

2 Integral Representations

In this section, we shall give the integral representations for f and $T_i f$, $i \geq 1$, $f \in \Lambda(f, \overline{\Omega})$, where

$$\Lambda(f,\overline{\Omega}) = \Big\{ f | f \in C^\infty(\overline{\Omega}, C(V_{n,n})), \max_{x\in\overline{\Omega}} \left|D^j f(x)\right| =$$

$$= O(M^j)(j \to +\infty), \text{for some}\, M, 0 < M < +\infty \Big\}.$$

In [5], [24] the kernel functions

(5)

$$H_j^*(x) = \begin{cases} \dfrac{A_j}{\omega_n}\dfrac{\mathbf{x}^j}{\rho^n(x)}, & n \text{ is odd;} \\ \dfrac{A_j}{\omega_n}\dfrac{\mathbf{x}^j}{\rho^n(x)}, & 1 \leq j < n, \ \ n \text{ is even;} \\ \dfrac{A_{j-1}}{2\omega_n}\log(\mathbf{x}^2), & j = n, \ \ n \text{ is even;} \\ \dfrac{A_{n-1}}{2\omega_n} C_{l,0}\mathbf{x}^l \left(\log(\mathbf{x}^2) - 2\displaystyle\sum_{i=0}^{l-1}\frac{C_{i+1,0}}{C_{i,0}}\right), & j = n + l, l > 0, \ \ n \text{ is even;} \end{cases}$$

are constructed for any $j \geq 1$, where $\mathbf{x} = \sum\limits_{k=1}^{n} x_k e_k$, $\rho(x) = \left(\sum\limits_{k=1}^{n} x_k^2\right)^{\frac{1}{2}}$, ω_n denotes the area of the unit sphere in $\mathcal{R}^n$, and

$$(6) \quad A_j = \frac{1}{2^{[\frac{j-1}{2}]}[\frac{j-1}{2}]! \prod\limits_{r=1}^{[\frac{j}{2}]}(2r-n)}, 1 \leq j < n (n\,\text{is even}), j \in N^*(n\,\text{is odd}),$$

$$C_{j,0}=\begin{cases}1, & j=0,\\ \dfrac{1}{2^{[\frac{j}{2}]}[\frac{j}{2}]!\prod\limits_{\mu=0}^{[\frac{j-1}{2}]}(n+2\mu)}, & j\in\mathbf{N}^*=\mathbf{N}\backslash\{0\}.\end{cases}\tag{7}$$

Lemma 1.(Higher order Cauchy-Pompeiu formula) (see [24])*Suppose that M is an n–dimensional differentiable compact oriented manifold contained in some open non empty subset $\Omega\subset\mathcal{R}^n$, $f\in C^{(r)}\left(\Omega,C\left(V_{n,n}\right)\right)$, $r\geq k$, moreover ∂M is given the induced orientation, for each $j=1,\cdots,k$, $H_j^*(x)$ is as above. Then, for $z\in\overset{\circ}{M}$*

(8)

$$f(z)=\sum_{j=0}^{k-1}(-1)^j\int\limits_{\partial M}H_{j+1}^*(x-z)\mathrm{d}\sigma_x D^j f(x)+(-1)^k\int\limits_{M}H_k^*(x-z)D^k f(x)\mathrm{d}x^N.$$

In the following, Ω is supposed to be an open non empty subset of $\mathcal{R}^n$ with a Liapunov boundary $\partial\Omega$. Denote

$$T_i f(z)=(-1)^i\int\limits_{\Omega}H_i^*(x-z)f(x)\mathrm{d}x^N\tag{9}$$

where $H_i^*(x)$ is denoted as in (5), $i\in N^*$, $f\in L^p(\Omega,C(V_{n,n}))$, $p\geq 1$. The operator T_1 is the Pompeiu operator T. Especially, we denote f as $T_0 f$.

In [25], it is shown that, if $f\in L^p(\Omega,C(V_{n,n})),p\geq 1$, then $Tf\in C^\alpha(\overline{\Omega},C(V_{n,n})),\alpha=\dfrac{p-n}{p}$. $T_k f$ provides a particular weak solution to the inhomogeneous equation $D^k\omega=f$(weak) in Ω. In this section, we shall show that, if $f\in\Lambda(f,\overline{\Omega})$, then $T_i f\in C^\infty(\Omega,C(V_{n,n}))$, $i\in\mathbf{N}^*$ and $T_k f$ provides a particular solution to the inhomogeneous equation $D^k\omega=f$ in Ω.

Theorem 1. *Let Ω be an open non empty bounded subset of $\mathcal{R}^n$ with a Liapunov boundary $\partial\Omega$, $f \in \Lambda(f, \overline{\Omega})$. Then, for $z \in \Omega$*

$$T_i f(z) = \sum_{j=0}^{\infty} (-1)^{j+i} \int_{\partial\Omega} H^*_{j+i+1}(x-z) \mathrm{d}\sigma_x D^j f(x), \ i \in \mathbf{N}. \tag{10}$$

Proof. Step 1. For $f \in \Lambda(f, \overline{\Omega})$, we shall firstly prove

$$\begin{aligned} T_i f(z) &= \sum_{j=0}^{k} (-1)^{j+i} \int_{\partial\Omega} H^*_{j+i+1}(x-z) \mathrm{d}\sigma_x D^j f(x) \\ &+ (-1)^{i+k+1} \int_{\Omega} H^*_{i+k+1}(x-z) D^{k+1} f(x) \mathrm{d}x^N, \end{aligned} \tag{11}$$

where $i, k \in \mathbf{N}$, $z \in \Omega$. It is obvious that (11) is the direct result of Lemma 1 for $i = 0$.

For $i \geq 1$, in view of the properties of the kernel functions of $H^*_j(x-z)$

(12)

$$D\left[H^*_{j+1}(x-z)\right] = \left[H^*_{j+1}(x-z)\right] D = H^*_j(x-z), \ x \in \mathcal{R}^n_z, \text{ for any } j \geq 1.$$

Combining Stokes formulas with (12), we have

(13)

$$\begin{aligned} (-1)^i \int_{\Omega\backslash B(z,\varepsilon)} H^*_i(x-z) f(x) \mathrm{d}x^N &= \sum_{j=0}^{k} (-1)^{j+i} \int_{\partial(\Omega\backslash B(z,\varepsilon))} H^*_{j+i+1}(x-z) \mathrm{d}\sigma_x D^j f(x) \\ &+ (-1)^{i+k+1} \int_{\Omega\backslash B(z,\varepsilon)} H^*_{i+k+1}(x-z) D^{k+1} f(x) \mathrm{d}x^N. \end{aligned}$$

For $i \geq 1$ and $j \geq 0$, it is easy to check that,

$$\lim_{\varepsilon \to 0} \int_{\partial B(z,\varepsilon)} H^*_{j+i+1}(x-z) \mathrm{d}\sigma_x D^j f(x) = 0. \tag{14}$$

In view of the weak singularity of the kernel functions and (14), taking limits as $\varepsilon \to 0$ in (13), (11) holds.

Step 2. For $f \in \Lambda(f, \overline{\Omega})$, we shall show that

$$\lim_{k\to\infty} \max_{z\in\overline{\Omega}} \left| \int_{\Omega} H^*_{i+k+1}(x-z) D^{k+1} f(x) \mathrm{d}x^N \right| = 0. \tag{15}$$

Since $f \in \Lambda(f, \overline{\Omega})$, then there exist constants C_0, M, $0 < C_0, M < +\infty$, and $N \in \mathbf{N}^*$, such that for any $k \geq N$

$$\max_{x\in\overline{\Omega}} \left| D^k f(x) \right| \leq C_0 M^k. \tag{16}$$

Case 1. n is odd. For any $k \geq N$, we have

$$\left| \int_{\Omega} H^*_{i+k+1}(x-z) D^{k+1} f(x) \mathrm{d}x^N \right| \leq 2^n A_{i+k+1} C_0 V(\Omega) M^{k+1} \delta^{i+k+1-n}, \tag{17}$$

where $\delta = \sup\limits_{x_1,x_2\in\Omega} \rho(x_1 - x_2)$, $V(\Omega)$ denotes the volume of Ω. It is obvious that the series

$$\sum_{k=1}^{\infty} 2^n A_{i+k+1} C_0 V(\Omega) M^{k+1} \delta^{i+k+1-n} \tag{18}$$

converges. Then

$$\lim_{k\to\infty} 2^n A_{i+k+1} C_0 V(\Omega) M^{k+1} \delta^{i+k+1-n} = 0, \tag{19}$$

thus (15) holds.

Case 2. n is even. In view of (5) and (7), it can be similarly proved that (15) holds.

Combining (11) with (15), taking limits $k \to \infty$ in (11), (10) follows.

By Theorem 1, we have

Corollary 1. *Suppose that f is k-regular in a domain U in $\mathcal{R}^n$, Ω is an open non empty bounded subset of U with a Liapunov boundary $\partial\Omega$. Then, for $z \in \Omega$*

$$(20) \qquad T_i f(z) = \sum_{j=0}^{k-1} (-1)^{j+i} \int_{\partial\Omega} H^*_{j+i+1}(x-z) \mathrm{d}\sigma_x D^j f(x), \ i \in \mathbf{N}.$$

Remark 1. *For $i = 0$, (20) is exactly the higher order Cauchy integral formula which has been obtained in [5, 24]. Analogous higher order Cauchy integral formula can be also found in [2, 3, 12].*

Corollary 2. *Let Ω be an open non empty bounded subset of $\mathcal{R}^n$ with a Liapunov boundary $\partial\Omega$, $f \in \Lambda(f, \overline{\Omega})$. Then, for $z \in \Omega$*

$$(21) \qquad D[T_{i+1} f] = T_i f, \ i \in \mathbf{N}.$$

Remark 2. *Corollary 2 implies that $T_k f$ provides a particular solution to the inhomogeneous equation $D^k \omega = f$ in Ω for $f \in \Lambda(f, \overline{\Omega})$. Especially, suppose U is a domain in $\mathcal{R}^n$, Ω is an open non empty bounded subset of U with a Liapunov boundary $\partial\Omega$, f is regular in U, then $T_k f$ is $(k+1)$-regular in Ω. This result gives an improved result in [2, 25] under the assumption of $f \in \Lambda(f, \overline{\Omega})$.*

Corollary 3. *Let U be a domain in $\mathcal{R}^n$, Ω be an open non empty bounded subset of U with a Liapunov boundary $\partial\Omega$, f be a solution of equation $Lu = 0$ in U, where $Lu = Du + uh$, $h = \sum_{i=1}^{n} h_i e_i, h_i \in \mathcal{R}$ or h be a real (complex) number. Then for $z \in \Omega$*

$$(22) \qquad T_i f(z) = \sum_{j=0}^{\infty} (-1)^{j+i} \int_{\partial\Omega} H^*_{j+i+1}(x-z) \mathrm{d}\sigma_x D^j f(x), \ i \in \mathbf{N}.$$

Proof. Obviously, if f is a solution of equation $Lu = 0$ in U, where $Lu = Du + uh$, $h = \sum\limits_{i=1}^{n} h_i e_i$ or h is a real (complex) number, then $f \in \Lambda(f, \overline{\Omega})$. By Theorem 1, the result follows.

Example 1. *Suppose* $u_i(x) = \sum\limits_{k=0}^{\infty} \frac{(\alpha x_i e_i)^k}{k!} \triangleq e^{\alpha x_i e_i}$, $i = 1, \cdots, n$, *where* α *is a real number. Clearly,* $Du_i(x) = \alpha u_i(x)$*. Thus for* $u_i(x)$, $z \in \Omega$, *by Corollary 3, (22) holds.*

Example 2. *Suppose* $h = \sum\limits_{i=1}^{n} h_i e_i, h_i \in \mathcal{R}$. *Denote* $R = |h| = \sqrt{\sum\limits_{i=1}^{n} h_i^2}$. *Obviously,* $e^{Rx_i e_i}$ *satisfies* $Du - Ru = 0$, *thus* $e^{Rx_i e_i}$ *is also a solution of the Helmholtz equation* $\triangle u - R^2 u = 0$. *Then* $e^{Rx_i e_i}(R - h)$ *is a solution of equation* $Du + uh = 0$. *For* $e^{Rx_i e_i}(R - h)$, $z \in \Omega$, *by Corollary 3, (22) holds.*

Ω is supposed to be an open non empty subset of $\mathcal{R}^n$ with a Liapunov boundary $\partial\Omega$. Denote

$$\Pi f(z) = \begin{cases} \int\limits_{\Omega} K(x - z) f(x) \mathrm{d}x^N, & z \in \Omega, \\ \lim\limits_{\substack{\xi \to z \\ \xi \in \Omega}} \int\limits_{\Omega} K(x - \xi) f(x) \mathrm{d}x^N & z \in \partial\Omega, \end{cases} \tag{23}$$

where

$$K(x) = \frac{1}{\omega_n} \left(\frac{(2 - n)e_1}{\rho^n(x)} - \frac{n\mathbf{x}e_1\mathbf{x}}{\rho^{n+2}(x)} \right), \ x \in \mathcal{R}_0^n. \tag{24}$$

$f \in H^\alpha(\overline{\Omega}, C(V_{n,n}))$, $0 < \alpha \leq 1$, Πf is a singular integral to be taken in the Cauchy principal sense. In [25], we have proved the existence and Hölder continuity of Πf in $\overline{\Omega}$.

For $u \in H^\alpha(\partial\Omega, C(V_{n,n}))$, $0 < \alpha \leq 1$, denote

$$(F_{\partial\Omega} u)(x) = \int\limits_{\partial\Omega} H_1^*(y - x) \mathrm{d}\sigma_y u(y), \ x \in \mathcal{R}^n \setminus \partial\Omega. \tag{25}$$

$$(S_{\partial\Omega}u)(x) = \int\limits_{\partial\Omega} H_1^*(y-x)\mathrm{d}\sigma_y u(y),\ x \in \partial\Omega. \tag{26}$$

$$\left(F_{\partial\Omega}^{+}u\right)(x) = \begin{cases} \left(F_{\partial\Omega}u\right)(x), & x \in \Omega^{+}, \\ \dfrac{1}{2}u(x) + \left(S_{\partial\Omega}u\right)(x) & x \in \partial\Omega. \end{cases} \tag{27}$$

Theorem 2. *Let Ω be an open non empty bounded subset of $\mathcal{R}^n$ with a Liapunov boundary $\partial\Omega$, $f \in C^1(\overline{\Omega}, C(V_{n,n}))$, Πf is defined as in (23). Then*

$$\Pi f(z) = \left(F_{\partial\Omega}^{+}(\alpha e_1 \alpha f)\right)(z) + T\left(e_1 D\left[f\right]\right)(z) - \frac{2-n}{n}e_1 f(z),\ z \in \overline{\Omega}, \tag{28}$$

where $\alpha(x)$ denotes the unit outer normal of $\partial\Omega$.

Proof. For $z \in \Omega$, by Stokes formula, we have,

$$\begin{aligned} \Pi f(z) &= \lim_{\varepsilon\to 0} \int\limits_{\Omega\setminus B(z,\varepsilon)} K(x-z) f(x)\mathrm{d}x^N \\ &= \lim_{\varepsilon\to 0} \int\limits_{\Omega\setminus B(z,\varepsilon)} \left[H_1^*(x-z)e_1\right] Df(x)\mathrm{d}x^N \\ &= \lim_{\varepsilon\to 0} \int\limits_{\partial(\Omega\setminus B(z,\varepsilon))} H_1^*(x-z)e_1\mathrm{d}\sigma_x f(x) + T\left(e_1 D\left[f\right]\right)(z) \\ &= \int\limits_{\partial\Omega} H_1^*(x-z)e_1\mathrm{d}\sigma_x f(x) + T\left(e_1 D\left[f\right]\right)(z) - \frac{2-n}{n}e_1 f(z). \end{aligned} \tag{29}$$

For $z \in \partial\Omega$, taking limits in (29), (28) follows.

Corollary 4. *Let Ω be an open non empty bounded subset of $\mathcal{R}^n$ with a Liapunov boundary $\partial\Omega$, $f \in \Lambda(f, \overline{\Omega})$, Πf is defined as in (23). Then in Ω*

$$D\left[\Pi f\right] = e_1 D\left[f\right] + \frac{n-2}{n} D\left[e_1 f\right]. \tag{30}$$

Corollary 5. *Suppose that f is regular in a domain U in $\mathcal{R}^n$, Ω is an open non empty bounded subset of U with a Liapunov boundary $\partial\Omega$. Πf is defined as in (23). Then in Ω*

$$\triangle\, [\Pi f] = 0, \tag{31}$$

where $\triangle$ is the Laplace operator.

3 Some applications

In this section, we shall give some applications of the higher order Cauchy-Pompeiu formula. The solutions of Dirichlet problems as well as the inhomogeneous equations $D^k u = f$ are obtained. In the sequel, K_n denotes the unit ball in $\mathcal{R}^n$ $(n \geq 3)$, more clearly,

$$K_n = \{x | x = (x_1, x_2, \cdots, x_n) \in \mathcal{R}^n, |x| < 1\}.$$

Denote

(32)

$$G(y,x) = \frac{1}{\rho^{n-2}(y-x)} - \frac{1}{|y|^{n-2}\rho^{n-2}(\frac{y}{|y|^2} - x)},\ x \in K_n,\ y \in \overline{K_n},\ x \neq y.$$

Remark 3. *$G(y,x)$ has the following properties:*

(1) $\triangle_x G(y,x) = 0,\ x \in K_n \setminus \{y\}$.

(2) $G(y,x) = G(x,y),\ x, y \in K_n,\ x \neq y$.

(3) $G(y,x) = 0,\ y \in \partial K_n,\ x \in K_n$.

Theorem 3. *Suppose $f \in C^2(\overline{K_n}, C(V_{n,n}))$, then for $x \in K_n$*

$$f(x) = \frac{1}{\omega_n}\int\limits_{\partial K_n} \frac{1-|x|^2}{\rho^n(y-x)} f(y)\mathrm{d}S_y + \frac{1}{(2-n)\omega_n}\int\limits_{K_n} G(y,x)\triangle_y f(y)\mathrm{d}y^N. \tag{33}$$

Proof. By Lemma 1, for $x \in K_n$, we have

(34)

$$f(x) = \frac{1}{\omega_n}\int\limits_{\partial K_n}\frac{\mathbf{y}-\mathbf{x}}{\rho^n(y-x)}\mathrm{d}\sigma_y f(y) - \frac{1}{(2-n)\omega_n}\int\limits_{\partial K_n}\frac{1}{\rho^{n-2}(y-x)}\mathrm{d}\sigma_y D[f](y)$$
$$+\frac{1}{(2-n)\omega_n}\int\limits_{K_n}\frac{1}{\rho^{n-2}(y-x)}\triangle_y f(y)\mathrm{d}y^N.$$

By Stokes formula, for $x \in K_n$ and $x \neq 0$, we have

(35)

$$0 = \frac{1}{\omega_n}\int\limits_{\partial K_n}\frac{\mathbf{y}-\dfrac{\mathbf{x}}{|x|^2}}{\rho^n(y-\dfrac{x}{|x|^2})}\mathrm{d}\sigma_y f(y) - \frac{1}{(2-n)\omega_n}\int\limits_{\partial K_n}\frac{1}{\rho^{n-2}(y-\dfrac{x}{|x|^2})}\mathrm{d}\sigma_y D[f](y)$$
$$+\frac{1}{(2-n)\omega_n}\int\limits_{K_n}\frac{1}{\rho^{n-2}(y-\dfrac{x}{|x|^2})}\triangle_y f(y)\mathrm{d}y^N.$$

(35) can be rewritten as

$$0 = \frac{1}{\omega_n}\int\limits_{\partial K_n}\frac{|x|^2\left(\mathbf{y}-\dfrac{\mathbf{x}}{|x|^2}\right)}{|x|^n\rho^n(y-\dfrac{x}{|x|^2})}\mathrm{d}\sigma_y f(y) - \tag{36}$$
$$-\frac{1}{(2-n)\omega_n}\int\limits_{\partial K_n}\frac{1}{|x|^{n-2}\rho^{n-2}(y-\dfrac{x}{|x|^2})}\mathrm{d}\sigma_y D[f](y)+$$
$$+\frac{1}{(2-n)\omega_n}\int\limits_{K_n}\frac{1}{|x|^{n-2}\rho^{n-2}(y-\dfrac{x}{|x|^2})}\triangle_y f(y)\mathrm{d}y^N.$$

In view of

$$|x|^k\rho^k(y-\frac{x}{|x|^2}) = |y|^k\rho^k(\frac{y}{|y|^2}-x),\ \ k \in \mathbf{N}^*, \tag{37}$$

combining (34), (36) with (37), (33) follows.

For $x = 0$, by Stokes formula and (34), (33) still holds. Thus the result is proved.

Remark 4. *Suppose $f \in C^2(\overline{K_n}, C(V_{n,n}))$, moreover, f is harmonic in K_n. Then for $x \in K_n$*

$$f(x) = \frac{1}{\omega_n} \int\limits_{\partial K_n} \frac{1-|x|^2}{\rho^n(y-x)} f(y) \mathrm{d}S_y. \tag{38}$$

(38) is exactly the Poisson expression of harmonic functions.

Theorem 4. *The solution of the Dirichlet problem for the Poisson equation in the unit ball K_n*

$$\triangle u = f \text{ in } K_n, \;\; u = \gamma \text{ on } \partial K_n,$$

for $f \in \Lambda(f, \overline{K_n})$ and $\gamma \in C(\partial K_n, C(V_{n,n}))$ is uniquely given by

$$u(x) = \frac{1}{\omega_n} \int\limits_{\partial K_n} \frac{1-|x|^2}{\rho^n(y-x)} \gamma(y) \mathrm{d}S_y + \frac{1}{(2-n)\omega_n} \int\limits_{K_n} G(y,x) f(y) \mathrm{d}y^N. \tag{39}$$

Proof. It can be directly proved by Corollary 2, Theorem 3, Remark 3 and Remark 4.

Lemma 2. *(see [26]) If f is k-regular in an open neighborhood Ω of the origin, then in a suitable open ball $\overset{\circ}{B}(0,R) \subset \Omega$*

$$f(x^N) = f(0) + \sum_{p=1}^{\infty} \sum_{j=0}^{k-1} \sum_{(l_1,\cdots,l_{p-j})} C_{j,p-j} \mathbf{x}^j V_{l_1,\cdots,l_{p-j}}(x^N) C_{l_1,\cdots,l_{p-j}}, \tag{40}$$

$C_{j,p-j}$ and $C_{l_1,\cdots,l_{p-j}}$ are constants which are suitably chosen.

By Lemma 2 and Corollary 2, we have

Theorem 5. *The solutions of inhomogeneous equations in the unit ball K_n*

$$D^k u = f \text{ in } K_n,$$

for $f \in \Lambda(f, \overline{K_n})$ *are given in a suitable open ball* $\overset{\circ}{B}(0,R) \subset K_n$ *by*

$$u = C_0 + \sum_{p=1}^{\infty}\sum_{j=0}^{k-1}\sum_{(l_1,\cdots,l_{p-j})} C_{j,p-j}\mathbf{x}^j V_{l_1,\cdots,l_{p-j}}(x^N)C_{l_1,\cdots,l_{p-j}} + T_k f. \tag{41}$$

C_0, $C_{j,p-j}$ *and* $C_{l_1,\cdots,l_{p-j}}$ *are constants which are suitably chosen.*

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