General Mathematics Vol. 13, No. 3 (2005), 31–36

The Identity of Three Classes of Polynomials

Andrei Vernescu

Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

It is well-known that, if $f \in \mathbb{R}[X]$, $\deg(f) = n \geq 2$ and Df divides f, then f is a scalar multiple of the n-th power of a monic polynomial of first degree, $X + a$, with a certain $a \in \mathbb{R}$ (it can be proved solving a simple differential equation which contains the associated polynomial function of f and its derivative). The converse assertion is obvious. In this paper, in the main result, we will show that, adding a simple supplimentar normating condition, the two classes defined by the mentioned properties also coincide with the class of the polynomials f which are reciprocal simultaneously with Df; but it results that $a = 1$. This result also will be considered in the general situation of the polynomials of $K[X]$, where K is an infinite commutative field an we will use only the formal derivative D. Finally we will pass in the umbral calculus and we will transpose the result in the case of a certain delta operator Q, in relation to its basic sequence $(p_n)_n$.

2000 Mathematics Subject Classification: 05A10, 12D05

Key words and phrases: Polynomial, formal derivative, reciprocal polynomial.

1. Let K be any infinite commutative field and $K^* = K\backslash\{0\}$; we will consider the divisibility in $K[X]$ in the usual sense. We will use the formal derivative, defined for any polynomial $f = a_nX^n + a_{n-1}X^{n-1} + \ldots + a_0$ (with $a_n \neq 0$) by the formula $Df = na_nX^n + (n-1)a_{n-1}X^{n-1} + \ldots + a_1$, for this formal derivative the usual properties also being valid.

A polynomial f with $\deg(f) \geq 1$ is called to be a reciprocal polynomial if the equalities $a_k = a_{n-k}$ are verified for any $k = 0, 1, \ldots, n$. For any reciprocal polynomial, we have $a_0 \neq 0$ (being equal to a_n) then (because $a_0 = f(0)$), we have $f(0) \neq 0$.

We present now the main result.

Theorem 1. *Let $f \in K[X]$ be, with $\deg(f) = n \geq 2$, and $a \in K^*$. Then the following affirmation are equivalent:*

(a) The polynomial Df divides f and $f(0) = \dfrac{(Df)(0)}{n} = a$.

(b) $f = a(X+1)^n$.

(c) The polynomial f reciprocal, DF also is reciprocal and $f(0) = a$.

Proof. (a) $\Longrightarrow$ (b) Because Df divides f, it exists $q \in K[X]$ such that:

$$f = (Df)q. \tag{1}$$

It results $\deg(q) = 1$, then it is $\alpha, \beta \in K$, $\alpha \neq 0$ such that $q = \alpha X + \beta$. Considering the coefficient of X^n of the both parts of the equality (1), we obtain $a_n = na_n\alpha$, then $\alpha = \dfrac{1}{n}$. Considering the free terms of the both parts of the same equality, it results $a_0 = a_1\beta$, or equivalent $f(0) = (Df)(0) \cdot \beta$. So, because one of hypothesis, we obtain $a = na\beta$, and so we also find $\beta = \dfrac{1}{n}$. So the equality (1) can be writen:

$$nf = (Df) \cdot (X+1). \tag{1'}$$

We fill now identify the coefficients of X^k from the two parts of $(1')$. We obtain:

$$na_k = (k+1)a_{k+1} + ka_k,$$

an equality which is true for any $k \geq 1$, but also for $k = 0$. It results:

$$(k+1)a_{k+1} = (n-k)a_k,$$

or, passing k in j,

$$(j+1)a_{j+1} = (n-j)a_j. \tag{2}$$

So, we obtain:
$j = 0 \quad \Rightarrow 1 \cdot a_1 = n \cdot a_0$
$j = 2 \quad \Rightarrow 2 \cdot a_2 = (n-1) \cdot a_1$
$\vdots$
$j = k-1 \Rightarrow k \cdot a_k = (n-k+1)a_{k-1}.$
Multiplying all these equalities, we obtain:

$$1 \cdot 2 \cdot \ldots \cdot k \cdot a_k = n(n-1) \cdot \ldots \cdot (n-k+1)a_0$$

and so, because $a_0 = a$, we have:

$$a_k = \binom{n}{k} a.$$

Therefore:

$$f = \sum_{k=0}^{n} a_k X^k = a \sum_{k=0}^{n} \binom{n}{k} X^k = a(X+1)^n$$

and we have obtained (b).

(b) implies (a) Obvious.

(b) implies (c) Obvious.

(c) implies (b) Because the polynomial f is reciprocal we have the equalities:

$$a_k = a_{n-k} \qquad (k = 0, 1, 2, \ldots, n). \tag{3}$$

Taking into account the expression of Df, the fact that Df also is reciprocal conducts us to equalize the coefficients of X^k and X^{n-k-1} (of Df). We obtain:

$$(k+1)a_{k+1} = (n-k)a_k \qquad (k = 0, 1, 2, \ldots, n-1). \tag{4}$$

Introducing a_{n-k} from (3) in (4), we obtain:

$$(k+1)a_{k+1} = (n-k)a_k, \tag{5}$$

i.e. we have again find the relation (3). And so, as in the proof of the implication (a)$\Longrightarrow$ (b), we obtain:

$$a_k = \binom{n}{k} a$$

and so

$$f = a(X+1)^n,$$

i.e. the point (b). The theorem 1 is proved.

So the three classes of polynomials considerated in the theorem coincide.

Also, we remark that if a polynomial f is reciprocal together with its derivative Df, then it is reciprocal together its successive derivatives Df, $D^2 f, \ldots, D^{n-1} f$.

2. We remember here some elements of umbral calculus.

It is known that a sequence of polynomials $(p_n)_n$ is said to be of binomial type if p_n is of degree n and the following equalities

$$p_n(u+v) = \sum_{k=0}^{n} \binom{n}{k} p_k(u) p_{n-k}(v) \tag{1.1}$$

are satisfied identically in u and v, for any non-negative integer n. We have: $p_0 = 1$ and $p_n(0) = 0$ for $n \geq 1$.

A simple example of polynomials of binomial type is represented by the monomials $e_n(x) = x^n$, $n \in \mathbb{N}$.

Let us denote by E^a the shift operator, defined by $(E^a f)(x) = f(x+a)$. An operator T which commutes with all shift operators is called a *shift-invariant operator*, that is $TE^a = E^a T$.

A *delta operator* Q is a shift-invariant operator for which Qe_1 is a non zero constant. Such operators possesse many of the properties of *the derivative operator* D, for which we have $De_n = ne_{n-1}$.

Here are some examples of delta operators: the forward difference Δ_h, the prederivative operator $D_h = \Delta/h$, the backward difference ∇_h and the central difference δ_h.

It is easy to see that: (i) for every delta operator Q we have $Qc = 0$, where c is a constant; (ii) if p_n is a polynomial of degree n, then, Qp_n is a polynomial of degree $n-1$.

A sequence of polynomials (p_n) is called by I.M.Sheffer [3] and Gian-Carlo Rota and his collaborators [1], [2], a sequence of *basic polynomials for a delta operator* Q if we have $p_0(x) = 1$, $p_n(0) = 0$, $(n \geq 1)$, while $Qp_n = np_{n-1}$.

J.F. Steffensen [4] observed that the property of $e_n(x) = x^n$ to be of binomial type can be extended to an arbitrary sequence of basic polynomials associated to a delta operator.

The following two results can be easily proved (see [1], pag. 182 – 183):

1) If (p_n) is a basic sequence of polynomials for a delta operator, then it is of binomial type;

2) If (p_n) is of binomial type, then it is a basic sequence for some delta operator.

By induction can be easily proved that every delta operator has a unique sequence of basic polynomials associated with it.

Examples (i) if $Q = D$ then $p_n(x) = x^n$;

(ii) if $Q = D_h = \Delta_h/h$ then:

$p_n(x) = x^{[n,h]} = x(x-h)\dots(x-(n-1)h)$.

3. The theorem 1 can be transposed in a more general context as following.

Theorem 2. Let be $f \in \mathbb{R}[X]$, $\deg(f) = n \geq 2$, $a \in \mathbb{R}^*$ and Q a delta operator. Then the following affirmations are equivalent.:

(a) The polynomial Qf divides the polynomial f and

$$f(0) = \frac{(Qf)(0)}{n} = a.$$

(b) $f = a \cdot (X+1)^n$.
(c) The polynomial f is reciprocal, Qf also is reciprocal, and $f(0) = a$.

References

[1] R. Mullin and G.-C. Rota, *On the foundations of combinatorial theory*, III. *Theory of binomial enumeration*. Graph Theory and its Applications, Academic Press, New York, 1970, 167 - 213.

[2] G.-C. Rota, *Finite Operator Calculus*. Academic Press, New York, 1975.

[3] I.M. Sheffer, *Some properties of polynomial sets of type zero*. Duke Math. J., **5**(1939), 590 - 622.

[4] J. F. Steffensen, *Interpolation*, Chelsea, New York, 1927.

Valahia University, Târgovişte
Department of Mathematics
Bd. Unirii, nr. 118,
130082 Târgovişte
E-mail: *avernescu@rdslink.ro*