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On a subclass of n-close to convex functions associated with some hyperbola

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

In this paper we define a subclass of n-close to convex functions associated with some hyperbola and we obtain some properties regarding this class.

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$, $A = \{f \in \mathcal{H}(U) : \ f(0) = f'(0) - 1 = 0\}$ and $S = \{f \in A : \ f \text{ is univalent in } U\}$.

We recall here the definition of the well - known class of close to convex functions:

$$CC = \left\{ f \in A \,:\, \text{exists } g \in S^*\,,\ Re\frac{zf'(z)}{g(z)} > 0,\ z \in U \right\}\,.$$

Let consider the Libera-Pascu integral operator $L_a : A \to A$ defined as:

$$(1) \qquad f(z) = L_a F(z) = \frac{1+a}{z^a} \int\limits_0^z F(t) \cdot t^{a-1} dt \ , \quad a \in \mathbb{C} \ , \quad Re\ a \geq 0.$$

For $a = 1$ we obtain the Libera integral operator, for $a = 0$ we obtain the Alexander integral operator and in the case $a = 1, 2, 3, ...$ we obtain the Bernardi integral operator.

Let D^n be the S'al'agean differential operator (see [6]) $D^n : A \to A$, $n \in \mathbb{N}$, defined as:

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

$$D^n f(z) = D(D^{n-1} f(z))$$

We observe that if $f \in S$, $f(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$, $z \in U$ then $D^n f(z) = z + \sum\limits_{j=2}^{\infty} j^n a_j z^j$.

The purpose of this note is to define a subclass of n-close to convex functions associated with some hyperbola and to obtain some estimations for the coefficients of the series expansion and some other properties regarding this class.

2 Preliminary results

Definition 1. (see [7]) *A function $f \in S$ is said to be in the class $SH(\alpha)$ if it satisfies*

$$\left| \frac{zf'(z)}{f(z)} - 2\alpha\left(\sqrt{2} - 1\right) \right| < Re \left\{ \sqrt{2} \frac{zf'(z)}{f(z)} \right\} + 2\alpha\left(\sqrt{2} - 1\right),$$

for some α $(\alpha > 0)$ and for all $z \in U$.

Definition 2. (see [2]) *Let $f \in S$ and $\alpha > 0$. We say that the function f is in the class $SH_n(\alpha)$, $n \in \mathbb{N}$, if*

$$\left|\frac{D^{n+1}f(z)}{D^n f(z)} - 2\alpha\left(\sqrt{2}-1\right)\right| < Re\left\{\sqrt{2}\,\frac{D^{n+1}f(z)}{D^n f(z)}\right\} + 2\alpha\left(\sqrt{2}-1\right),\ z \in U\,.$$

Remark 1. *Geometric interpretation: If we denote with p_α the analytic and univalent functions with the properties $p_\alpha(0) = 1$, $p'_\alpha(0) > 0$ and $p_\alpha(U) = \Omega(\alpha)$, where $\Omega(\alpha) = \{w = u + i \cdot v \,:\, v^2 < 4\alpha u + u^2\,,\ u > 0\}$ (note that $\Omega(\alpha)$ is the interior of a hyperbola in the right half-plane which is symmetric about the real axis and has vertex at the origin), then $f \in SH_n(\alpha)$ if and only if $\frac{D^{n+1}f(z)}{D^n f(z)} \prec p_\alpha(z)$, where the symbol $\prec$ denotes the subordination in U. We have $p_\alpha(z) = (1+2\alpha)\sqrt{\frac{1+bz}{1-z}} - 2\alpha\,,\ b = b(\alpha) = \frac{1+4\alpha-4\alpha^2}{(1+2\alpha)^2}$ and the branch of the square root $\sqrt{w}$ is chosen so that $Im\,\sqrt{w} \geq 0$. If we consider $p_\alpha(z) = 1 + C_1 z + \dots$, we have $C_1 = \frac{1+4\alpha}{1+2\alpha}$.*

Remark 2. *If we denote by $D^n g(z) = G(z)$,we have: $g \in SH_n(\alpha)$ if and only if $G \in SH(\alpha) = SH_0(\alpha)$.*

Theorem 1. (see [2]) *If $F(z) \in SH_n(\alpha)$, $\alpha > 0$, $n \in \mathbb{N}$, and $f(z) = L_a F(z)$, where L_a is the integral operator defined by (1), then $f(z) \in SH_n(\alpha)$, $\alpha > 0$, $n \in \mathbb{N}$.*

Definition 3. (see [1]) *Let $f \in A$ and $\alpha > 0$. We say that the function f is in the class $CCH(\alpha)$ with respect to the function $g \in SH(\alpha)$ if*

$$\left|\frac{zf'(z)}{g(z)} - 2\alpha\left(\sqrt{2}-1\right)\right| < Re\left\{\sqrt{2}\,\frac{zf'(z)}{g(z)}\right\} + 2\alpha\left(\sqrt{2}-1\right),\ z \in U\,.$$

Remark 3. *Geometric interpretation: $f \in CCH(\alpha)$ with respect to the function*
$g \in SH(\alpha)$ if and only if $\frac{zf'(z)}{g(z)}$ take all values in the convex domain $\Omega(\alpha)$, where $\Omega(\alpha)$ is defined in Remark 1.

Theorem 2. (see [1]) *If* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ *belong to the class* $CCH(\alpha)$, $\alpha > 0$, *with respect to the function* $g(z) \in SH(\alpha)$, $\alpha > 0$, $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$, *then*

$$|a_2| \le \frac{1+4\alpha}{1+2\alpha}\ ,\ |a_3| \le \frac{(1+4\alpha)(11+56\alpha+72\alpha^2)}{12(1+2\alpha)^3}\ .$$

The next theorem is result of the so called "admissible functions method" due to P.T. Mocanu and S.S. Miller (see [3], [4], [5]).

Theorem 3. *Let* q *be convex in* U *and* $j : U \to \mathbb{C}$ *with* $Re[j(z)] > 0$, $z \in U$. *If* $p \in \mathcal{H}(U)$ *and satisfied* $p(z) + j(z) \cdot zp'(z) \prec q(z)$, *then* $p(z) \prec q(z)$.

3 Main results

Definition 4. *Let* $f \in A$, $n \in \mathbb{N}$ *and* $\alpha > 0$. *We say that the function* f *is in the class* $CCH_n(\alpha)$, *with respect to the function* $g \in SH_n(\alpha)$, *if*

$$\left|\frac{D^{n+1}f(z)}{D^n g(z)} - 2\alpha\left(\sqrt{2}-1\right)\right| < Re\left\{\sqrt{2}\,\frac{D^{n+1}f(z)}{D^n g(z)}\right\} + 2\alpha\left(\sqrt{2}-1\right)\ ,\ z \in U\ .$$

Remark 4. *Geometric interpretation:* $f \in CCH_n(\alpha)$, *with respect to the function* $g \in SH_n(\alpha)$, *if and only if* $\frac{D^{n+1}f(z)}{D^n g(z)} \prec p_\alpha(z)$, *where the symbol* $\prec$ *denotes the subordination in* U *and* p_α *is defined in Remark 1.*

Remark 5. *If we denote* $D^n f(z) = F(z)$ *and* $D^n g(z) = G(z)$ *we have:*

$f \in CCH_n(\alpha)$, *with respect to the function* $g \in SH_n(\alpha)$, *if and only if* $F \in CCH(\alpha)$, *with respect to the function* $G \in SH(\alpha)$ *(see Remark 2).*

Theorem 4. *Let* $\alpha > 0$, $n \in \mathbb{N}$ *and* $f \in CCH_n(\alpha)$, $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$, *with respect to the function* $g \in SH_n(\alpha)$, *then*

$$|a_2| \le \frac{1}{2^n} \cdot \frac{1+4\alpha}{1+2\alpha}\ ,\ |a_3| \le \frac{1}{3^n} \cdot \frac{(1+4\alpha)(11+56\alpha+72\alpha^2)}{12(1+2\alpha)^3}\ .$$

Proof. If we denote by $D^n f(z) = F(z)$, $F(z) = \sum_{j=2}^{\infty} b_j z^j$, we have (using Remark 5) from the above series expansions we obtain $|a_j| \leq \frac{1}{j^n} \cdot |b_j|$, $j \geq 2$. Using the estimations from the Theorem 2 we obtain the needed results.

Theorem 5. *Let $\alpha > 0$ and $n \in \mathbb{N}$. If $F(z) \in CCH_n(\alpha)$, with respect to the function $G(z) \in SH_n(\alpha)$, and $f(z) = L_a F(z)$, $g(z) = L_a G(z)$, where L_a is the integral operator defined by (1), then $f(z) \in CCH_n(\alpha)$, with respect to the function $g(z) \in SH_n(\alpha)$.*

Proof. By differentiating (1) we obtain $(1+a)F(z) = af(z) + zf'(z)$ and $(1+a)G(z) = ag(z) + zg'(z)$.

By means of the application of the linear operator D^{n+1} we obtain

$$(1+a)D^{n+1}F(z) = aD^{n+1}f(z) + D^{n+1}(zf'(z))$$

or

$$(1+a)D^{n+1}F(z) = aD^{n+1}f(z) + D^{n+2}f(z)$$

Similarly, by means of the application of the linear operator D^n we obtain

$$(1+a)D^n G(z) = aD^n g(z) + D^{n+1} g(z)$$

Thus

$$\frac{D^{n+1}F(z)}{D^n G(z)} = \frac{D^{n+2}f(z) + aD^{n+1}f(z)}{D^{n+1}g(z) + aD^n g(z)} =$$

$$= \frac{\frac{D^{n+2}f(z)}{D^{n+1}g(z)} \cdot \frac{D^{n+1}g(z)}{D^n g(z)} + a \cdot \frac{D^{n+1}f(z)}{D^n g(z)}}{\frac{D^{n+1}g(z)}{D^n g(z)} + a} \tag{2}$$

With notations $\frac{D^{n+1}f(z)}{D^n g(z)} = p(z)$ and $\frac{D^{n+1}g(z)}{D^n g(z)} = h(z)$, by simple calculations, we have

$$\frac{D^{n+2}f(z)}{D^{n+1}g(z)} = p(z) + \frac{1}{h(z)} \cdot zp'(z)$$

Thus from (2) we obtain

$$\frac{D^{n+1}F(z)}{D^nG(z)} = \frac{h(z)\cdot\left(zp'(z)\cdot\dfrac{1}{h(z)}+p(z)\right)+a\cdot p(z)}{h(z)+a} =$$

$$(3) \qquad = p(z) + \frac{1}{h(z)+a}\cdot zp'(z)$$

From Remark 4 we have $\dfrac{D^{n+1}F(z)}{D^nG(z)} \prec p_\alpha(z)$ and thus, using (3), we obtain

$$p(z) + \frac{1}{h(z)+a} zp'(z) \prec p_\alpha(z)\,.$$

We have from Remark 1 and from the hypothesis $Re\dfrac{1}{h(z)+a} > 0\,,\ z \in U\,.$ In this conditions from Theorem 3 we obtain $p(z) \prec p_\alpha(z)$ or $\dfrac{D^{n+1}f(z)}{D^ng(z)} \prec p_\alpha(z)$. This means that $f(z) = L_aF(z) \in CCH_n(\alpha)\,,$ with respect to the function $g(z) = L_aG(z) \in SH_n(\alpha)$ (see Theorem 1).

Theorem 6. *Let $a \in \mathbb{C}$, $Re\, a \geq 0$, $\alpha > 0$, and $n \in \mathbb{N}$. If $F(z) \in CCH_n(\alpha)$, with respect to the function $G(z) \in SH_n(\alpha)\,,$ $F(z) = z + \sum_{j=2}^{\infty} a_j z^j$, and $g(z) = L_aG(z)\,,$ $f(z) = L_aF(z)\,,$ $f(z) = z + \sum_{j=2}^{\infty} b_j z^j$, where L_a is the integral operator defined by (1), then*

$$|b_2| \leq \left|\frac{a+1}{a+2}\right| \cdot \frac{1}{2^n}\cdot\frac{1+4\alpha}{1+2\alpha}\ ,\ |b_3| \leq \left|\frac{a+1}{a+3}\right|\cdot\frac{1}{3^n}\cdot\frac{(1+4\alpha)(11+56\alpha+72\alpha^2)}{12(1+2\alpha)^3}\,.$$

Proof. From $f(z) = L_aF(z)$ we have $(1+a)F(z) = af(z) + zf'(z)\,.$ Using the above series expansions we obtain

$$(1+a)z + \sum_{j=2}^{\infty}(1+a)a_j z^j = az + \sum_{j=2}^{\infty} ab_j z^j + z + \sum_{j=2}^{\infty} jb_j z^j$$

and thus $b_j(a+j) = (1+a)a_j\,,\ j \geq 2$. From the above we have $|b_j| \leq \left|\frac{a+1}{a+j}\right| \cdot |a_j|\,,\ j \geq 2$. Using the estimations from Theorem 4 we obtain the needed results.

For $a = 1$, when the integral operator L_a become the Libera integral operator, we obtain from the above theorem:

Corollary 1. *Let $\alpha > 0$ and $n \in \mathbb{N}$. If $F(z) \in CCH_n(\alpha)$, with respect to the function $G(z) \in SH_n(\alpha)\,,\ F(z) = z + \sum_{j=2}^{\infty} a_j z^j$, and $g(z) = L(G(z))\,,$ $f(z) = L(F(z))\,,\ f(z) = z + \sum_{j=2}^{\infty} b_j z^j$, where L is Libera integral operator defined by $L(H(z)) = \frac{2}{z}\int_0^z H(t)dt\,,$ then*

$$|b_2| \leq \frac{1}{2^{n-1}} \cdot \frac{1+4\alpha}{3+6\alpha}\,,\ |b_3| \leq \frac{1}{3^n} \cdot \frac{(1+4\alpha)(11+56\alpha+72\alpha^2)}{24(1+2\alpha)^3}\,.$$

Theorem 7. *Let $n \in \mathbb{N}$ and $\alpha > 0$. If $f \in CCH_{n+1}(\alpha)$ then $f \in CCH_n(\alpha)$.*

Proof. With notations $\frac{D^{n+1}f(z)}{D^n g(z)} = p(z)$ and $\frac{D^{n+1}g(z)}{D^n g(z)} = h(z)$ we have (see the proof of the Theorem 5):

$$\frac{D^{n+2}f(z)}{D^{n+1}g(z)} = p(z) + \frac{1}{h(z)} \cdot zp'(z)\,.$$

From $f \in CCH_{n+1}(\alpha)$ we obtain (see Remark 4) $p(z) + \frac{1}{h(z)} \cdot zp'(z) \prec p_\alpha(z)$. Using the Remark 1 we have $Re\frac{1}{h(z)} > 0\,,\ z \in U$, and from Theorem 3 we obtain $p(z) \prec p_\alpha(z)$ or $f \in CCH_n(\alpha)$.

Remark 6. *From the above theorem we obtain $CCH_n(\alpha) \subset CCH_0(\alpha) = CCH(\alpha)$ for all $n \in \mathbb{N}$.*

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