General Mathematics Vol. 13, No. 3 (2005), 47–60

A General Schlicht Integral Operator

Eugen Drăghici

Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

Let A be the class of analytic functions f in the open complex unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$, with $f(0) = 0, f'(0) = 1$ and $f(z)/z \neq 0$ in U. Let define the integral operator $I : A \to A$, $I(f) = F$, where:

$$F(z) = \left[(\alpha+\beta+1)\int_0^z f^{\alpha}(u)g^{\beta}(u)\right]^{1/(\alpha+\beta+1)}, \quad z \in U$$

With suitable conditions on the constants α and β and on the function $g \in A$, the author shows that F is analytic and univalent (or schlicht) in U. Additional results are also obtained, such as a new generalization of Becker's condition of univalence and improvements of some known results.

2000 Mathematics Subject Classification: 30C45.

Keywords: univalence, subordination chain.

1 Introduction

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the complex unit disc and let A be the class of analytic functions in U of the form:

$$f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \cdots$$

and with $f(z)/z \neq 0$ for all $z \in U$.

Univalence of complex functions is an important property, but, unfortunately, it is difficult and in many cases impossible to show directly that a certain complex function is univalent. For this reason, many authors found different types of sufficient conditions of univalence. One of these conditions of univalence is the well–known criterion of Ahlfors and Becker ([1] and [7]), which states that the function $f \in A$ is univalent if:

$$(1) \qquad (1 - |z|^2) \left| \frac{zf'(z)}{f(z)} \right| \leq 1$$

There are many generalizations of this criterion, such those obtained in [4], [5], [6] and [9]. In this paper, as an additional result, we will also obtain a new generalization of the above–mentioned univalence criterion. But, the principal result deals with finding sufficient conditions on the constants α and β and on the function $g \in A$ so that the function:

$$(2) \qquad F(z) = \left[(\alpha + \beta + 1) \int_0^z f^\alpha(u) g^\beta(u) du \right]^{1/(\alpha+\beta+1)}, \quad z \in U$$

is univalent.The result improves also former results obtained in [3], [4], [5], [6] and [7].

2 Preliminaries

For proving our principal result we will need the following definitions and lemma:

Definition 1. *If f and g are analytic functions in U and g is univalent, then we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$ if $f(0) = g(0)$ and $f(U) \subset g(U)$.*

Definition 2. *A function $L(z,t)$, $z \in U$, $t \geq 0$ is called a Lőwner chain or a subordination chain if:*

(i) $L(\cdot, t)$ *is analytic and univalent in U for all $t \geq 0$.*

(ii) $L(z, \cdot)$ *is continuously differentiable in $[0, \infty)$ for all $t \geq 0$.*

(iii) $L(z, s) \prec L(z, t)$ *for all real s and t with $0 \leq s < t$.*

Let $0 < r \leq 1$. We denote by U_r the set: $U_r = \{z \in \mathbb{C} : |z| < r\}$.

Lemma 1. (see [8], [9]) *Let $0 < r_0 \leq 1$, $t \geq 0$ and $a_1(t) \in \mathbb{C} \setminus \{0\}$. Let:*

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$$

be analytic in U_{r_0} for all $t \geq 0$, locally absolutely continuous in $[0, \infty)$ locally uniform with respect to U_{r_0}. For almost all $t \geq 0$ suppose that:

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \quad z \in U_{r_0} \tag{3}$$

where $p(z,t)$ is analytic in the unit disc U and $\operatorname{Re} p(z) > 0$ in U for all $t \geq 0$.If:

$$\lim_{t\to\infty} |a_1(t)| = \infty$$

and $\{L(z,t)/a_1(t)\}$ *forms a normal family in* U_{r_0}*, then, for each* $t \geq 0$*,* $L(z,t)$ *has an analytic and univalent extension to the whole unit disc* U *and is a Lőwner chain.*

Lemma 1 is a variant of the well–known theorem of Pommerenke ([8]) and it's proof can be found in [9].

3 Principal result

Let B be the class of analytic functions p in U with $p(0)=1$ and $p(z)\neq 0$ for all $z \in U$.

Theorem 1. *Let* $f, g \in A$*,* $p \in B$ *and* α, β, γ *and* δ *complex numbers satisfying:*

$$\operatorname{Re}\frac{\gamma}{\alpha+\beta+1} > \frac{1}{2} \tag{4}$$

$$\operatorname{Re}(\alpha+\beta+1) > 0 \tag{5}$$

$$\operatorname{Re}\gamma > 0 \tag{6}$$

$$\left|\frac{\delta+1}{\gamma p(z)} - 1\right| < 1,\ z \in U \tag{7}$$

$$\left|\frac{\delta+1}{\alpha+\beta+1} - 1\right| < 1 \tag{8}$$

and, for all $z \in U$*:*

$$\left|\frac{1-\gamma}{\gamma} + \frac{1+\delta-p(z)}{\gamma p(z)}|z|^{2\gamma} + \frac{1-z^{2\gamma}}{\gamma}\left[\alpha\frac{zf'(z)}{f(z)} + \beta\frac{zg'(z)}{g(z)} + \frac{zp'(z)}{p(z)}\right]\right| \leq 1 \tag{9}$$

Then, the function F *defined by* (2) *is analytic and univalent in* U*.*

Proof. Let :

$$h(u) = \left[\frac{f(u)}{u}\right]^{\alpha}\left[\frac{g(u)}{u}\right]^{\beta}$$

where the powers are considered with their principal branches. The function h does not vanish in U because f and g are in A.Let define now the function:

$$h_1(z,t) = \frac{\alpha+\beta+1}{(\mathrm{e}^{-t}z)^{\alpha+\beta+1}}\int_0^{\mathrm{e}^{-t}z} h(u)u^{\alpha+\beta}du = 1 + b_1 z + \cdots$$

where $t \geq 0$ and $z \in U$. We consider now the power development of h:

$$h(u) = 1 + \sum_{n=1}^{\infty} c_n u^n \ , \ u \in U.$$

We denote:

$$\phi(w) = \frac{\alpha+\beta+1}{w^{\alpha+\beta+1}}\int_0^{w} h(u)u^{\alpha+\beta}du = 1 + \sum_{n=1}^{\infty} c_n \frac{\alpha+\beta+1}{n+\alpha+\beta+1} w^n.$$

From (5) we have that $\mathrm{Re}(\alpha+\beta+1) > 0$ and, consequently:
$\mathrm{Re}(\alpha+\beta+1 > -n/2$ for all $n \in \mathbb{N}$. It follows immediately that:

$$\mathrm{Re}\,\frac{n}{n+2(\alpha+\beta+1)} > 0 \ , \ n \in \mathbb{N}$$

and hence:

$$\left|\frac{\alpha+\beta+1}{n+\alpha+\beta+1}\right| < 1.$$

Taking into account that h is analytic in U, we deduce that:

$$1 + \sum_{n=1}^{\infty} c_n \frac{\alpha+\beta+1}{n+\alpha+\beta+1} w^n$$

converges locally uniformly in U, and, thus, ϕ is analytic in U. Because for every $t \geq 0$ and for every $z \in U$ we have that $\mathrm{e}^{-t}z \in U$ we deduce that $\phi(\mathrm{e}^{-t}z) = h_1(z,t)$ is analytic in U for all $t \geq 0$. Let now:

$$m = \frac{\alpha+\beta+1}{\delta+1}$$

$$h_2(z,t) = p(\mathrm{e}^{-t}z)h(\mathrm{e}^{-t}z)\ ,\ z \in U\ ,\ t \geq 0$$

$$h_3(z,t) = h_1(z,t) + m(\mathrm{e}^{2\gamma t} - 1)h_2(z,t)\ ,\ z \in U\ ,\ t \geq 0.$$

Suppose now that $h_3(0,t_1) = 0$ for a certain positive rel number t_1, that is $1 + m(\mathrm{e}^{2\gamma t_1} - 1) = 0$, or:

$$\mathrm{e}^{2\gamma t_1} = \frac{m-1}{m} = \frac{\alpha+\beta-\delta}{\alpha+\beta+1}. \tag{10}$$

From (6) we have that $|\mathrm{e}^{2\gamma t_1}| = \mathrm{e}^{2t_1 \operatorname{Re}\gamma} \geq 1$ and from (8) we deduce that $\left|\frac{\alpha+\beta-\delta}{\alpha+\beta+1}\right| < 1$. It follows immediately that (10) is false and then, we have:

$$h_3(0,t) \neq 0 \ \text{ for all } \ t \geq 0 \tag{11}$$

Let now suppose that for all r with $0 < r \leq 1$ it exists at least one $t_r \geq 0$ so that $h_3(z,t_r)$ has at least one zero in $U_r = \{z \in \mathbb{C} : |z| < r\}$. We choose $r = 1, 1/2, 1/3, \ldots$ and form a sequence $(t_n)_{n\in\mathbb{N}}$ so that $h_3(z,t_n)$ has at least one zero in $U_{1/n}$.

If $(t_n)_{n\in\mathbb{N}}$ is bounded, we can find a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ of $(t_n)_{n\in\mathbb{N}}$ that converges to $\tau_0 \geq 0$. Because h_3 is continuously with respect to t we obtain:

$$\lim_{k\to\infty} h_3(z,t_{n_k}) = h_3(z,\tau_0) \ \text{ for all } \ z \in U.$$

But in this case $h_2(\cdot,\tau_0)$ has at least one zero in every disc U_{1/n_k}. If we let now $k \to \infty$ we deduce that $h_3(0,\tau_0) = 0$, which contradicts (11).

If the sequence $(t_n)_{n\in\mathbb{N}}$ is umbounded we can consider, without loss of generality, that $\lim_{n\to\infty} t_n = \infty$. We have now:

$$h_3(z,t) = h_1(z,t) + m(\mathrm{e}^{2\gamma t} - 1)h_2(z,t) = \phi(\mathrm{e}^{-t}z) + m(\mathrm{e}^{2\gamma t} - 1)h_2(z,t)$$

Because $\phi(0) = 1$ we deduce that $M = \max_{z\in\overline{U}} |\phi(\mathrm{e}^{-t}z)| > 0$. Because $p(0)h(0) = 1$, there exists $r_1 \in (0,1]$ so that $p(w)h(w) \neq 0$ in $\overline{U}_{r_1}$. Then, $h_2(w,t) = p(\mathrm{e}^{-t}z)h(\mathrm{e}^{-t}z)$ do not vanish in $\overline{U}_{r_1}$ for every $t \geq 0$ and, thus, we have: $K = \min_{w\in\overline{U}_{r_1}} |h_2(w,t)| > 0$. From (5) we deduce that $m \neq 0$ and thus, $|m| > 0$. It follows immediately that:

$$\lim_{t\to\infty} \left|1 - \mathrm{e}^{2\gamma t}\right| = \lim_{t\to\infty} \mathrm{e}^{2t\,\mathrm{Re}\,\gamma}\sqrt{\mathrm{e}^{-4t\,\mathrm{Re}\,\gamma} - 2\mathrm{e}^{-2t\,\mathrm{Re}\,\gamma}\cos 2t\,\mathrm{Im}\,\gamma + 1} = \infty$$

because $\mathrm{Re}\,\gamma > 0$.

Hence, for sufficiently large t we have:

$$(12) \qquad |m|\left|1 - \mathrm{e}^{2\gamma t}\right||h_2(z,t)| > |m|\left|1 - \mathrm{e}^{2\gamma t}\right| K > M + 1 > \left|\phi(\mathrm{e}^{-t}z) + 1\right|$$

In the same time we have:

$$\begin{aligned}|h_3(z,t)| &= \left|h_1(z,t) - m\left(1 - \mathrm{e}^{2\gamma t}\right)h_2(z,t)\right| \geq \\ &\geq \left||h_1(z,t)| - |m|\left|1 - \mathrm{e}^{2\gamma t}\right||h_2(z,t)|\right|\end{aligned}$$

From (12) it follows immediately that $|h_3(z,t)| > 1$ for all $z \in U_{r_1}$ and for sufficiently large t. Thus, it exists $N \in \mathbb{N}$ so that $h_3(\cdot, t_n)$ does not vanish in U_{r_1} for all $n > N$. For $n \in [0, N]$ we have that $h_3(z, t_n)$ does not vanish in U_{r2} where:

$$r_2 = \min\{r_{t_n} : h_3(z,t) \neq 0, z \in U_{r_{t_n}}, t \geq 0, n \in [0,N]\}.$$

If we let now $r_0 = \min\{r_1, r_2\}$ we have that $h_3(\cdot, t_n)$ does not vanish in U_{r_0} for every $n \in \mathbb{N}$. It follows that the supposition of the nonexistence of a positive real number $r_0 < 1$ with the property that $h_3(z,t) \neq 0$ for all $t \geq 0$ and all $z \in U_{r_0}$ is false. Hence, we can choose $r_0 \in (0,1]$ so that $h_3(z,t) \neq 0$

for all $t \geq 0$ and all $z \in U_{r_0}$.
Let $h_4(z,t)$ be the uniform branch of $[h_3(z,t)]^{1/(\alpha+\beta+1)}$ which takes the value $[1+m\,(\mathrm{e}^{2\gamma t}-1)]^{1/(\alpha+\beta+1)}$ at the origin. Let us define:

$$L(z,t) = \mathrm{e}^{-t} z h_4(z,t) \tag{13}$$

which is analytic for all $t \geq 0$.If $L(z,t) = a_1(t)z + a_2(z)z^2 + \cdots$, it is clear that $L(0,t) = 0$ for every $t \geq 0$ and:

$$a_1(t) = \mathrm{e}^{-t}\left[1 + m\left(\mathrm{e}^{2\gamma t} - 1\right)\right]^{1/(\alpha+\beta+1)}.$$

From the above written equations we can formally write:

$$\begin{aligned} L(z,t) = [L_1(z,t)]^{1/(\alpha+\beta+1)} = [(\alpha+\beta+1)\int_0^{\mathrm{e}^{-t}z} f^{\alpha}(u) g^{\beta}(u) du + \\ + m(\mathrm{e}^{2\gamma t}-1)\mathrm{e}^{-t} z f^{\alpha}(\mathrm{e}^{-t}z) g^{\beta}(\mathrm{e}^{-t}z) p(\mathrm{e}^{-t}z)]^{1/(\alpha+\beta+1)}. \end{aligned} \tag{14}$$

By simple calculations we obtain:

$$a_1(t) = (c+1)^{-\frac{1}{\alpha+\beta+1}} \mathrm{e}^{\frac{2\gamma-\alpha-\beta-1}{\alpha+\beta+1}} \left[\alpha+\beta+1-(\alpha+\beta-c)\mathrm{e}^{-2\gamma t}\right]^{\frac{1}{\alpha+\beta+1}}.$$

Thus, $\mathrm{e}^t a_1(t) = h_4(0,t) = [h_3(0,t)]^{1/(\alpha+\beta+1)}$ with the choosen uniform branch. Because $h_3(\cdot,t)$ does not vanish in U_{r_0} for all $t \geq 0$, we obtain that $a_1(t) \neq 0$ for every $t \geq 0$. If we let $t \to \infty$, from (4) and (6) we easily obtain:

$$\lim_{t\to\infty} |a_1(t)| = \infty.$$

Because $h_4(\cdot,t)$ is analytic in U_{r_0} for every $t \geq 0$, we deduce that $L(z,t) = \mathrm{e}^{-t} z h_4(z,t)$ is also analytic in U_{r_0} for all $t \geq 0$.The family $\{L(z,t)/a_1(t)\}_{t\geq 0}$ consists of analytic functions in U_{r_0}. Hence, this family is uniformly bounded in U_{r_1}, where $0 < r_1 \leq r_0$. By applying Montelş theorem we have that $\{L(z,t)/a_1(t)\}$ forms a normal family in U_{r_1}. Let denote:

$$J(z,t) = m(e^{2\gamma t}-1)\left[\alpha\frac{e^{-t}zf'(e^{-t}z)}{f(e^{-t}z)} + \beta\frac{e^{-t}zg'(e^{-t}z)}{g(e^{-t}z)} + \frac{e^{-t}zp'(e^{-t}z)}{p(e^{-t}z)}\right]p(e^{-t}z) \tag{15}$$

From (14) we obtain:

$$\frac{\partial L(z,t)}{\partial t} = \frac{1}{\alpha+\beta+1}\left[L_1(z,t)\right]^{-\frac{\alpha+\beta}{\alpha+\beta+1}} e^{-t}zf^{\alpha}(e^{-t}z)g^{\beta}(e^{-t}z)\cdot$$
$$\cdot\left[2\gamma m e^{2\gamma t}p(e^{-t}z) - m(e^{2\gamma t}-1)p(e^{-t}z) - \alpha - \beta - 1 - J(z,t)\right]$$

It is clear that $\partial L(z,t)/\partial t$ is analytic in U_{r_2}, where $0 < r_2 \leq r_1$. Consequently, $L(z,t)$ is locally absolutely continuous and we have also:

$$\frac{\partial L(z,t)}{\partial z} = \frac{1}{\alpha+\beta+1}\left[L_1(z,t)\right]^{-\frac{\alpha+\beta}{\alpha+\beta+1}} e^{-t}zf^{\alpha}(e^{-t}z)g^{\beta}(e^{-t}z)\cdot$$
$$\cdot\left[m(e^{2\gamma t}-1)p(e^{-t}z) + \alpha + \beta + 1 + J(z,t)\right]$$

Let:

$$p_1(z,t) = \frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t} = \frac{m(e^{2\gamma t}-1)p(e^{-t}z) + \alpha+\beta+1+J(z,t)}{(2\gamma-1)me^{2\gamma t}p(e^{-t}z) + mp(e^{-t}z)}$$

Consider now the function:

$$w(z,t) = \frac{p_1(z,t)-1}{p_1(z,t)+1}$$

Further calculations show that:

$$w(z,t) = \frac{m(1-\gamma)e^{2\gamma t}p(e^{-t}z) - mp(e^{-t}z) + \alpha+\beta+1+J(z,t)}{\gamma m e^{2\gamma t}p(e^{-t}z)}$$

It is clear that $w(\cdot,t)$ is analytic in U_{r_2} for all $t \geq 0$. Hence, $w(\cdot,t)$ has an analytic extension $\tilde{w}(\cdot,t)$.

Let now $t = 0$. Taking into account that $m = (\alpha + \beta + 1)/(\delta + 1)$, we easily obtain from (15):

$$\tilde{w}(z, 0) = -1 + \frac{c+1}{\gamma p(z)}$$

and from (7) it follows immediately that $|\tilde{w}(z, 0)| < 1$.

Let now $t > 0$. We observe that $\tilde{w}(\cdot, t)$ is analytic in $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ because if $t \geq 0$, for every $z \in \overline{U}$ we have that $\mathrm{e}^{-t}z \in U$. In this case we have:

$$|\tilde{w}(z, t)| = \max_{z \in \overline{U}} |\tilde{w}(z, t)| = \max_{|z| = 1} |\tilde{w}(z, t)| = |\tilde{w}(\mathrm{e}^{\mathrm{i}\theta}, t)|$$

with $\theta \in \mathbb{R}$. Let $v = \mathrm{e}^{-t}\mathrm{e}^{\mathrm{i}\theta} \in U$. After simple calculations we obtain:

$$\tilde{w}(\mathrm{e}^{\mathrm{i}\theta}, t) = \frac{1-\gamma}{\gamma} + \frac{\alpha + \beta + 1 - mp(v)}{\gamma mp(v)}|v|^{2\gamma} + \\ + \frac{1 - |v|^{2\gamma}}{\gamma}\left[\alpha\frac{vf'(v)}{f(v)} + \beta\frac{vg'(v)}{g(v)} + \frac{vp'(v)}{p(v)}\right]$$

But:

$$\frac{\alpha + \beta + 1 - mp(v)}{\gamma mp(v)} = \frac{\delta + 1 - p(v)}{\gamma p(v)}$$

and from (9) we deduce that $|\tilde{w}(\mathrm{e}^{\mathrm{i}\theta}, t)| \leq 1$ and hence, $|\tilde{w}(z, t)| < 1$ in U for all $t \geq 0$. From the definition of w and $\tilde{w}$ we deduce that $p_1(\cdot, t)$ has an analytic extension $\tilde{p}_1(\cdot, t)$ to the whole disc U for all $t \geq 0$ and $\mathrm{Re}\,\tilde{p}_1(z, t) > 0$ in U for all $t \geq 0$. By applying **Lemma1** we obtain that $L(z, t)$ is a subordination chain and thus, $L(z, 0) = F(z)$ is analytic and univalent in U and the proof of the theorem is complete.

Remark 1. *We can write a variant of* **Theorem 1** *with* $\gamma \in \mathbb{R}$. *In this case, condition (8) can be replaced by:*

(16) $$1 - \frac{\delta + 1}{\alpha + \beta + 1} \notin [1, \infty)$$

However, condition (8) was necessary only for showing that $h_2(0,t) \neq 0$ for all $t \geq 0$. But if $\gamma \in \mathbb{R}$ then $h_2(0,t) = 0$ is equivalent to $e^{2\gamma t} = (m-1)/m \in \mathbb{R}$. Bat this last equality is impossible because $e^{2\gamma t} > 1$ and $(m-1)/m \notin [1,\infty)$.

4 Some particular cases

If we let in **Theorem 1** $\gamma = 1$ and $p(z) = 1$ for all $z \in U$, then we obtain, using **Remark 1** also, the following result:

Corollary 1. *If $f, g \in A$ and α, β and δ are complex numbers satisfying:*

$$|\alpha + \beta| < 1 \tag{17}$$

$$|\delta| < 1 \tag{18}$$

$$1 - (\delta + 1)/(\alpha + \beta + 1) \notin [1, \infty) \tag{19}$$

$$\left| c|z|^2 + (1 - |z|^2)\left[\alpha \frac{zf'(z)}{f(z)} + \beta \frac{zg'(z)}{g(z)}\right]\right| \leq 1\ ,\ z \in U \tag{20}$$

then the function F defined in (2) *is analytic and univalent in U.*

If in **Corollary 1** we let $\delta = \alpha + \beta$ we obtain **Theorem 1** from [5] and if we let additionally $g(z) = z$ for all $z \in U$ we obtain **Theorem 1** from [4]. For $\beta = -1$ in this last theorem we obtain **Theorem 1** from [3].

From **Theorem 1** we can obtain many other results by choosing properly the constants. An interesting example can be obtained if we let $\alpha + \beta = \omega$, $p(z) = 1$ and $g(z) = f(z)[f'(z)]^{1/\beta}$ for all $z \in U$ in **Theorem 1**. For the power we choose the principal branch and obtain:

Corollary 2. *If $f \in A$ and γ, δ and ω are complex numbers satisfying:*

$$\operatorname{Re} \frac{2\gamma}{\omega + 1} > 1 \tag{21}$$

$$\operatorname{Re} \gamma > 0 \ , \ \left| \frac{\delta + 1}{\gamma} - 1 \right| < 1 \ , \ \operatorname{Re} \omega > -1 \tag{22}$$

$$\left| \frac{\delta + 1}{\omega + 1} - 1 \right| < 1 \tag{23}$$

and for all $z \in U$:

$$\left| \frac{1-\gamma}{\gamma} + \frac{\delta}{\gamma} |z|^{2\gamma} + \frac{\omega}{\gamma} (1 - |z|^{2\gamma}) \frac{zf'(z)}{f(z)} + \frac{1 - |z|^{2\gamma}}{\gamma} \frac{zf''(z)}{f'(z)} \right| \leq 1 \tag{24}$$

then f is univalent in u.

If we let in **Corollary 2** $\gamma = 1$ and use also **Remark 1** we obtain a generalization of the well–known criterion of univalence of L.V.Ahlfors and J.Becker ([1], [2]), given in (1):

Corollary 3. *If $f \in A$, δ and $\omega \in \mathbb{C}$ satisfie:*

$$|\delta| < 1 \tag{25}$$

$$|\omega| < 1 \tag{26}$$

$$\frac{\omega - \delta}{\delta + 1} \notin [1, \infty) \tag{27}$$

$$\left| \delta |z|^2 + \omega (1 - |z|^2) \frac{zf'(z)}{f(z)} + (1 - |z|^2) \frac{zf''(z)}{f'(z)} \right| \leq 1 \ , \ z \in U \tag{28}$$

then f is univalent in U.

For $\delta = \omega = 0$ we obtain from **Corollary 3** the criterion of univalence of Ahlfors and Becker.

For $\delta = \omega = (1-\alpha)/\alpha$, conditions (25) and (26) are equivalent to: $\mathrm{Re}\,\alpha > 1/2$ and we obtain the result from [6].

If in **Corollary 2** we let $\omega = 0$ and $\gamma = (m+1)/2$, $m \in \mathbb{R}$ we obtain the result from [7].

References

[1] L.V.Ahlfors, *Sufficient conditions for quasiconformal extension*, Princeton Annales of Math. Studies **79**(1974), 23–29.

[2] J.Becker, *Lőwnersche Differentialgleichung und quasiconform vorsetzbare schlichte Functionen*, J. Reine Angew. Math. **225**(1972), 23–43. MR. 45 8828.

[3] D.Blezu, E.Drăghici, H.Ovesea, N.N.Pascu. *Sufficient conditions for univalence of a large class of functions*, Studia Univ. Babeş–Bolyai (Cluj–Romania), Math. XXXV, **4**, 1990, 41–47.

[4] E.Drăghici, *A generalization of Becker's univalence criterion*, Studia Univ. Babeş–Bolyai (Cluj–Romania), Math. XXXV, **2**, 1990, 41–47.

[5] E.Drăghici, *An improvement of Becker's condition of univalence*, Mathematica (Cluj–Romania), Tome 34(57) No. **2**, 1992, 139–144.

[6] S.Moldoveanu, N.N.Pascu, *A sufficient condition for univalence*, Seminar of Geometric Function Theory, Preprint No. **2**, Braşov–Romania, 1991, 57–62.

[7] N.N.Pascu, V.Pescar, *An integral operator which preserves the univalency*, Seminar of Geometric Function Theory, Preprint, Braşov–Romania, 1991, 91–98.

[8] Ch.Pommerenke, *Univalent functions*, Vanderhoek & Ruprecht, Gőttingen, 1975.

[9] S.Rusheweyh. *An extension of Becker's univalence condition*, Math. Annalen **220**(1976), 285–290.

Department of Mathematics
Faculty Of Sciences
"Lucian Blaga" University of Sibiu
Str Dr. I. Raţiu, nr. 7, et. III
550012 - Sibiu, Romania.
E-mail: *eugen.draghici@ulbsibiu.ro*